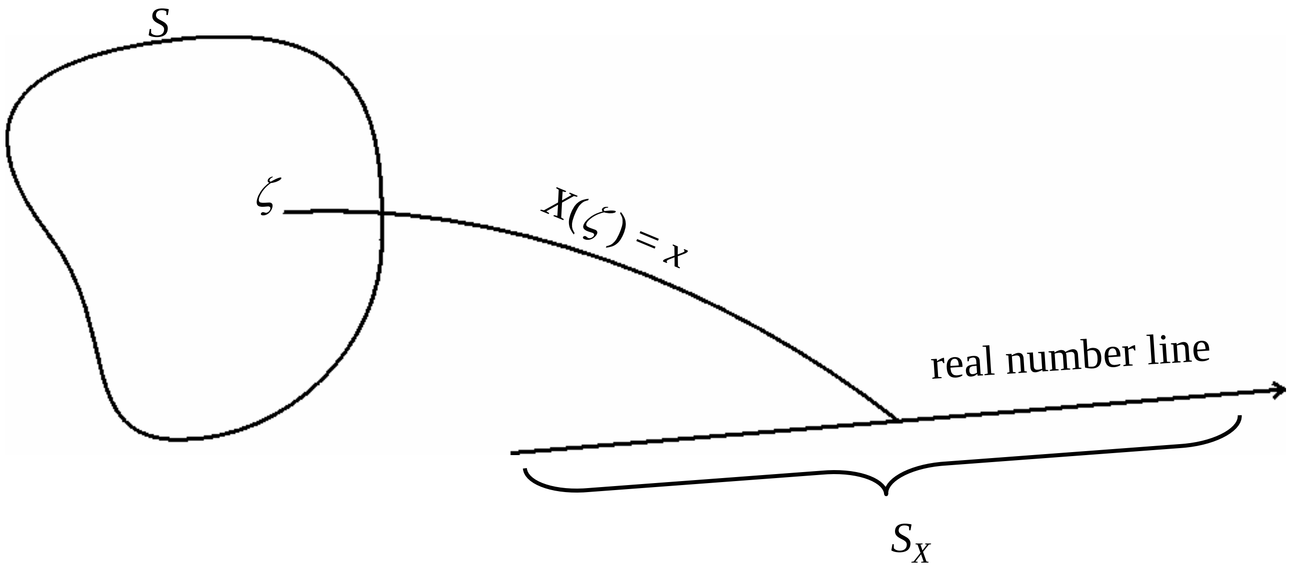
Random variables

Some random experiments may yield a sample space whose elements (events) are numbers, but some do not. For mathematical purposes, it is desirable to have numbers associated with the outcomes.

A *random variable* X is a function that assigns a real number, $X(\zeta)$, to each outcome ζ in the sample space of a random experiment.

The sample space S is the **domain** of the random variable and the set S_X of all values taken on by X is the **range** of the random variable. Note that $S_X \subset \mathsf{R}$, R is set of all real numbers.



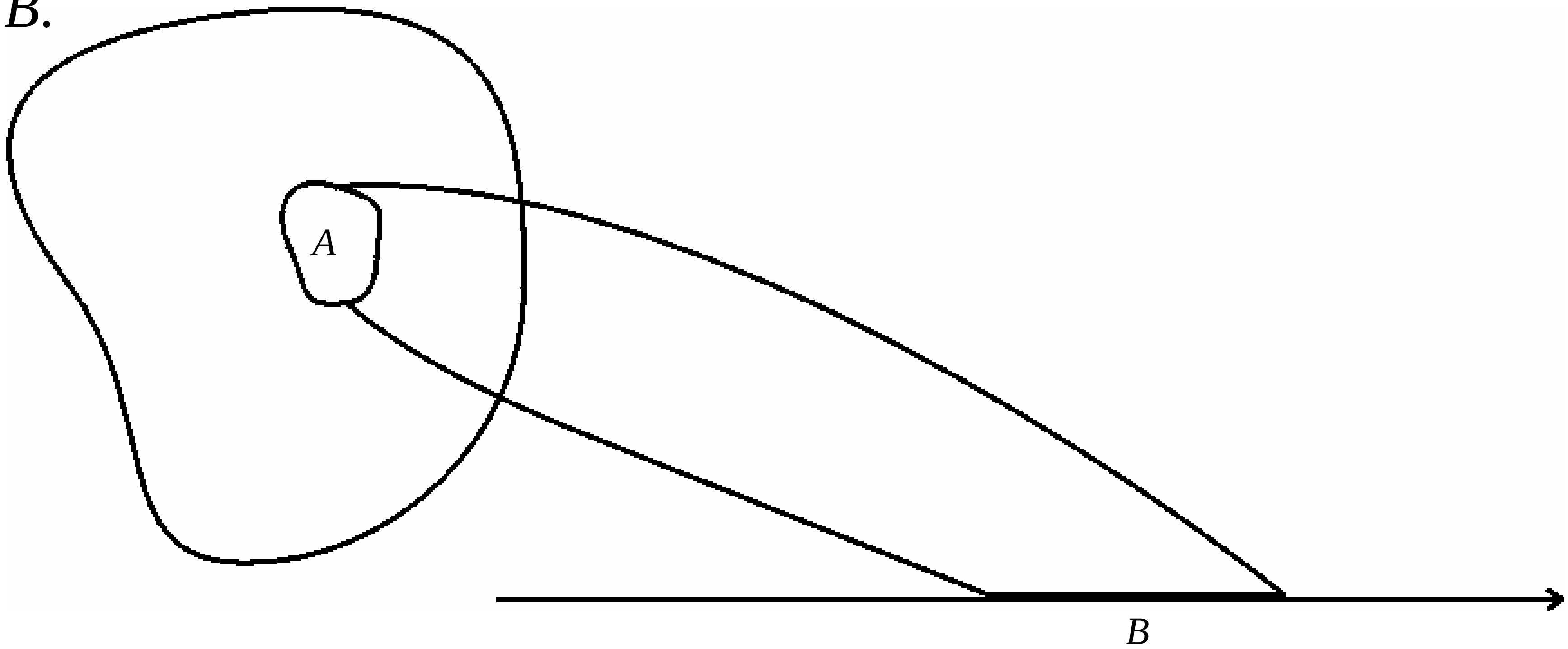
Example A random experiment of tossing 3 fair coins.
Sample space S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.
Let X be the number of heads; then S_X = {0, 1, 2, 3}.

eg.

$$X(THH) = 2; P[X = 0] = \frac{1}{8}, P[X = 1] = \frac{3}{8}, P[X = 2] = \frac{3}{8}, P[X = 3] = \frac{1}{8}.$$

Equivalent events

Let A be the set of outcomes ζ in S that leads to the set of values $X(\zeta)$ in B.



$$A = X^{-1}(B) = \{\zeta \in S : X(\zeta) \in B\}$$

eg. in the above coins tossing example,

$X^{-1}(\{2, 3\})$ $= \{HHT, HTH, THH, HHH\}$

= set of **all** preimages of elements in $B = \{2, 3\}$.

Since event B in S_X occurs whenever event A in S occurs, and vice versa. Hence $P[B] = P[A] = P[\{\zeta: X(\zeta) \text{ in } B\}]$. A and B are called *equivalent events* with respect to X.

If we assign probabilities in this manner, then the probabilities assigned to subsets of the real line will satisfy the three axioms of probability.

1. $P[B] \geq 0$ for all $B \subset S_X$.
2. $P[S_X] = 1$.
3. If B_1 and B_2 are mutually exclusive, then

$$P[B_1 \cup B_2] = P[B_1] + P[B_2].$$

In the tossing coins experiment, we observe

$$P[X \leq 0] = \frac{1}{8}, P[X \leq 1] = \frac{1}{2}, P[X \leq 2] = \frac{7}{8}, P[X \leq 3] = 1.$$

Hence, $P[X \leq x]$ is a number whose value depends on x, and so it is a function of x.

Example

Consider the random experiment of tossing 3 coins

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

X = no of heads in the 3 coins, $S_X = \{0, 1, 2, 3\}$

$A_1 = \{HTT, TTT\}$

$A_2 = \{HHT, HTH, THH, TTT\}$

$A_3 = \{HTT, THT, TTH, TTT\}$

$X(A_1) = \{0, 1\}$ = set of all values taken by $X(\zeta)$, $\zeta \in A_1$

$X(A_2) = \{0, 2\}$

$X^{-1}(\{0, 1\})$ = set of all preimages of elements in $\{0, 1\}$

$= \{HTT, THT, TTH, TTT\} = A_3$.

Note that A_3 and $\{0, 1\}$ are equivalent events since

$$\frac{1}{2} = P[A_3] = P[X = 0 \quad \text{or} \quad X = 1].$$

Note that $A_3 \subset S$ and $\{0, 1\} \subset S_X$.

Consider another random variable:

$$Y = \text{number of heads} - \text{number of tails}$$

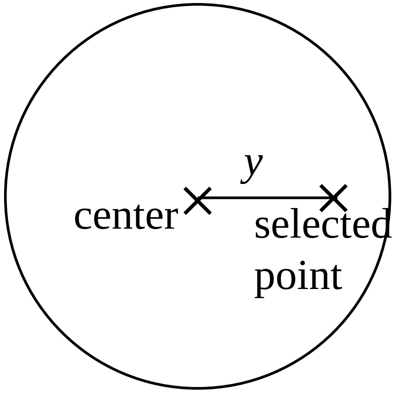
then Y can assume the values –3, –1, 1 and 3.

Now, $Y^{-1}(\{-3, -1\}) = \{TTT, HTT, THT, TTH\}$, so $\{TTT, HTT, THT, TTH\}$, and $\{-3, -1\}$ are equivalent events.

Example

A point is selected at random from inside the unit circle centered at the origin. Let Y be the random variable representing the distance of the point from the origin.

(a) $S_Y = \{y: 0 \le y \le 1\}$ = range of Y.



(b) The equivalent event in the sample space S for the event $\{Y \le y\}$ in S_Y is that the selected point falls inside the region centered at the origin and with radius y.

(c) $P[Y \le y]$ $(0 \le y \le 1)$

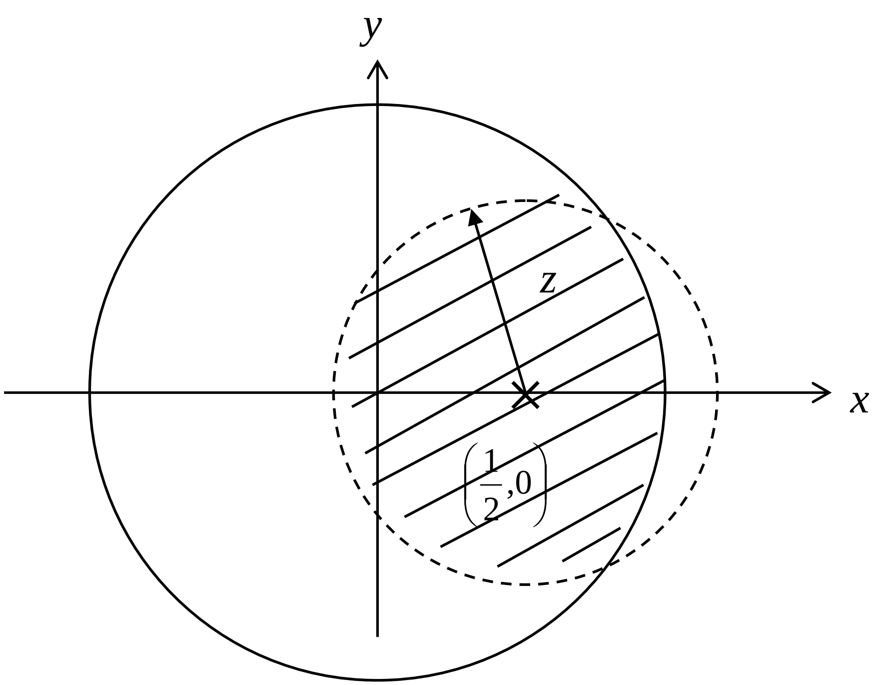
= probability of selecting a point inside the unit circle, and whose distance is less than or equal to y $= \dfrac{\pi y^2}{\pi} = y^2$.

Let Z be the random variable representing the distance of the selected point from $\left(\frac{1}{2},0\right)$.

(a) $S_Z = \left\{z : 0 \le z \le \frac{3}{2}\right\}$

(b) The equivalent event in S for the event $\{Z \le z\}$ is the region formed by the intersection of the circles:

$$\begin{cases} x^2 + y^2 \le 1 \\ \left(x - \frac{1}{2}\right)^2 + y^2 \le z^2 \end{cases}$$



Cumulative distribution function (cdf)

The cdf of a random variable X is defined as

$$F_X(x) = P[X \le x], \qquad -\infty < x < \infty.$$

Axioms of probability $\Rightarrow$ following properties of cdf

1. $0 \le F_X(x) \le 1$

2. $\lim_{x\to\infty} F_X(x) = 1$ (sure event)

3. $\lim_{x\to-\infty} F_X(x) = 0$ (impossible event)

4. $F_X(x)$ is a *non-decreasing function* of x

This is obvious since for $x_2 > x_1$, we have

$$P[X \le x_1] \le P[X \le x_2].$$

5. $F_X(x)$ is continuous from the right i.e. for $h > 0$

$$F_X(b) = \lim_{h \to 0^+} F_X(b+h) = F_X(b^+)$$

Example The tossing coins experiment again, where X = number of heads appearing in tossing 3 coins.

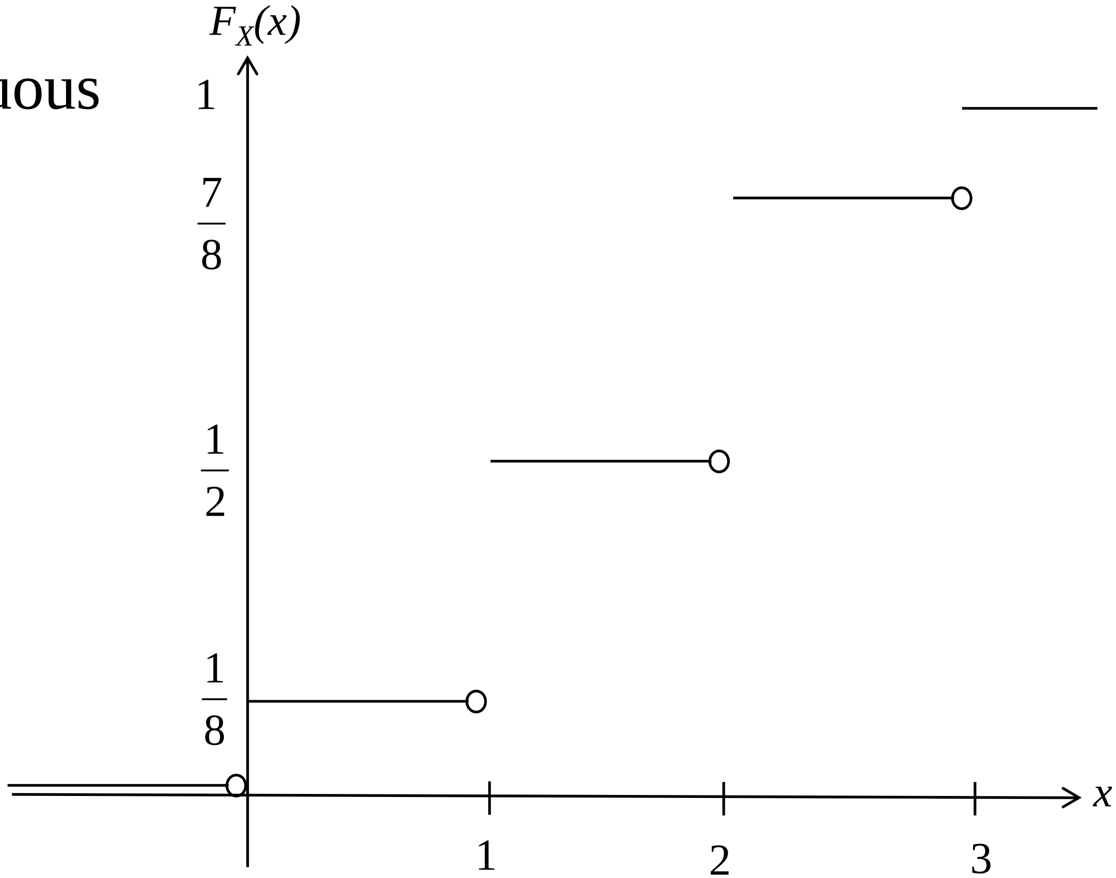
Take $h > 0$ and $h \to 0^+$,

$$F_X(1-h) = P[X \le 1-h] = P\{0 \text{ head}\} = \frac{1}{8}$$

$$F_X(1) = P[X \le 1] = P\{0 \text{ or } 1 \text{ head}\} = \frac{1}{2}$$

$$F_X(1+h) = P[X \le 1+h] = \frac{1}{2}.$$

Hence, the cdf of X is continuous from the right.



Define the unit step function:

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0, \end{cases}$$

$$F_X(x) = \frac{1}{8}u(x) + \frac{3}{8}u(x-1) + \frac{3}{8}u(x-2) + \frac{1}{8}u(x-3).$$

The jump at $x = 0$ is given by $P[X = 0]$, and similarly, for the jump at $x = 1$, 2 and 3.

6. $P[a < X \le b] = F_X(b) - F_X(a)$
since $\{X \le a\} \cup \{a < X \le b\} = \{X \le b\}$,
and $\{X \le a\}$ and $\{a < X \le b\}$ are mutually exclusive
so $F_X(a) + P[a < X \le b] = F_X(b)$.

Suppose we take $a = b - h$, $h > 0$,

$P[b - h < X \le b] = F_X(b) - F_X(b - h)$.

As $h \to 0^+$, $P[X = b] = F_X(b) - F_X(b^-)$.

The probability that X takes on the special value b is given by the magnitude of the jump of the cdf $F_X(x)$ at b.

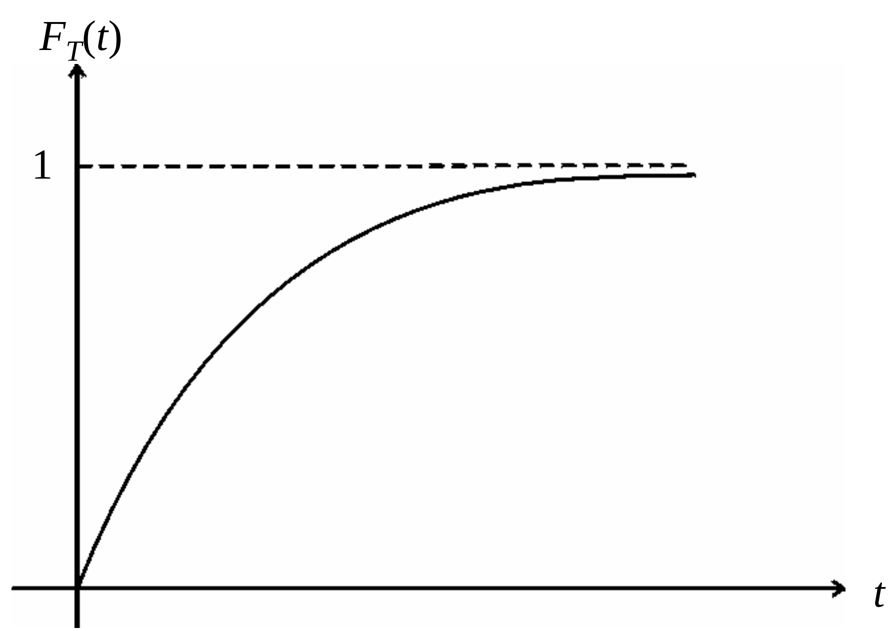
- If the cdf is continuous at b, then the event $\{X = b\}$ has probability zero (essentially).
- If the cdf is continuous at $x = a$ and $x = b$, then $P[a < X < b]$, $P[a \le X < b]$, $P[a < X \le b]$, $P[a \le X \le b]$ have the same value.

7. $P[X > x] = 1 - F_X(x)$.

Example Let T be the random variable which equals the life of a diode. Suppose the cdf of T takes the form

$$F_T(t) = P[T \le t]$$
$$= \begin{cases} 0 & t < 0 \\ 1 - e^{-\mu t} & t \ge 0 \end{cases} = u(t)(1 - e^{-\mu t}),$$

then the probability that the diode fails between times a and b is $P[a < T \le b] = P[T \le b] - P[T \le a] = e^{-\mu a} - e^{-\mu b}$.



Three types of random variables

1. *Discrete random variable*

The cdf is a right-continuous, staircase function of x with jumps at a countable set of points x_1, x_2, …

$$F_X(x) = \sum_k P_X(x_k)u(x - x_k)$$

where $P_X(x_k) = P[X = x_k]$ gives the magnitude of the jump at $X = x_k$ in the cdf.

2. *Continuous random variable*

The cdf $F_X(x)$ is continuous everywhere, so

$$P[X = x] = 0 \text{ for all } x.$$

3. *Random variable of mixed type*

The cdf has jumps on a countable set of points and also increases continuously over at least one interval of values of x

$$F_X(x) = p\,\underbrace{F_1(x)}_{\substack{\text{cdf of a}\\\text{discrete random}\\\text{variable}}} + (1-p)\,\underbrace{F_2(x)}_{\substack{\text{cdf of a}\\\text{continuous random}\\\text{variable}}}, \qquad 0 < p < 1.$$

Example Let X be the time instant that a customer in a queue is being served. We have: X is zero if the system is idle and exponentially distributed if the system is busy.

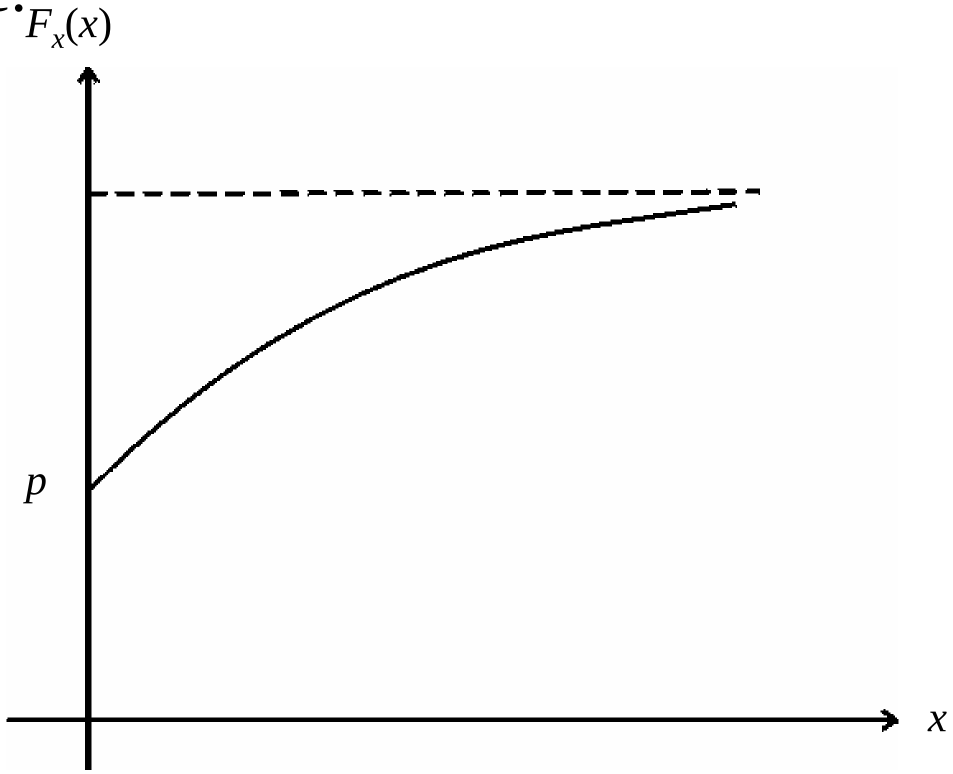
$$P[X \le x] = P[X \le x \mid \text{idle}]P[\text{idle}] + P[X \le x \mid \text{busy}]P[\text{busy}]$$

p = probability that the system is idle.

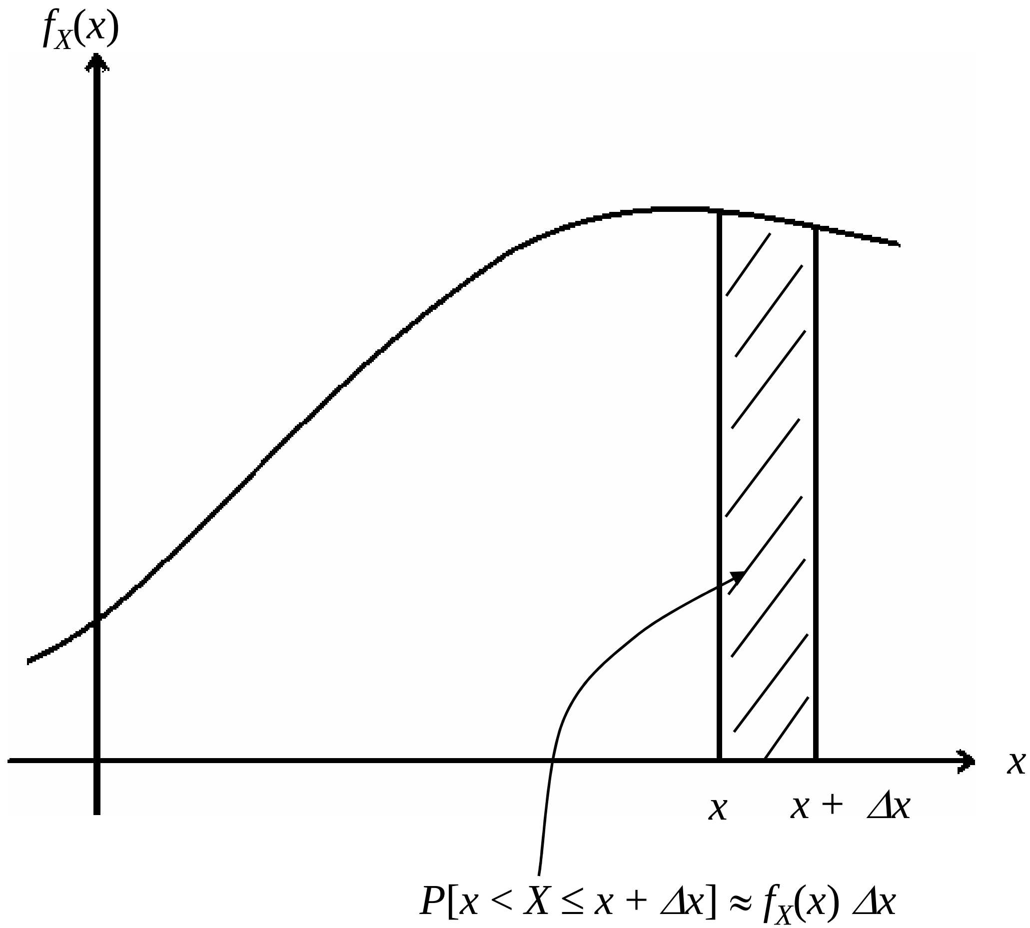
$$X = pX_{\text{idle}} + (1-p)X_{\text{busy}}.$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ p + (1-p)(1-e^{-\lambda x}) & x \ge 0 \end{cases}$$

$$= pu(x) + (1-p)u(x)(1-e^{-\lambda x})$$



X_{idle} is a discrete random variable with $P[X_{\text{idle}} = 0] = 1$ so that $F_{X_{\text{idle}}}(x) = u(x)$; X_{busy} is continuous with $F_{X_{\text{busy}}}(x) = u(x)\,(1- e^{-\lambda x})$.



Probability density function

pdf, if exists, is defined as $f_X(x) = \dfrac{dF_X(x)}{dx}$

since

$$P[x < X \leq x + \Delta x]$$

$$= F_X(x+\Delta x) - F_X(x) = \frac{F_X(x+\Delta x) - F_X(x)}{\Delta x}\Delta x.$$

Properties of pdf

1. $f_X(x) \geq 0$ since cdf is a non-decreasing function of x

2. $F_x(x) = \int_{-\infty}^{x} f_X(t)\, dt$

   Proof: From $f_X(x) = \dfrac{d}{dx} F_X(x),$ we obtain

$$F_X(x) = \int_{c}^{x} f_X(t)\, dt.$$

   The constant c is determined by $F_X(-\infty) = 0$, given $c = -\infty$.

3. $P[a < X \leq b] = \int_{a}^{b} f_X(t)\, dt$

   Proof:

$$P[a < X \leq b] = F_X(b) - F_X(a)$$

$$= \int_{-\infty}^{b} f_X(t)\, dt - \int_{-\infty}^{a} f_X(t)\, dt = \int_{a}^{b} f_X(t)\, dt$$

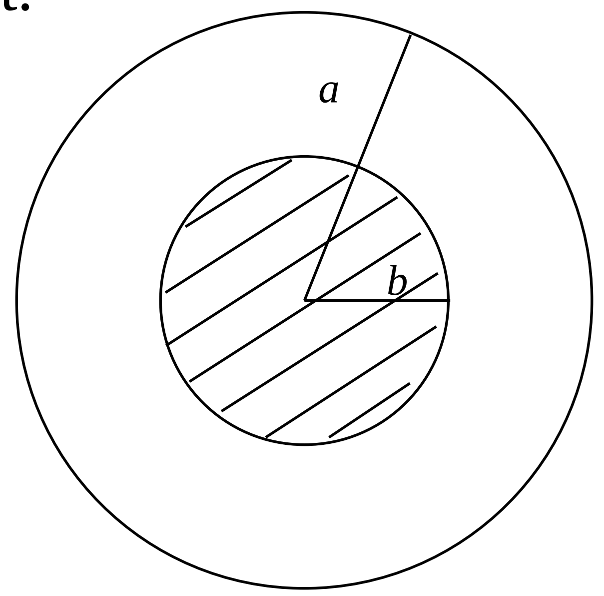
4. $1 = \int_{-\infty}^{\infty} f_X(t)\, dt$

Example Let radius of bull-eye = b and radius of target = a.

Probability of the dart striking between r and $r + dr$ is

$$P[r \le R \le r + dr] = C\left[1 - \left(\frac{r}{a}\right)^2\right] dr.$$

R = distance of hit from the center of the target.



The density function takes the form

$$f_R(r) = C\left[1 - \left(\frac{r}{a}\right)^2\right].$$

How to determine C?

Assume that the target is always hit:

$$C\int_0^a 1 - \left(\frac{r}{a}\right)^2 dr = 1 \Rightarrow C = \frac{3}{2a},$$

probability of hitting bull-eye $= P[0 \le R \le b] = \int_0^b f_R(r)\, dr = \dfrac{b(3a^2 - b^2)}{2a^3}.$

pdf for a discrete random variable

The delta function $\delta(x)$ is related to $u(x)$ via

$$\delta(x) = \frac{d}{dx} u(x) \text{ or } u(x) = \int_{-\infty}^{x} \delta(t)\, dt. \text{ Note that } \int_{-\infty}^{\infty} \delta(t) dt = 1.$$

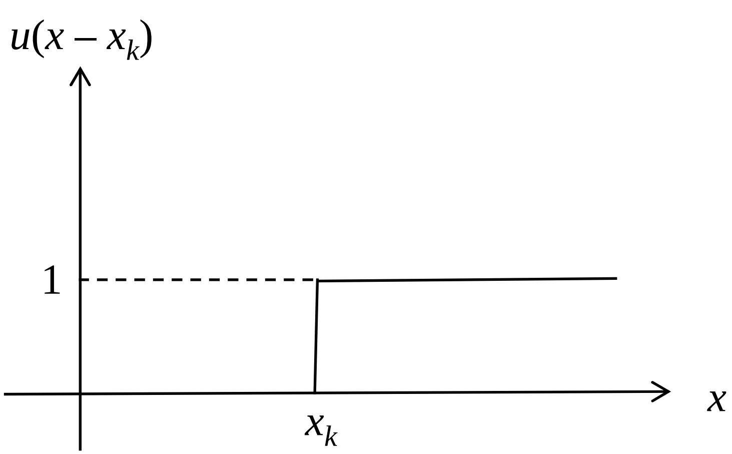
Recall that

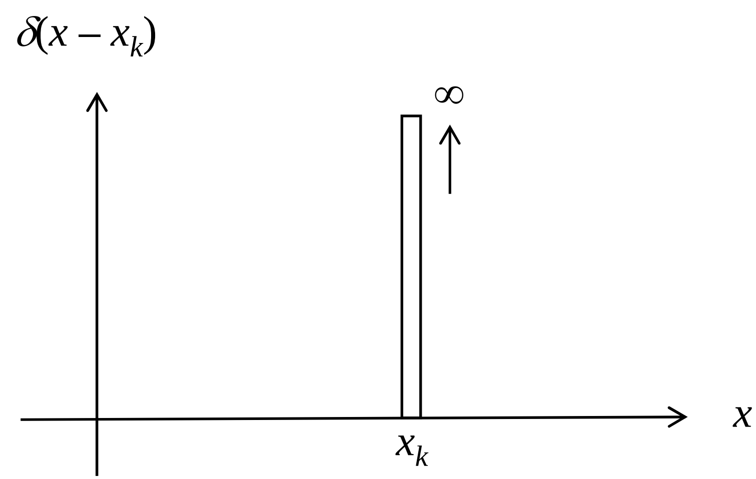
$$F_X(x) = \sum_k P_X(x_k) u(x - x_k)$$

probability mass function

According to $F_X(x) = \int_{-\infty}^{x} f_x(t)\, dt,$ we then have

$$f_X(x) = \sum_k P_X(x_k) \delta(x - x_k), \text{ where } \delta(x - x_k) = \begin{cases} \infty & \text{when } x = x_k \\ 0 & \text{otherwise} \end{cases}.$$





Example The coins tossing experiment

cdf:

$$F_X(x) = \frac{1}{8}u(x) + \frac{3}{8}u(x-1) + \frac{3}{8}u(x-2) + \frac{1}{8}u(x-3).$$

pdf:

$$f_X(x) = \frac{1}{8}\delta(x) + \frac{3}{8}\delta(x-1) + \frac{3}{8}\delta(x-2) + \frac{1}{8}\delta(x-3).$$

$$P[1 < X \le 2] = \int_{1^+}^{2} f_X(x)\,dx = P[X = 2] = \frac{3}{8}.$$

Note that the delta function located at 1 is excluded but the delta function located at 2 is included.

Similarly,

$$P[2 \le X < 3] = \int_{2}^{3^-} f_X(x)\,dx = P[X = 2] = \frac{3}{8}.$$

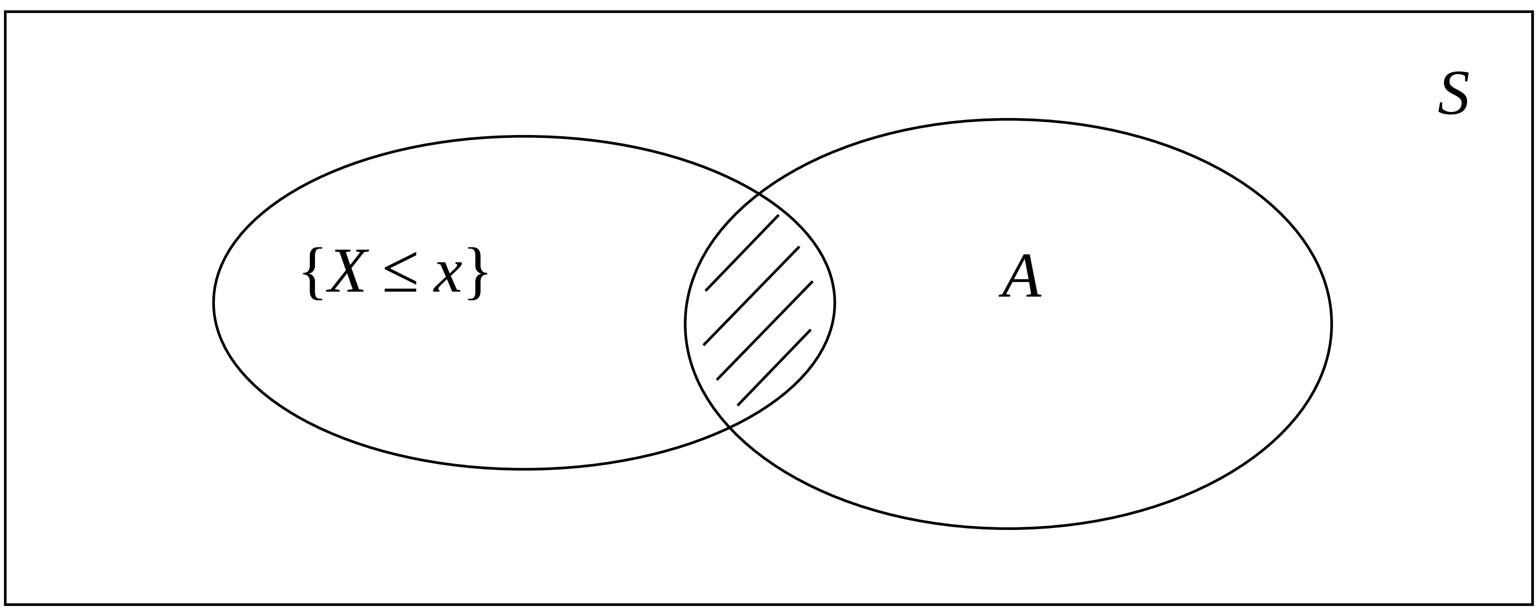
Conditional cdf of X given A

$$F_X(x \mid A) = \frac{P[\{X \le x\} \cap A]}{P[A]} \quad \text{if} \quad P[A] > 0.$$

- cdf of X with reference to the reduced sample space A

Conditional pdf of X given A

$$f_X(x \mid A) = \frac{d}{dx} F_X(x \mid A).$$



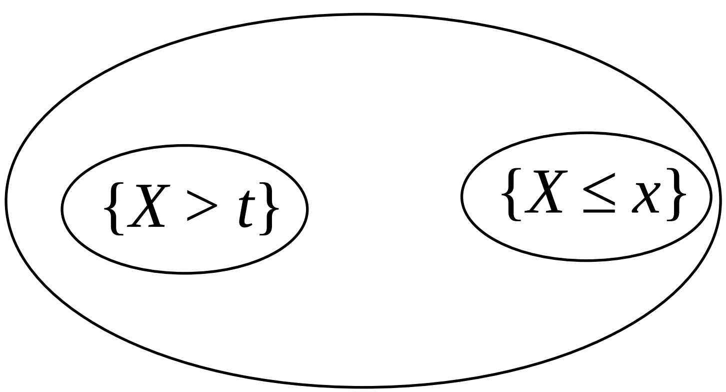
Example

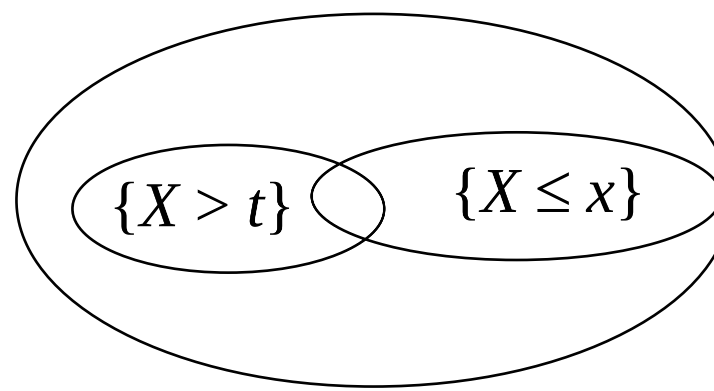
The lifetime X of a machine has a continuous cdf, $F_X(x)$. Find the conditional cdf and pdf given the event $A = \{X > t\}$, that is, the machine is still working at time t.

Conditional cdf

$$F_X(x \mid X > t) = P[X \le x \mid X > t]$$
$$= \frac{P[\{X \le x\} \cap \{X > t\}]}{P[X > t]}.$$

(i) $x \le t$ (ii) $x > t$





Note that

$$\{X \le x\} \cap \{X > t\} = \begin{cases} \phi & x \le t \\ \{t < X \le x\} & x > t \end{cases},$$

so

$$F_X(x \mid X > t) = \begin{cases} 0 & x \le t \\ \dfrac{F_X(x) - F_X(t)}{1 - F_X(t)} & x > t \end{cases}.$$

Conditional pdf is found by differentiating $F_X(x \mid X > t)$ with respect to x

$$f_X(x \mid X > t) = \begin{cases} 0 & x \le t \\ \dfrac{f_X(x)}{1 - F_X(t)} & x > t \end{cases}.$$

Note that $F_X(x \mid X>t)$ is continuous at $x = t$, but $f_X(x \mid X>t)$ has a jump at $x = t$.

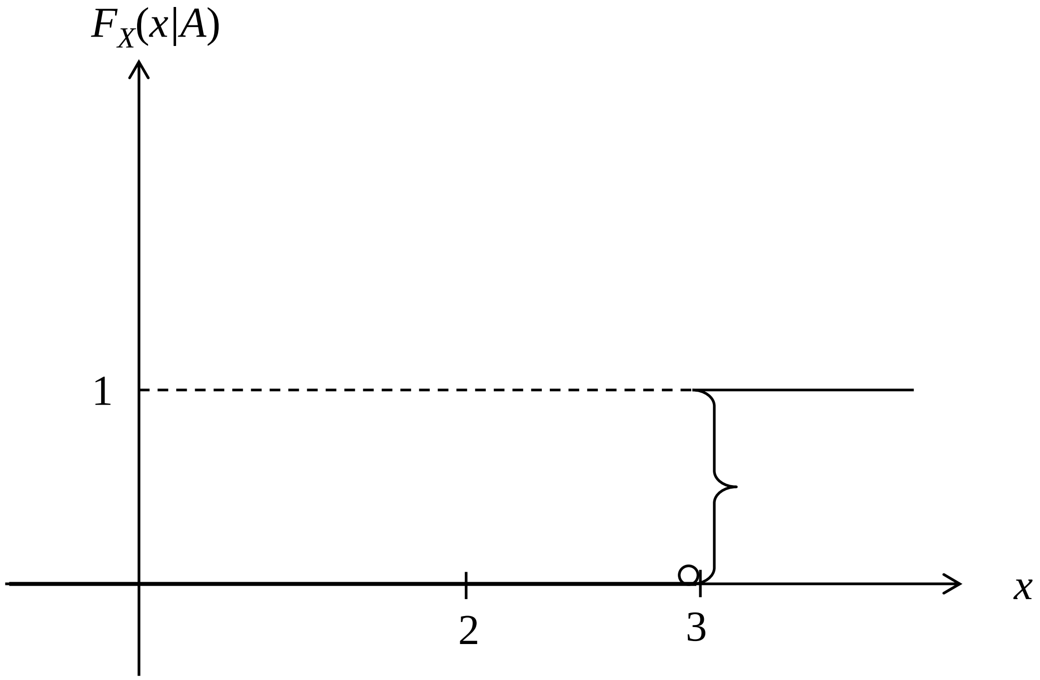
Example Tossing of 3 coins; X = number of heads; $A = \{X > 2\}$.

$$F_X(x \mid A) = F_X(x \mid X > 2)$$

$$= \frac{P[\{X \leq x\} \cap \{X > 2\}]}{P[X > 2]}$$

$$= \begin{cases} 0 & \text{if} \quad x \leq 2 \\ \dfrac{P[2 < X \leq x]}{1 - P[X \leq 2]} & \text{if} \quad x > 2 \end{cases}$$

$$= \begin{cases} 0 & \text{if} \quad x < 3 \\ 1 & \text{if} \quad x \geq 3 \end{cases}.$$



Note that $P[X > 2] = P[X = 3] = \dfrac{1}{8}$ and

$$P[2 < X \leq x] = \begin{cases} 0 & \text{if} \quad x < 3 \\ \dfrac{1}{8} & \text{if} \quad x \geq 3 \end{cases}.$$