Poisson Random Variable

Counting the number of occurrences of an event in a certain time period or a certain region in space, e.g. counts of emissions from radioactive substances.

The pmf for a Poisson random variable N is

$$P[N = k] = \frac{\alpha^k}{k!} e^{-\alpha}, \quad k = 0, 1, 2, \cdots$$

where α is the average number of event occurrences in a specified time interval or region in space.

Note that $\sum_{k=0}^{\infty} \frac{\alpha^k}{k!} e^{-\alpha} = 1$.

Approximation of binomial probabilities by Poisson probabilities

When n is large and p is small and $\alpha = np$ is finite, then

$$p_k = {}_nC_k p^k (1-p)^{n-k} \approx \frac{\alpha^k}{k!} e^{-\alpha}, \quad k = 0, 1, 2, \cdots.$$

Proof Consider

$$\begin{aligned} \frac{p_{k+1}}{p_k} &= \frac{k!(n-k)!p}{(k+1)!(n-k-1)!q} = \frac{(n-k)p}{(k+1)q}, \quad \text{(with } q = 1-p\text{)} \\ &= \frac{\left(1-\frac{k}{n}\right)\alpha}{(k+1)\left(1-\frac{\alpha}{n}\right)} \longrightarrow \frac{\alpha}{k+1} \quad \text{as } n \to \infty. \end{aligned}$$

Hence, the limiting probabilities satisfy

$$p_{k+1} = \frac{\alpha}{k+1} p_k \quad k = 0, 1, 2, \cdots$$

$$p_1 = \frac{\alpha}{1} p_0, \quad p_2 = \frac{\alpha}{2} p_1 = \frac{\alpha^2}{2!} p_0, \cdots, p_k = \frac{\alpha^k}{k!} p_0, \quad k = 0, 1, 2 \cdots.$$

To determine p_0, we use $\sum_{k=0}^{\infty} p_k = p_0 \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} = p_0 e^{\alpha} = 1$, so that $p_0 = e^{-\alpha}$.

Example

Suppose 300 misprints are distributed randomly throughout a book of 500 pages. Find the probability that a given page contains exactly 2 misprints.

Solution

We view the number of misprints on one page as the number of successes in a sequence of Bernoulli trials. Here, there are 300 misprints; so number of trials $n = 300$. Let p be the probability that a particular misprint (considered as a particular trial) falls on a given page.

Here, $p = 1/500$ and $\alpha = np = 0.6$.

Hence, $P[N = 2] = \dfrac{(0.6)^2 e^{-0.6}}{2!} = \dfrac{(0.36)(0.549)}{2} \approx 0.0988.$

Extension of the question

Suppose we assume M misprints to be randomly distributed on N pages, find the probability that k misprints occur on ℓ particular pages.

Solution

Consider ℓ particular pages among N pages ($\ell \ll N$), the probability that a given misprint falls onto these ℓ pages is ℓ/N. This is one trial experiment associated with one misprint. There are M misprints which are falling on these ℓ particular pages at random. This can be treated as M trials.

In the terminology of binomial experiment, number of trials is $n = M$ and probability of success is $p = \ell/N$. The corresponding average number of occurrences is $\alpha = np = \dfrac{M\ell}{N}$. We then have

$$P[X = k] = e^{-\alpha}\frac{\alpha^k}{k!},$$

where $X =$ number of misprints on ℓ pages.

Example (White blood-cell count)

The white blood-cell count of a healthy individual can average as low as $6,000$ per mm^3. To detect a white blood cell deficiency, a 0.001 mm^3 drop of blood is taken and the number of white blood cells X is found.

Viewed as a Poisson process: discrete event of interest is the occurrence of a white cell.

Query

How to find the parameter α, which in this case corresponds to the average number of white blood cells within a droplet of blood of volume 0.001mm^3?

Here, $\alpha = 6,000 \times 0.001 = 6$, that is a healthy individual should be expected to have 6 white cells on average in one drop of blood.

Suppose in the test, not more than two white blood cells are found, is there evidence of a white-cell deficiency? Consider the probability of not having more than 2 white blood cells of a healthy individual:

$$P[X \leq 2] = \sum_{k=0}^{2} \frac{e^{-6}6^k}{k!} = e^{-6} + \frac{e^{-6} \cdot 6^1}{1} + \frac{e^{-6} \cdot 6^2}{2} = 0.062.$$

The probability of mis-judgement of a healthy individual to have a white-cell deficiency is 6.2%.

Variance of a Poisson random variable (with parameter α)

$$\begin{aligned} E[N] &= \sum_{k=0}^{\infty} k P_N(k) = \sum_{k=0}^{\infty} k \frac{e^{-\alpha}\alpha^k}{k!} = e^{-\alpha} \sum_{k=1}^{\infty} \frac{\alpha(\alpha^{k-1})}{(k-1)!} \\ &= e^{-\alpha}\alpha \underbrace{\sum_{\ell=0}^{\infty} \frac{\alpha^\ell}{\ell!}}_{e^\alpha} = \alpha, \quad \ell = k-1. \end{aligned}$$

$$\sum_{k=0}^{\infty} k(k-1)P_N(k) = \sum_{k=2}^{\infty} \frac{e^{-\alpha}\alpha^{k-2}\alpha^2}{(k-2)!} = \alpha^2 e^{-\alpha} \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} = \alpha^2, m = k-2.$$

Note that $E[N^2] - E[N] = \sum_{k=0}^{\infty} k^2 P_N(k) - \sum_{k=0}^{\infty} k P_N(k)$.

$$\begin{aligned} \text{VAR}(N) &= E[N^2] - E[N]^2 = (E[N^2] - E[N]) + (E[N] - E[N]^2) \\ &= \alpha^2 + \alpha - \alpha^2 = \alpha. \end{aligned}$$

Interestingly, the mean and variance of N have the same value, namely, α.

Maximum of $P[N=k]$ when N is a Poisson random variable

(1) For $\alpha < 1, P[N=k]$ is a maximum at $k=0$.

(2) For $\alpha \geq 1, P[N=k]$ is maximum at floor(α).

If α is a positive integer, then $P[N=k]$ is maximum at both $k=\alpha$ and $k=\alpha-1$.

Proof

The ratio $P[N=k]/P[N=k-1]=\alpha/k$, which decreases with increasing k. For $\alpha < 1, \alpha/k < 1$ for $k \leq 1$ so that $P[N=k]$ attains its maximum value at $k=0$. Suppose $\alpha \geq 1$ and α is not an integer, then the ratio is greater than 1 at $k=$ floor(α) and becomes less than 1 at floor(α) $+ 1$. When α happens to be an integer, $P[N=\alpha]/P[N=\alpha-1]=1$.

Example

Requests for telephone connection arrive at a rate of λ calls per second. It is known that the number of requests during a time period of t seconds is a Poisson random variable.

Solution

The average number of call requests in a t-second period is $\alpha = \lambda t$.

$$\begin{aligned} P[N(t) = 0] &= \frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-\lambda t} \\ P[N(t) \geq n] &= 1 - P[N(t) < n] \\ &= 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}. \end{aligned}$$

Waiting time

Find the probability density of the random variable Y, the *waiting time* until the first success. Here, the number of successes is a value of the discrete random variable N having the Poisson distribution with $\alpha = \lambda y$, $\lambda =$ number of occurrences per unit time.

$$
\begin{aligned}
F_Y(y) &= P[Y \leq y] = 1 - P[Y > y] \\
&= 1 - P[\text{zero success in a time interval of } y] \\
&= 1 - e^{-\lambda y} \quad \text{for } y \geq 0; \\
F_Y(y) &= 0 \quad \text{for } y < 0 \\
f_Y(y) &= \begin{cases} \lambda e^{-\lambda y} & y \geq 0 \\ 0 & y < 0 \end{cases} .
\end{aligned}
$$

Example

At a certain location on a highway, the number of cars exceeding the speed limit per half hour is 8.4. What is the probability of a waiting time of less than 5 minutes between cars exceeding the speed limit?

Solution

Using half an hour as one unit of time.

Now 5 minutes $= 1/6$ of one unit of time, and $f_Y(y) = 8.4e^{-8.4y}, \quad \lambda = 8.4$.

$$\text{Required probability} = P\left[0 \leq Y \leq \frac{1}{6}\right] = \int_0^{1/6} 8.4e^{-8.4y}\, dy$$

$$= 1 - e^{-1.4} \approx 0.75.$$