Poisson Random Variable

Counting the number of occurrences of an event in a certain time period or a certain region in space, e.g. counts of emissions from radioactive substances.

The pmf for a Poisson random variable $N$ is

$$P[N = k] = \frac{\alpha^k}{k!}e^{-\alpha}, \quad k = 0, 1, 2, \ldots$$

where $\alpha$ is the average number of event occurrences in a specified time interval or region in space.

Note that $\sum_{k=0}^{\infty} \frac{\alpha^k}{k!}e^{-\alpha} = 1$. 
Approximation of binomial probabilities by Poisson probabilities

When \( n \) is large and \( p \) is small and \( \alpha = np \) is finite, then

\[
p_k = nC_k p^k (1-p)^{n-k} \approx \frac{\alpha^k}{k!} e^{-\alpha}, \quad k = 0, 1, 2, \ldots.
\]

**Proof** Consider

\[
\frac{p_{k+1}}{p_k} = \frac{k!(n-k)!p}{(k+1)!(n-k-1)!q} = \frac{(n-k)p}{(k+1)q}, \quad \text{(with } q = 1 - p)\]

\[
= \frac{(1 - \frac{k}{n}) \alpha}{(k+1) \left(1 - \frac{\alpha}{n}\right)} \longrightarrow \frac{\alpha}{k + 1} \quad \text{as } n \to \infty.
\]

Hence, the limiting probabilities satisfy

\[
p_{k+1} = \frac{\alpha}{k + 1} p_k \quad k = 0, 1, 2, \ldots
\]

\[
p_1 = \frac{\alpha}{1} p_0, \quad p_2 = \frac{\alpha}{2} p_1 = \frac{\alpha^2}{2!} p_0, \ldots, p_k = \frac{\alpha^k}{k!} p_0, \quad k = 0, 1, 2, \ldots.
\]

To determine \( p_0 \), we use

\[
\sum_{k=0}^{\infty} p_k = p_0 \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} = p_0 e^\alpha = 1, \quad \text{so that } p_0 = e^{-\alpha}.
\]
Example

Suppose 300 misprints are distributed randomly throughout a book of 500 pages. Find the probability that a given page contains exactly 2 misprints.

Solution

We view the number of misprints on one page as the number of successes in a sequence of Bernoulli trials. Here, there are 300 misprints; so number of trials $n = 300$. Let $p$ be the probability that a particular misprint (considered as a particular trial) falls on a given page.

Here, $p = 1/500$ and $\alpha = np = 0.6$.

Hence, $P[N = 2] = \frac{(0.6)^2e^{-0.6}}{2!} = \frac{(0.36)(0.549)}{2} \approx 0.0988$. 
Extension of the question

Suppose we assume $M$ misprints to be randomly distributed on $N$ pages, find the probability that $k$ misprints occur on $\ell$ particular pages.

Solution

Consider $\ell$ particular pages among $N$ pages ($\ell \ll N$), the probability that a given misprint falls onto these $\ell$ pages is $\ell/N$. This is one trial experiment associated with one misprint. There are $M$ misprints which are falling on these $\ell$ particular pages at random. This can be treated as $M$ trials.

In the terminology of binomial experiment, number of trials is $n = M$ and probability of success is $p = \ell/N$. The corresponding average number of occurrences is $\alpha = np = \frac{M\ell}{N}$. We then have

$$P[X = k] = e^{-\alpha} \frac{\alpha^k}{k!},$$

where $X =$ number of misprints on $\ell$ pages.
Example (White blood-cell count)

The white blood-cell count of a healthy individual can average as low as 6,000 per mm$^3$. To detect a white blood cell deficiency, a 0.001 mm$^3$ drop of blood is taken and the number of white blood cells $X$ is found.

Viewed as a Poisson process: discrete event of interest is the occurrence of a white cell.

Query

How to find the parameter $\alpha$, which in this case corresponds to the average number of white blood cells within a droplet of blood of volume 0.001mm$^3$?
Here, \( \alpha = 6,000 \times 0.001 = 6 \), that is a healthy individual should be expected to have 6 white cells on average in one drop of blood.

Suppose in the test, not more than two white blood cells are found, is there evidence of a white-cell deficiency? Consider the probability of not having more than 2 white blood cells of a healthy individual:

\[
P[X \leq 2] = \sum_{k=0}^{2} \frac{e^{-6} 6^k}{k!} = e^{-6} + \frac{e^{-6} \cdot 6^1}{1} + \frac{e^{-6} \cdot 6^2}{2} = 0.062.
\]

The probability of mis-judgement of a healthy individual to have a white-cell deficiency is 6.2%.
Variance of a Poisson random variable (with parameter $\alpha$)

$$E[N] = \sum_{k=0}^{\infty} kP_N(k) = \sum_{k=0}^{\infty} ke^{-\alpha}\frac{\alpha^k}{k!} = e^{-\alpha} \sum_{k=1}^{\infty} \frac{\alpha^{k-1}}{(k-1)!}$$

$$= e^{-\alpha}\alpha \sum_{\ell=0}^{\infty} \frac{\alpha^{\ell}}{\ell!} = \alpha, \quad \ell = k - 1.$$

$$\sum_{k=0}^{\infty} k(k-1)P_N(k) = \sum_{k=2}^{\infty} \frac{e^{-\alpha}\alpha^{k-2}\alpha^2}{(k-2)!} = \alpha^2 e^{-\alpha} \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} = \alpha^2, m = k - 2.$$

Note that $E[N^2] - E[N] = \sum_{k=0}^{\infty} k^2P_N(k) - \sum_{k=0}^{\infty} kP_N(k)$.


$$= \alpha^2 + \alpha - \alpha^2 = \alpha.$$

Interestingly, the mean and variance of $N$ have the same value, namely, $\alpha$. 
Maximum of $P[N = k]$ when $N$ is a Poisson random variable

(1) For $\alpha < 1$, $P[N = k]$ is a maximum at $k = 0$.

(2) For $\alpha \geq 1$, $P[N = k]$ is maximum at floor($\alpha$).

If $\alpha$ is a positive integer, then $P[N = k]$ is maximum at both $k = \alpha$ and $k = \alpha - 1$.

Proof

The ratio $P[N = k]/P[N = k - 1] = \alpha/k$, which decreases with increasing $k$. For $\alpha < 1$, $\alpha/k < 1$ for $k \leq 1$ so that $P[N = k]$ attains its maximum value at $k = 0$. Suppose $\alpha \geq 1$ and $\alpha$ is not an integer, then the ratio is greater than 1 at $k = \text{floor}(\alpha)$ and becomes less than 1 at floor($\alpha$) + 1. When $\alpha$ happens to be an integer, $P[N = \alpha]/P[N = \alpha - 1] = 1$. 
Example

Requests for telephone connection arrive at a rate of $\lambda$ calls per second. It is known that the number of requests during a time period of $t$ seconds is a Poisson random variable.

Solution

The average number of call requests in a $t$-second period is $\alpha = \lambda t$.

\[
P[N(t) = 0] = \frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-\lambda t}
\]

\[
P[N(t) \geq n] = 1 - P[N(t) < n] = 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}.
\]
Waiting time

Find the probability density of the random variable $Y$, the waiting time until the first success. Here, the number of successes is a value of the discrete random variable $N$ having the Poisson distribution with $\alpha = \lambda y$, $\lambda =$ number of occurrences per unit time.

$$F_Y(y) = P[Y \leq y] = 1 - P[Y > y]$$
$$= 1 - P[\text{zero success in a time interval of } y]$$
$$= 1 - e^{-\lambda y} \quad \text{for } y \geq 0;$$

$$F_Y(y) = 0 \quad \text{for } y < 0$$

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & y \geq 0 \\ 0 & y < 0 \end{cases}.$$
Example

At a certain location on a highway, the number of cars exceeding the speed limit per half hour is 8.4. What is the probability of a waiting time of less than 5 minutes between cars exceeding the speed limit?

Solution

Using half an hour as one unit of time.

Now 5 minutes = $1/6$ of one unit of time, and $f_Y(y) = 8.4e^{-8.4y}, \quad \lambda = 8.4$.

Required probability $= P\left[0 \leq Y \leq \frac{1}{6}\right] = \int_0^{1/6} 8.4e^{-8.4y} \, dy$

$= 1 - e^{-1.4} \approx 0.75$. 