Poisson process

Events occur at random instants of time at an average rate of λ events per second.

e.g. arrival of a customer to a service station or breakdown of a component in some system.

Let $N(t)$ be the number of event occurrences in $[0, t], N(t)$ is a nondecreasing, integer valued, continuous-time random process.

Suppose $[0, t]$ is divided into n subintervals of width $\Delta t = \dfrac{t}{n}$.

Two assumptions:

1. The probability of more than one event occurences in a subinterval is negligible compared to the probability of observing one or zero events. That is, outcome in each subinterval is a Bernoulli trial.

2. Whether or not an event occurs in a subinterval is independent of the outcomes in other intervals. That is, these Bernoulli trials are independent.

These two assumptions together imply that the counting process $N(t)$ can be approximated by the binomial counting process that counts the number of successes in the n Bernoulli trials.

If the probability of an event occurrence in each subinterval is p, then the expected number of event occurrences in $[0, t]$ is np. Since events occur at the rate λ events per second, then

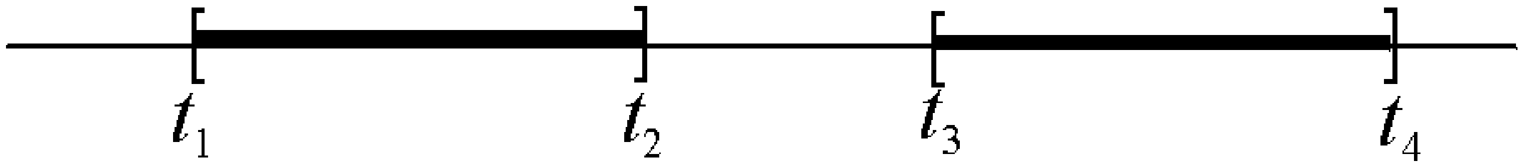
$$\lambda t = np.$$

Let $n \to \infty, p \to 0$ while $\lambda t = np$ remains fixed, the binomial distribution approaches a Poisson distribution with parameter λt.

$$p[N(t) = k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0, 1, \cdots.$$

The Poisson process $N(t)$ inherits properties of independent and stationary increments from the underlying binomial process. Hence, the pmf for the number of event occurrences in **any** interval of length t is given by the above formula.

1. Independent increments for non-overlapping intervals



$[t_1, t_2]$ and $[t_3, t_4]$ are non-overlapping time intervals

$N[t_2] - N[t_1] =$ increment over the interval $[t_1, t_2]$

$N[t_4] - N[t_3] =$ increment over the interval $[t_3, t_4]$.

If $N(t)$ is a Poisson process, then

$$N[t_2] - N[t_1] \text{ and } N[t_4] - N[t_3] \text{ are independent.}$$

2. Stationary increments property

Increments in intervals of the same length have the same distribution regardless of when the interval begins.

$$\begin{aligned} P[N(t_2) - N(t_1) = k] &= P[N(t_2 - t_1) - N(0) = k] \\ &= P[N(t_2 - t_1) = k] \qquad (N(0) = 0) \\ &= \frac{e^{-\lambda(t_2 - t_1)}[\lambda(t_2 - t_1)]^k}{k!}, \quad k = 0, 1, \cdots. \end{aligned}$$

For $t_1 < t_2$, the joint pmf is

$$\begin{aligned} P[N(t_1) = i, N(t_2) = j] &= P[N(t_1) = i]P[N(t_2) - N(t_1) = j - i] \\ &= P[N(t_1) = i]P[N(t_2 - t_1) = j - i] \\ &= \frac{(\lambda t_1)^i e^{-\lambda t_1}}{i!} \frac{[\lambda(t_2 - t_1)]^{j-i} e^{-\lambda(t_2 - t_1)}}{(j - i)!}. \end{aligned}$$

Use of the independent increments property leads to

$$\begin{aligned}
C_N(t_1,t_2) &= E[(N(t_1)-\lambda t_1)(N(t_2)-\lambda t_2)], \quad \text{assuming } t_1 \le t_2 \\
&= E[(N(t_1)-\lambda t_1)\{N(t_2)-N(t_1)-\lambda(t_2-t_1)+N(t_1)-\lambda t_1\}] \\
&= E[(N(t_1)-\lambda t_1)\{N(t_2)-N(t_1)-\lambda(t_2-t_1)\}]+E[(N(t_1)-\lambda t_1)^2] \\
&= E[N(t_1)-\lambda t_1]E[N(t_2)-N(t_1)-\lambda(t_2-t_1)]+\mathrm{VAR}[N(t_1)] \\
&= \mathrm{VAR}[N(t_1)] = \lambda t_1 = \lambda \min(t_1,t_2).
\end{aligned}$$

$\boxed{\text{Example}}$ Inquiries arrive at the rate 15 inquiries per minute as a Poisson process. Find the probability that in a 1-minute period, 3 inquiries arrive during the first 10 seconds and 2 inquiries arrive during the last 15 seconds.

Solution The arrival rate in seconds is $\lambda = \frac{15}{60} = \frac{1}{4}$. The probability of interest is

$$
\begin{aligned}
& P[N(10) = 3, N(60) - N(45) = 2] \\
= \; & P[N(10) = 3]P[N(60) - N(45) = 2] \quad \text{(independent increments)} \\
= \; & P[N(10) = 3]P[N(60 - 45) = 2] \quad \text{(stationary increments)} \\
= \; & \frac{\left(\frac{10}{4}\right)^3 e^{-10/4}}{3!} \cdot \frac{\left(\frac{15}{4}\right)^2 e^{-15/4}}{2!}.
\end{aligned}
$$

Consider the time T between event occurrences in a Poisson process. The probability that the inter-event time T exceeds t seconds is equivalent to no event occurring in t seconds (that is, no event in n Bernoulli trials)

$$\begin{aligned} P[T > t] &= P[\text{no event in } t \text{ seconds}] \\ &= (1-p)^n = \left(1 - \frac{\lambda t}{n}\right)^n \to e^{-\lambda t}, \text{ as } n \to \infty. \end{aligned}$$

The random variable T is an exponential random variable with parameter λ. Since the times between event occurrences in the underlying binomial process are independent geometric random variables, the sequence of interevent times in a Poisson process is composed of independent random variables. The interevent times in a Poisson process form an iid sequence of exponential random variables with mean $1/\lambda$.

Example

Show that the inter-event times in a Poisson process with rate λ are independent and identically distributed exponential random variables with parameter λ.

Solution

Let $Z_1, Z_2, \cdots$ be the random variables representing the length of inter-event times. First, note that $\{Z_1 > t\}$ happens if and only if no event occurs in $[0, t]$ and thus

$$P[Z_1 > t] = P[X(t) = 0] = e^{-\lambda t}.$$

Since $P[X(t) = k] = \dfrac{(\lambda t)^k}{k!} e^{-\lambda t}$, so $F_{Z_1}(t) = 1 - e^{-\lambda t}$. Hence, Z_1 is an exponential random variable with parameter λ. Note that

$$\begin{aligned} \{Z_2 > t | Z_1 = \tau\} &= \{\text{No event occur in } [\tau, \tau + t]\} \\ &= \{X(\tau + t) - X(\tau) = 0\}. \end{aligned}$$

Let $f_1(t)$ be the pdf of Z_1. By the rule of total probabilities, we have

$$\begin{aligned} P[Z_2 > t] &= \int_0^\infty P[Z_2 > t | Z_1 = \tau] f_1(\tau)\ d\tau \\ &= \int_0^\infty P[X(\tau + t) - X(\tau) = 0] f_1(\tau)\ d\tau \\ &= \int_0^\infty P[X(t) = 0] f_1(\tau)\ d\tau \text{ by stationary increments} \\ &= e^{-\lambda t} \int_0^\infty f_1(\tau)\ d\tau = e^{-\lambda t}. \end{aligned}$$

Therefore, Z_2 is also an exponential random variable with parameter λ and it is independent of Z_1. Repeating the same argument, we conclude that $Z_1, Z_2, \cdots$ are iid exponential random variables with parameter λ.

Occurrence of nth event

Write t_j as the random time corresponding to the occurence of the j^{th} event, $j = 1, 2, \cdots$. Let T_j denote the iid exponential interarrival times, then $T_j = t_j - t_{j-1}, \quad t_0 = 0$.

$$\begin{aligned} S_n &= \text{time at which the } n\text{th event occurs in a Poisson process} \\ &= T_1 + T_2 + \cdots + T_n. \end{aligned}$$

$\boxed{\text{Example}}$ With $\lambda = 1/4$ inquiries per second, find the mean and variance of the time until the arrival of the 10th inquiry.

$$E[S_{10}] = 10E[T] = \frac{10}{\lambda} = 40 \text{ sec}$$

$$\text{VAR}[S_{10}] = 10\text{VAR}[T] = \frac{10}{\lambda^2} = 160 \text{ sec}^2.$$

$\boxed{\text{Example}}$ Messages arrive at a computer from two telephone lines according to independent Poisson processes of rates λ_1 and λ_2, respectively.

(a) Find the probability that a message arrives first on line 1.

(b) Find the pdf for the time until a message arrives on either line.

(c) Find the pmf for $N(t)$, the total number of messages that arrive in an interval of length t.

Solution

(a) Let X_1 and X_2 be the number of messages from line 1 and line 2 in time t, respectively.

Probability that a message arrives first on line 1

$$= P[X_1 = 1 | X_1 + X_2 = 1] = \frac{P[X_1 = 1, X_2 = 0]}{P[X_1 + X_2 = 1]}.$$

Since X_1 and X_2 are independent Poisson processes, their sum $X_1 + X_2$ is a Poisson process with rate $\lambda_1 + \lambda_2$. Further, since X_1 and X_2 are independent,

$$P[X_1 = 1, X_2 = 0] = P[X_1 = 1]P[X_2 = 0]$$

$$\text{so } P[X_1 = 1 | X_1 + X_2 = 1] = \frac{P[X_1 = 1]P[X_2 = 0]}{P[X_1 + X_2 = 1]}$$

$$= \frac{e^{-\lambda_1 t}(\lambda_1 t) e^{-\lambda_2 t}(\lambda_2 t)^0}{e^{-(\lambda_1+\lambda_2)t}(\lambda_1+\lambda_2)t} = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

(b) Let T_i be the time until the first message arrives in line $i, i = 1, 2$; T_1 and T_2 are independent exponential random variables.

The time until the first message arrives at a computer $= T = \min(T_1, T_2)$.

$$\begin{aligned} P[T > t] &= P[\min(T_1, T_2) > t] = P[T_1 > t, T_2 > t] \\ &= P[T_1 > t]P[T_2 > t] \\ &= e^{-\lambda_1 t}e^{-\lambda_2 t} \end{aligned}$$

pdf of $T = f_T(t) = -\frac{d}{dt}P[T > t] = (\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)t}$.

(c) $N =$ total number of messages that arrive in an interval of time t

$= X_1 + X_2$.

It is known that the sum of independent Poisson processes remains to be Poisson. Hence,

$$P[N = n] = \frac{e^{-(\lambda_1+\lambda_2)t}[(\lambda_1 + \lambda_2)t]^n}{n!}.$$

$\boxed{\text{Example}}$

Show that given one arrival has occurred in the interval $[0, t]$, then the customer arrival time is uniformly distributed in $[0, t]$. Precisely, let X denote the arrival time of the single customer, then for $0 < x < t, P[X \leq x] = x/t$.

Solution

$$\begin{aligned}
P[X \leq x] &= P[N(x) = 1 | N(t) = 1] \\
&= \frac{P[N(x) = 1 \text{ and } N(t) = 1]}{P[N(t) = 1]} \\
&= \frac{P[N(x) = 1 \text{ and } N(t) - N(x) = 0]}{P[N(t) = 1]} \\
&= \frac{P[N(x) = 1]P[N(t) - N(x) = 0]}{P[N(t) = 1]} \\
&= \frac{\lambda x e^{-\lambda x} e^{-\lambda(t-x)}}{\lambda t e^{-\lambda t}} = \frac{x}{t}.
\end{aligned}$$

Random telegraph signal

Consider a random process $X(t)$ that assumes the values ± 1. Suppose that $X(0) = \pm 1$ with probability $\frac{1}{2}$ and $X(t)$ then changes polarity with each occurrence of an event in a Poisson process of rate α.

The figure shows a sample path of a random telegraph signal. The times between transitions X_j are iid exponential random variables. It can be shown that the random telegraph signal is equally likely to be ± 1 at any time $t > 0$.

Note that $P[X(t) = \pm 1] = P[X(t) = \pm 1 | X(0) = 1] P[X(0) = 1]$

$$+ P[X(t) = \pm 1 | X(0) = -1] P[X(0) = -1].$$

(i) $X(t)$ will have the same polarity as $X(0)$ only when an even number of events occurs in $(0, t]$.

$$\begin{aligned} P[X(t) = \pm 1 | X(0) = \pm 1] &= P[N(t) = \text{even integer}] \\ &= \sum_{j=0}^{\infty} \frac{(\alpha t)^{2j}}{(2j)!} e^{-\alpha t} \\ &= e^{-\alpha t} \frac{e^{\alpha t} + e^{-\alpha t}}{2} = \frac{1}{2}(1 + e^{-2\alpha t}). \end{aligned}$$

(ii) $X(t)$ and $X(0)$ will differ in sign if the number of events in t is odd

$$\begin{aligned} P[X(t) = \pm 1 | X(0) = \mp 1] &= \sum_{j=0}^{\infty} \frac{(\alpha t)^{2j+1}}{(2j+1)!} e^{-\alpha t} \\ &= e^{-\alpha t} \frac{e^{\alpha t} - e^{-\alpha t}}{2} = \frac{1 - e^{-2\alpha t}}{2}. \end{aligned}$$

Now, $P[X(t)=1] = \frac{1}{2}\left[\frac{1+e^{-2\alpha t}}{2} + \frac{1-e^{-2\alpha t}}{2}\right] = \frac{1}{2}$

and $P[X(t)=-1] = 1 - P[X(t)=1] = \frac{1}{2}$.

Next, $m_X(t) = 1P[X(t)=1] + (-1)P[X(t)=-1] = 0$

$$\begin{aligned}\text{VAR}[X(t)] &= E[X(t)^2] - m_X(t)^2 \\ &= 1^2 P[X(t)=1] + (-1)^2 P[X(t)=-1] = 1\end{aligned}$$

$$\begin{aligned}C_X(t_1,t_2) &= E[X(t_1)X(t_2)] \\ &= 1P[X(t_1)=X(t_2)] + (-1)P[X(t_1)\neq X(t_2)] \\ &= \frac{1}{2}[1+e^{-2\alpha|t_2-t_1|}] - \frac{1}{2}[1-e^{-2\alpha|t_2-t_1|}] = e^{-2\alpha|t_2-t_1|}.\end{aligned}$$

The autocovariance tends to zero when $|t_2-t_1| \to \infty$.

Example

Find $P[N(t-d)=j|N(t)=k]$ with $d>0$, where $N(t)$ is a Poisson process with rate λ.

Solution

$$
\begin{aligned}
& P[N(t-d)=j|N(t)=k] \\
= & \frac{P[N(t-d)=j, N(t)=k]}{P[N(t)=k]} \\
= & \frac{P[N(t-d)=j, N(t)-N(t-d)=k-j]}{P[N(t)=k]} \\
= & \frac{P[N(t-d)=j]P[N(t)-N(t-d)=k-j]}{P[N(t)=k]} \text{ (independent increments)} \\
= & \frac{P[N(t-d)=j]P[N(d)=k-j]}{P[N(t)=k]} \text{ (stationary increments)} \\
= & \frac{\frac{[\lambda(t-d)]^j e^{-\lambda(t-d)}}{j!} \; \frac{(\lambda d)^{k-j} e^{-\lambda d}}{(k-j)!}}{\frac{(\lambda t)^k e^{-\lambda t}}{k!}} \\
= & {}_kC_j \frac{[\lambda(t-d)]^j (\lambda d)^{k-j}}{(\lambda t)^k} = {}_kC_j \left(\frac{t-d}{t}\right)^j \left(\frac{d}{t}\right)^{k-j}.
\end{aligned}
$$

This is same as the probability of choosing j successes out of k trials, with probability of success $= \dfrac{t-d}{t}$. Conditional on k occurrences over $[0, t]$, we find the probability of j occurrences over $[0, t-d]$.

$\boxed{\text{Example}}$ Customers arrive at a soft drink dispensing machine according to a Poisson process with rate λ. Suppose that each time a customer deposits money, the machine dispenses a soft drink with probability p. Find the pmf for the number of soft drinks dispensed in time t. Assume that the machine holds an infinite number of soft drinks.

Solution

Let $N(t)$ be the number of soft drinks dispensed up to time t, and $X(t)$ be the number of customer arrivals up to time t.

$$
\begin{aligned}
P[N(t)=k] &= \sum_{n=k}^{\infty} P[N(t)=k|X(t)=n]P[X(t)=n] \\
&= \sum_{n=k}^{\infty} {}_nC_k\ p^k(1-p)^{n-k}\left[\frac{e^{-\lambda t}(\lambda t)^n}{n!}\right] \\
&= \sum_{m=0}^{\infty} {}_{m+k}C_k\ p^k(1-p)^m \frac{e^{-\lambda t}(\lambda t)^{m+k}}{(m+k)!}, \text{set } n=m+k \\
&= e^{-\lambda t}\left\{\sum_{m=0}^{\infty}\frac{[\lambda t(1-p)]^m}{m!}\right\}\frac{(\lambda pt)^k}{k!} \\
&= e^{-\lambda t}e^{\lambda t(1-p)}\frac{(\lambda pt)^k}{k!} = \frac{e^{-\lambda pt}(\lambda pt)^k}{k!}, \quad k=0,1,2,\cdots
\end{aligned}
$$

Conditional Expectation

The conditional expectation of Y given $X = x$ is given by

$$E[Y|x] = \int_{-\infty}^{\infty} y f_Y(y|x) \; dy.$$

When X and Y are both discrete random variables

$$E[Y|x] = \sum_{y_j} y_j P_Y(y_j|x).$$

On the other hand, $E[Y|x]$ can be viewed as a function of x:

$$g(x) = E[Y|x].$$

Correspondingly, this gives rise to the random variable: $g(X) = E[Y|X]$.

What is $E[E[Y|X]]$?

Note that $E[E[Y|X]] = \begin{cases} \displaystyle\int_{-\infty}^{\infty} E[Y|x] f_X(x)\ dx, & X \text{ is continuous} \\ \displaystyle\sum_{x_k} E[Y|x_k] P_X(x_k), & X \text{ is discrete} \end{cases}$.

Suppose X and Y are jointly continuous random variables

$$\begin{aligned} E[E[Y|X]] &= \int_{-\infty}^{\infty} E[Y|x] f_X(x)\ dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_Y(y|x)\ dy\ f_X(x)\ dx \\ &= \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{XY}(x,y)\ dx dy \\ &= \int_{-\infty}^{\infty} y f_Y(y)\ dy = E[Y]. \end{aligned}$$

Generalization $E[h(Y)] = E[E(h(Y)|X]]$ [in the above proof, change y to $h(y)$];

and in particular, $E[Y^k] = E[E[Y^k|X]]$.

Example

A customer entering a service station is served by serviceman i with probability $p_i, i = 1, 2, \cdots, n$. The time taken by serviceman i to service a customer is an exponentially distributed random variable with parameter α_i. Let I be the discrete random variable which assumes the value i if the customer is serviced by the ith serviceman, and let $P_I(i)$ denote the probability mass function of I. Let T denote the time taken to service a customer.

(a) Explain the meaning of the following formula

$$P[T \leq t] = \sum_{i=1}^{n} P_I(i) P[T \leq t | I = i].$$

Use it to find the probability density function of T.

(b) Use the conditional expectation formula

$$E[E[T|I]] = E[T]$$

to compute $E[T]$.

Solution

(a) From the conditional probability formula, we have

$$P[T \leq t, I = i] = P_I(i)P[T \leq t|I = i].$$

The marginal distribution function $P[T \leq t]$ is obtained by summing the joint probability values $P[T \leq t, I = i]$ for all possible values of i. Hence,

$$P[T \leq t] = \sum_{i=1}^{n} P_I(i)P[T \leq t|I = i].$$

Here, $P_I(i) = p_i$ and $P[T \leq t|I = i] = 1 - e^{-\alpha_i t}, t \geq 0$. The probability density function of T is given by

$$f_T(t) = \frac{d}{dt}P[T \leq t] = \begin{cases} \sum_{i=1}^{n} p_i \alpha_i e^{-\alpha_i t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

(b)

$$\begin{aligned} E[T] &= E[E[T|I]] = \sum_{i=1}^{n} P_I(i) E[T|I=i] \\ &= \sum_{i=1}^{n} p_i \int_0^{\infty} \alpha_i t e^{-\alpha_i t} \, dt \\ &= \sum_{i=1}^{n} \frac{p_i}{\alpha_i}. \end{aligned}$$

The mean service time is the weighted average of mean service times at different counters, where $\dfrac{1}{\alpha_i}$ is the mean service time for the ith serviceman.

Example Find the mean and variance of number of customer arrivals N during the service time T of a specific customer. Let $f_T(t)$ denote the pdf of T. Assume the customer arrivals follow the Poisson process.

Solution $E[N|T=t] = \lambda t, E[N^2|T=t] = \lambda t + \lambda^2 t^2$ where λ is the average number of customers per unit time.

$$\begin{aligned} E[N] &= \int_0^\infty E[N|T=t] f_T(t)\ dt = \int_0^\infty \lambda t f_T(t)\ dt = \lambda E[T] \\ E[N^2] &= \int_0^\infty E[N^2|T=t] f_T(t)\ dt = \int_0^\infty (\lambda t + \lambda^2 t^2) f_T(t)\ dt = \lambda E[T] + \lambda^2 E[T^2] \end{aligned}$$

$$\begin{aligned} \text{VAR}[N] = E[N^2] - E[N]^2 &= \lambda E[T] + \lambda^2 E[T^2] - \lambda^2 E[T]^2 \\ &= \lambda^2 \text{VAR}[T] + \lambda E[T]. \end{aligned}$$

$\boxed{\text{Example}}$

(a) Show that

$$\text{VAR}[X] = E[\text{VAR}[X|Y]] + \text{VAR}[E[X|Y]].$$

(b) Suppose that by any time t the number of people that have arrived at a train station is a Poisson variable with mean λt. If a train arrives at the station at a time that is uniformly distributed over $(0, T)$, what are the mean and variance of the number of passengers that enter the train?

Hint: Let Y denote the arrival time of the train. Knowing that $E[N(Y)|Y = t] = \lambda t$, compute $E[N(Y)]$ and $\text{VAR}[N(Y)]$.

Solution

Starting with

$$\text{Var}(X|Y) = E[X^2|Y] - (E[X|Y])^2$$

so

$$\begin{aligned} E[\text{Var}(X|Y)] &= E[E[X^2|Y]] - E[E[X|Y])^2] \\ &= E[X^2] - E[(E|Y])^2]. \end{aligned}$$

Since $E[E[X|Y]] = E[X]$, we have

$$\text{Var}(E[X|Y]) = E[(E[X|Y])^2] - (E[X])^2.$$

Hence, by adding the above two equations, we obtain the result.

Let $N(t)$ denote the number of arrivals by t, and let Y denote the time at which the train arrives. The random variable of interest is then $N(Y)$. Conditioning on $Y = t$, we have

$$\begin{aligned} E[N(Y)|Y = t] &= E[N(t)|Y = t] \\ &= E[N(t)] \quad \text{by the independence of } Y \text{ and } N(t) \\ &= \lambda t \quad \text{since } N(t) \text{ is Poisson with mean } \lambda t. \end{aligned}$$

Hence

$$E[N(Y)|Y] = \lambda Y$$

so taking expectations gives

$$E[N(Y)] = \lambda E[Y] = \frac{\lambda T}{2}.$$

To obtain Var$(N(Y))$, we use the conditional variance formula:

$$\begin{aligned}
\text{Var}(N(Y)|Y=t) &= \text{Var}(N(t)|Y=t) \\
&= \text{Var}(N(t)) \quad \text{by independence} \\
&= \lambda t
\end{aligned}$$

so

$$\begin{aligned}
\text{Var}(N(Y)|Y) &= \lambda Y \\
E[N(Y)|Y] &= \lambda Y.
\end{aligned}$$

Hence, from the conditional variance formula,

$$\begin{aligned}
\text{Var}(N(Y)) &= E[\lambda Y] + \text{Var}(\lambda Y) \\
&= \lambda \frac{T}{2} + \lambda^2 \frac{T^2}{12}
\end{aligned}$$

Note that we have used Var$(Y) = T^2/12$.