Markov Processes

In general, the probability structure of a random sequence (discrete parameter random process) is determined by the joint probabilities

$$P[X_0 = j_0, X_1 = j_1, \cdots, X_n = j_n].$$

If it happens that

$$P[X_n = j_n | X_{n-1} = j_{n-1}, \cdots, X_0 = j_0] = P[X_n = j_n | X_{n-1} = j_{n-1}], \tag{1}$$

that is, knowledge of $X_0, X_1, \cdots, X_{n-1}$ give no more information for predicting the value of X_n than does knowledge of X_{n-1} alone, then such a process is termed a *Markov process*.

With the Markov process property defined in Eq. (1), we have

$$\begin{aligned}
& P[X_n = j_n, X_{n-1} = j_{n-1}, \cdots, X_0 = j_0] \\
= \; & P[X_n = j_n | X_{n-1} = j_{n-1}, X_{n-2} = j_{n-2}, \cdots, X_0 = j_0] \\
& P[X_{n-1} = j_{n-1}, X_{n-2} = j_{n-2}, \cdots, X_0 = j_0].
\end{aligned}$$

By observing the Markov property, we write

$$\begin{aligned}
& P[X_n = j_n, X_{n-1} = j_{n-1}, \cdots, X_0 = j_0] \\
= \; & P[X_n = j_n | X_{n-1} = j_{n-1}] P[X_{n-1} = j_{n-1}, X_{n-2} = j_{n-2}, \cdots, X_0 = j_0].
\end{aligned}$$

Applying the same procedure to $P[X_{n-1} = j_{n-1}, X_{n-2} = j_{n-2}, \cdots, X_0 = j_0]$, we obtain

$$\begin{aligned}
& P[X_n = j_n, X_{n-1} = j_{n-1}, \cdots, X_0 = j_0] \\
= \; & P[X_n = j_n | X_{n-1} = j_{n-1}] P[X_{n-1} = j_{n-1} | X_{n-2} = j_{n-2}] \\
& P[X_{n-2} = j_{n-2}, \cdots, X_0 = j_0].
\end{aligned}$$

Deductively, we obtain

$$
\begin{aligned}
& P[X_n = j_n, X_{n-1} = j_{n-1}, \cdots, X_0 = j_0] \\
= \; & P[X_n = j_n | X_{n-1} = j_{n-1}] P[X_{n-1} = j_{n-1} | X_{n-2} = j_{n-2}] \cdots \\
& \qquad \underbrace{P[X_1 = j_1 | X_0 = j_0]}_{\text{one-step transition probability}} P[X_0 = j_0]. \qquad (2)
\end{aligned}
$$

The joint probability is determined in terms of the product of *one-step transition probabilities* and *initial state probability* $P[X_0 = j_0]$.

Markov chain — integer-valued Markov random process

$\boxed{\text{Example}}$ (moving average of Bernoulli sequence)

$$Y_n = \frac{1}{2}(X_n + X_{n-1})$$

where X_n's are members of an independent Bernoulli sequence with $p = \frac{1}{2}$. Consider the pmf of Y_n:

$$\begin{aligned} P[Y_n = 0] &= P[X_n = 0, X_{n-1} = 0] = \frac{1}{4} \\ P[Y_n = \frac{1}{2}] &= P[X_n = 0, X_{n-1} = 1] + P[X_n = 1, X_{n-1} = 0] = \frac{1}{2} \\ P[Y_n = 1] &= P[X_n = 1, X_{n-1} = 1] = \frac{1}{4}. \end{aligned}$$

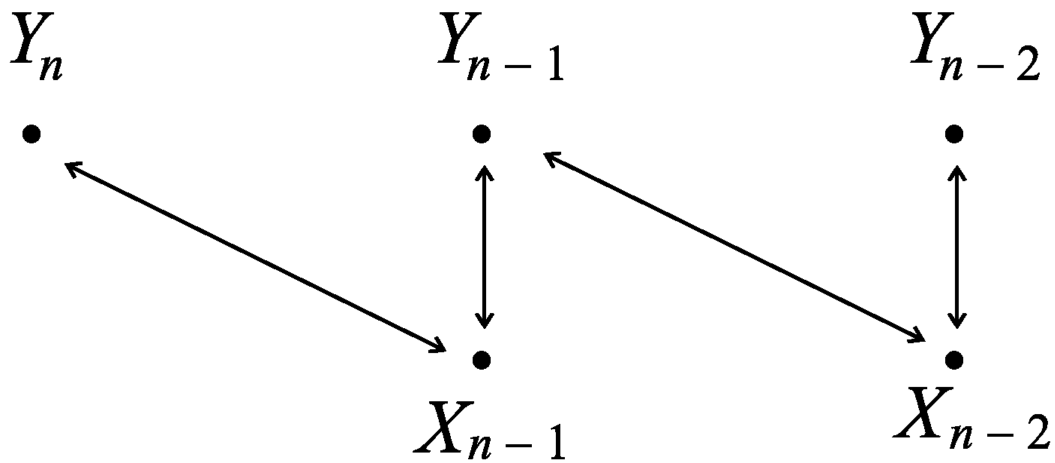
Now, consider

$$\begin{aligned} P\left[Y_n = 1 | Y_{n-1} = \frac{1}{2}\right] &= \frac{P\left[Y_n = 1, Y_{n-1} = \frac{1}{2}\right]}{P\left[Y_{n-1} = \frac{1}{2}\right]} \\ &= \frac{P[X_n = 1, X_{n-1} = 1, X_{n-2} = 0]}{\frac{1}{2}} = \frac{\left(\frac{1}{2}\right)^3}{\frac{1}{2}} = \frac{1}{4}. \end{aligned}$$

$$P\left[Y_n = 1 | Y_{n-1} = \frac{1}{2}, Y_{n-2} = 1\right] = \frac{P\left[Y_n = 1, Y_{n-1} = \frac{1}{2}, Y_{n-2} = 1\right]}{P\left[Y_{n-1} = \frac{1}{2}, Y_{n-2} = 1\right]}$$
$$= 0$$

since no sequence of X_n's leads to the sequence of Y_n as $1, \frac{1}{2}, 1$.

Hence, $P\left[Y_n = 1 | Y_{n-1} = \frac{1}{2}\right] \neq P\left[Y_n = 1 | Y_{n-1} = \frac{1}{2}, Y_{n-2} = 1\right]$ so that Y_n is not a Markov process. This is because both Y_{n-1} and Y_{n-2} depend on X_{n-2}.



Stationary transition mechanism

The transition probabilities do not depend on which particular step is being considered, that is,

$$P[X_n = j|X_{n-1} = i] = P[X_k = j|X_{k-1} = i]$$

independent of n and k.

Transition probability matrix

Suppose the state space of X_n is given by $S = \{1, 2, \cdots, m\}$; define a matrix P whose $(i, j)^{\text{th}}$ entry is given by

$$p_{ij} = P[X_n = j|X_{n-1} = i],$$

which is the same for all $n \geq 1$ due to the *stationary* property.

(i) P is a $m \times m$ matrix since i and j can assume m different values.

(ii) Sum of entries in any row is one since

$$\sum_{j=1}^{m} P[X_n = j | X_{n-1} = i] = 1.$$

This is because in the next move based on $X_{n-1} = i, X_n$ must end up in one of the values in the state space.

(iii) The joint pmf in Eq. (2) can be expressed as

$$P[X_0 = j_0, X_1 = j_1, \cdots, X_n = j_n] = P[X_0 = j_0] p_{j_0,j_1} p_{j_1,j_2} \cdots p_{j_{n-1},j_n}. \quad (3)$$

n-step transition probability matrix

Suppose we write $\pi_{n,j} = P[X_n = j]$ and define the row vector $\boldsymbol{\pi}_n = (\pi_{n,1} \cdots \pi_{n,m})$, where m is the number of possible states. We have

$$\pi_{1,j} = \sum_{k=1}^{m} \pi_{0,k} p_{k,j}, \quad j = 1, 2, \cdots, m.$$

In matrix form, we have

$$(\pi_{1,1} \cdots \pi_{1,m}) = (\pi_{0,1} \cdots \pi_{0,m}) \begin{pmatrix} p_{11} & \cdots & p_{1m} \\ p_{21} & & \vdots \\ \vdots & & \vdots \\ p_{m1} & & p_{mm} \end{pmatrix}_{m \times m},$$

that is, $\boldsymbol{\pi}_1 = \boldsymbol{\pi}_0 P$.

Since the transition matrix P is independent of n, so

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 P^n.$$

Here, P^n is the n-step transition probability matrix. Under the stationary assumption, the n-step transition probability matrix is the nth power of the one-step transition probability matrix.

$\boxed{\text{Example}}$

X_n can assume two states: {study, non-study}. If a student studies on the $(n-1)$th day, she will not study on the next day (nth day) with probability α. If she does not study on the $(n-1)$th day, she will study on the next day with probability β.

Let $X_n = \begin{cases} 1 & \text{if in study state} \\ 2 & \text{if in non-study state} \end{cases}$

$P[X_n = 1|X_{n-1} = 1] = 1 - \alpha; P[X_n = 2|X_{n-1} = 1] = \alpha;$

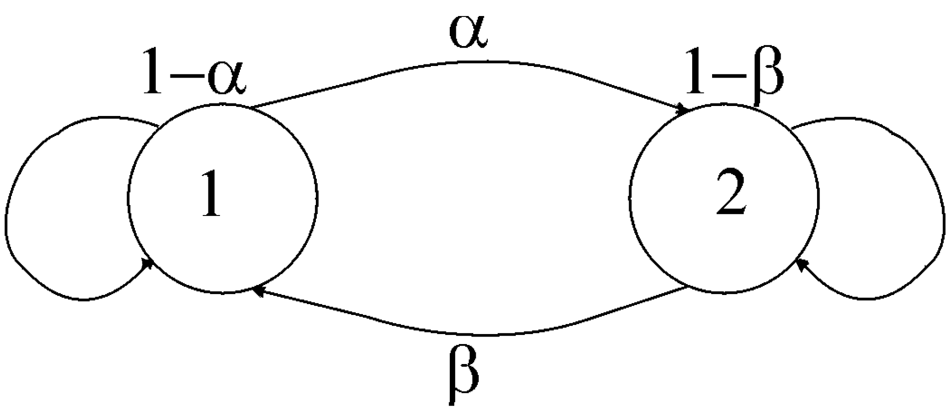
$P[X_n = 1|X_{n-1} = 2] = \beta; P[X_n = 2|X_{n-1} = 2] = 1 - \beta.$

The state transition matrix is $P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$, which is independent of n. [Though in real life, α and β may change possibly due to the approach of a test.]

From Eq. (3), we have

$$\begin{aligned}
& P[\text{non-study on 0th day, study on 1st day,} \\
& \quad \text{study on 2nd day, non-study on 3rd day}] \\
= \; & P[X_0 = 2, X_1 = 1, X_2 = 1, X_3 = 2] \\
= \; & P[X_0 = 2] p_{21} p_{11} p_{12} \\
= \; & P[X_0 = 2] \beta (1-\alpha) \alpha.
\end{aligned}$$

$$\begin{aligned}
(P[X_n = 1] \;\; P[X_n = 2]) &= (P[X_{n-1} = 1] \;\; P[X_{n-1} = 2]) \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \\
&= (P[X_0 = 1] \;\; P[X_0 = 2]) \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}^n .
\end{aligned}$$



How to find P^n?

Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the two independent eigenvectors of P, where $P\mathbf{x}_i = \lambda_i \mathbf{x}_i, i = 1, 2, \lambda_1$ and λ_2 are the eigenvalues. Form the matrix whose columns are $\mathbf{x}_1$ and $\mathbf{x}_2$.

Then $PX = X\Lambda$, where Λ is a diagonal matrix whose diagonal entries are λ_1 and λ_2. From $P = X\Lambda X^{-1}$, then $P^n = X\Lambda^n X^{-1}$. It can be shown that

$$\begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}^n = \frac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix} + \frac{(1-\alpha-\beta)^n}{\alpha+\beta}\begin{pmatrix} \alpha & -\alpha \\ -\beta & \beta \end{pmatrix}.$$

When $n \to \infty$, $P^n \to \dfrac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix}$ since $|1-(\alpha+\beta)| < 1$.

Therefore, as $n \to \infty$,

$$
\begin{aligned}
& (P[X_\infty = 1] \quad P[X_\infty = 2]) \\
= \; & \frac{1}{\alpha + \beta}(P[X_0 = 1] \quad P[X_0 = 2]) \begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix} \\
= \; & \left(\frac{\beta}{\alpha + \beta} \quad \frac{\alpha}{\alpha + \beta} \right) \text{ since } P[X_0 = 1] + P[X_0 = 2] = 1.
\end{aligned}
$$

We observe that at steady state, the state probabilities are independent of the initial values of the state probability. In fact, $\lim_{n\to\infty} \boldsymbol{\pi}_n = \lim_{n\to\infty} \boldsymbol{\pi}_{n-1} P$ so that $\boldsymbol{\pi}_\infty = \boldsymbol{\pi}_\infty P$, where $\boldsymbol{\pi}_\infty$ denotes the state probability row vector at steady state.

How to find $\boldsymbol{\pi}_\infty$?

Let $\boldsymbol{\pi}_\infty = (\pi_{\infty,1} \quad \pi_{\infty,2})$ so that

$$(\pi_{\infty,1} \quad \pi_{\infty,2}) = (\pi_{\infty,1} \quad \pi_{\infty,2}) \left(\begin{array}{cc} 1-\alpha & \alpha \\ \beta & 1-\beta \end{array} \right).$$

The first equation gives

$$\pi_{\infty,1} = (1-\alpha)\pi_{\infty,1} + \beta\pi_{\infty,2} \qquad \Leftrightarrow \qquad \alpha\pi_{\infty,1} = \beta\pi_{\infty,2}.$$

The second equation is redundant.

In addition, sum of probabilities $= 1$ so that $\pi_{\infty,1} + \pi_{\infty,2} = 1$.

The solution for $\pi_{\infty,1}$ and $\pi_{\infty,2}$ are

$$\pi_{\infty,1} = \frac{\beta}{\alpha+\beta} \quad \text{and} \quad \pi_{\infty,2} = \frac{\alpha}{\alpha+\beta}.$$

Additional observations

1. Why $\lim_{n\to\infty} P^n = \frac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix}$ has identical rows?

   Hint Steady state probability vector $\boldsymbol{\pi}_\infty$ should be independent of $\boldsymbol{\pi}_0$. Therefore, $\boldsymbol{\pi}_\infty$ must be equal to either one of the rows of $\lim_{n\to\infty} P^n$.

2. Interpretation of $\alpha\pi_{\infty,1} = \beta\pi_{\infty,2}$ as a global balance of steady state probabilities.

   "There is α portion of steady state probability $\pi_{\infty,1}$ flowing out of state 1 and β portion of $\pi_{\infty,2}$ flowing into state 1".

   The global balance concept leads to efficient valuation of $\boldsymbol{\pi}_\infty$.

3. *Behaviors of P^n*

$$P^n = \frac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix} + \frac{(1-\alpha-\beta)^n}{\alpha+\beta}\begin{pmatrix} \alpha & -\alpha \\ -\beta & \beta \end{pmatrix}.$$

First row of $P^n = \left(\frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n \quad \frac{\alpha}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n \right)$.

Second row of $P^n = \left(\frac{\beta}{\alpha+\beta} - \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^n \quad \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^n \right)$.

For finite n, note that the two rows of P^n are different. However, as $n \to \infty$, the two rows tend to the same vector

$$\left(\frac{\beta}{\alpha+\beta} \quad \frac{\alpha}{\alpha+\beta} \right).$$

This must be the steady state pmf $\boldsymbol{\pi}_\infty$.

Recall that $\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 P^n$; $\boldsymbol{\pi}_\infty$ is governed by $\boldsymbol{\pi}_\infty = \boldsymbol{\pi}_\infty P$. Therefore, $\boldsymbol{\pi}_n$ depends on $\boldsymbol{\pi}_0$ but $\boldsymbol{\pi}_\infty$ is independent of $\boldsymbol{\pi}_0$.

In particular, with the choice of

(i) $\boldsymbol{\pi}_0 = (1 \quad 0)$; then $\boldsymbol{\pi}_n =$ first row of P^n.

(ii) $\boldsymbol{\pi}_0 = (0 \quad 1)$; then $\boldsymbol{\pi}_n =$ second row of P^n.

The two rows must tend to the same row vector as $n \to \infty$ since the dependence of $\boldsymbol{\pi}_n$ on $\boldsymbol{\pi}_0$ diminishes as n gets larger.

A salesman's territory consists of three cities, A, B and C. He never sells in the same city on successive days. If he sells in city A, then the next day he sells in city B. However, if he sells in either B or C, then the next day he is twice as likely to sell in city A as in the other city. In the long run, how often does he sell in each of the cities?

The transition matrix of the problem is as follows:

$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

How to find $\boldsymbol{\pi}_\infty$?

$$(\pi_{\infty,1} \quad \pi_{\infty,2} \quad \pi_{\infty,3}) \begin{pmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} = (\pi_{\infty,1} \quad \pi_{\infty,2} \quad \pi_{\infty,3})$$

Take the second and third equations

$$\frac{2}{3}\pi_{\infty,1} + \frac{1}{3}\pi_{\infty,3} = \pi_{\infty,2}$$
$$\frac{1}{3}\pi_{\infty,2} = \pi_{\infty,3}$$

and together with

$$\pi_{\infty,1} + \pi_{\infty,2} + \pi_{\infty,3} = 1,$$

we obtain

$$\boldsymbol{\pi}_\infty = \begin{pmatrix} \frac{2}{5} & \frac{9}{20} & \frac{3}{20} \end{pmatrix}.$$

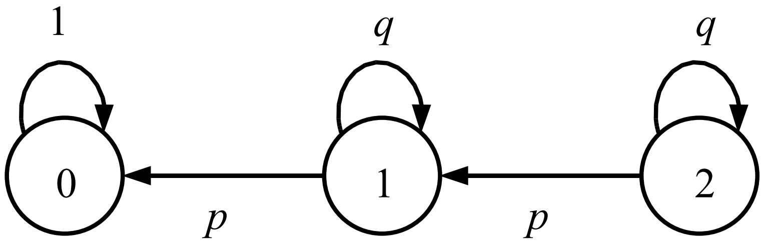
In the long run, he sells 40% of the time in city A, 45% of the time in B and 15% of the time in C.

Example

On the zeroth day, a house has two new light bulbs in reserve.

$p =$ probability that the light bulb fails on a given day.

$Y_n =$ number of new light bulbs left in the house at the end of day n.



State transition diagram

One-step transition probability matrix is given by

$$P = \begin{pmatrix} 1 & 0 & 0 \\ p & q & 0 \\ 0 & p & q \end{pmatrix}$$

We deduce that

$$P^n = \begin{pmatrix} 1 & 0 & 0 \\ 1 - q^n & q^n & 0 \\ 1 - q^n - npq^{n-1} & npq^{n-1} & q^n \end{pmatrix}$$

$$\begin{aligned} p_{33}(n) = P[\text{no new light bulbs needed in } n \text{ days}] = q^n \\ p_{32}(n) = P[\text{one light bulbs needed in } n \text{ days}] = npq^{n-1} \\ p_{31}(n) = 1 - p_{33}(n) - p_{32}(n). \end{aligned}$$

For $q < 1$ and $n \to \infty$, $P^\infty \to \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

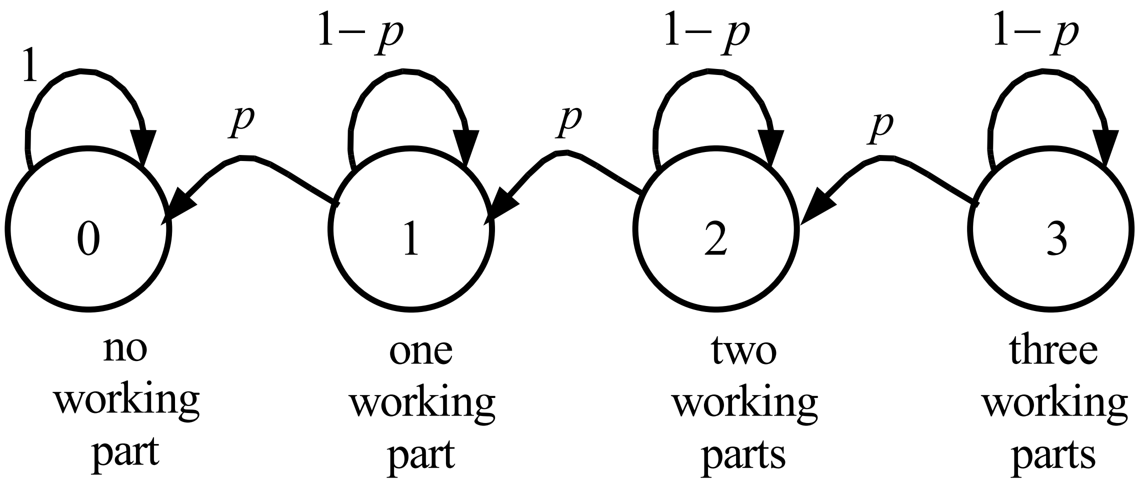
$$\boldsymbol{\pi}_0 = \underbrace{(0 \quad 0 \quad 1)}_{\substack{\text{start with two} \\ \text{light bulbs}}} \quad ; \quad \boldsymbol{\pi}_\infty \to (0 \quad 0 \quad 1) \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \underbrace{(1 \quad 0 \quad 0)}_{\substack{\text{eventually run out} \\ \text{of light bulb}}}$$

Why $p_{31}(n)$ is not given by ${}_nC_2p^2q^{n-2}$?

On each day, we perform a binomial experiment with probability of "failure" p. If "failure" occurs, we move down one state. However, once we are in state "0", we remain in state "0" even "failure" occurs in further trials. Starting at state "2" initially, after n trials, we move to state "0" if there are 2 or more "failure" in these n trials. Hence

$$\begin{aligned} P[Y_n = 1] &= {}_nC_2p^2q^{n-2} + {}_nC_3p^3q^{n-3} + \cdots + {}_nC_np^n \\ &= (p+q)^n - {}_nC_1pq^{n-1} - q^n \\ &= 1 - {}_nC_1pq^{n-1} - q^n. \end{aligned}$$

$\boxed{\text{Example}}$ Three working parts. With probability p, the working part may fail after each day of operation. There is no repair.



one-step transition probability matrix is given by

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ p & q & 0 & 0 \\ 0 & p & q & 0 \\ 0 & 0 & p & q \end{pmatrix}, \quad q = 1 - p.$$

Applying similar arguments, we obtain the n-step transition probability matrix

$$\begin{aligned} P^n &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1-q^n & q^n & 0 & 0 \\ 1-q^n-npq^{n-1} & npq^{n-1} & q^n & 0 \\ 1-{}_nC_2p^2q^{n-2}-npq^{n-1}-q^n & {}_nC_2p^2q^{n-2} & npq^{n-1} & q^n \end{pmatrix} \\ &\overset{n\to\infty}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The steady state pmf is $(1 \quad 0 \quad 0 \quad 0)$. This is obvious since all parts will not be working eventually.

There are only flows of probabilities into state 0 but no out-flow of probabilities from state 0.

$\boxed{\text{Example}}$

A machine consists of two parts that fail and are repaired independently. A working part fails during any given day with probability a. A part that is not working is repaired by the next day with probability b.

$$\text{Define} \quad Y_n = \begin{cases} 1 & \text{if there is no working part} \\ 2 & \text{if there is one working part} \\ 3 & \text{if there are two working parts} \end{cases}$$

$$P[Y_n = 1|Y_{n-1} = 1] = (1-b)^2, P[Y_n = 2|Y_{n-1} = 1] = 2b(1-b),$$
$$P[Y_n = 3|Y_{n-1} = 1] = b^2,$$
$$P[Y_n = 1|Y_{n-1} = 2] = a(1-b), P[Y_n = 2|Y_{n-1} = 2] = (1-a)(1-b) + ab,$$
$$P[Y_n = 3|Y_{n-1} = 2] = b(1-a),$$
$$P[Y_n = 1|Y_{n-1} = 3] = a^2, P[Y_n = 2|Y_{n-1} = 3] = 2a(1-a),$$
$$P[Y_n = 3|Y_{n-1} = 3] = (1-a)^2.$$

Why $P[Y_n = 2 | Y_{n-1} = 2] = (1-a)(1-b) + ab$*?*

On the $(n-1)$th day, there is only one working part. On the next day, there will be only one working part if and only if

(i) the working part does not fail and the failing part has not been repaired; the associated probability is $(1-a)(1-b)$.

(ii) the working part fails and the earlier failing part has been repaired; the associated probability is ab.

The transition probability matrix is found to be

$$P = \begin{pmatrix} (1-b)^2 & 2b(1-b) & b^2 \\ a(1-b) & (1-a)(1-b)+ab & b(1-a) \\ a^2 & 2a(1-a) & (1-a)^2 \end{pmatrix}.$$

The sum of entries in each row must be one.

How to find the steady state pmf?

Write $\boldsymbol{\pi}_\infty = (\pi_{\infty,1} \quad \pi_{\infty,2} \quad \pi_{\infty,3})$; the governing equation is given by

$$\boldsymbol{\pi}_\infty = \boldsymbol{\pi}_\infty P.$$

$$\begin{cases} \pi_{\infty,1} = (1-b)^2\pi_{\infty,1} + a(1-b)\pi_{\infty,2} + a^2\pi_{\infty,3} \\ \pi_{\infty,3} = b^2\pi_{\infty,1} + b(1-a)\pi_{\infty,2} + (1-a)^2\pi_{\infty,3} \\ \pi_{\infty,1} + \pi_{\infty,2} + \pi_{\infty,3} = 1 \end{cases}.$$

The first two equations are obtained by equating the first and the last entries in $\boldsymbol{\pi}_\infty$ and $\boldsymbol{\pi}_\infty P$. The last equation is obtained by observing that the sum of all probabilities must be one.

Remark

1. Why we drop the second equation arising from $\boldsymbol{\pi}_\infty = \boldsymbol{\pi}_\infty P$?

   There are only two non-redundant equations obtained from equating $\boldsymbol{\pi}_\infty$ and $\boldsymbol{\pi}_\infty P$. We choose to drop the one which has the most complexity.

2. Interpretation of the first equation from the perspective of global balance.

   Consider state 1

   portion of $\pi_{\infty,1}$ flowing out of state 1 is $1-(1-b)^2$

   portion of $\pi_{\infty,2}$ flowing into state 1 is $a(1-b)$

   portion of $\pi_{\infty,3}$ flowing into state 1 is a^2.

   In balance, in-flow probabilities = out-flow probabilities:

$$a(1-b)\pi_{\infty,2} + a^2\pi_{\infty,3} = [1-(1-b)^2]\pi_{\infty,1}.$$

Sale of aquariums — Inventory control problem

The manager takes inventory and place order at the end of each week. He will order 3 new aquariums if all of the current inventory has been sold. If one or more aquariums remain in stock, no new units are ordered.

Question

Suppose the store only sells an average of one aquarium per week, is this policy adequate to guard against potential lost sales of aquariums?

Formulation

Let S_n = supply of aquariums at the beginning of week n;

D_n = demand for aquariums during week n.

Assumption: Number of potential buyers in one week will have a Poisson distribution with mean one so that $P[D_n = k] = e^{-1}/k!, \quad k = 0, 1, 2, \cdots.$

If $D_{n-1} < S_{n-1}$, then $S_n = S_{n-1} - D_{n-1}$; if $D_{n-1} \geq S_{n-1}$, then $S_n = 3$.

How to find $P[D_n > S_n]$!

Assume $S_0 = 3$ and the state space of S_n is $\{1, 2, 3\}$.

$P[D_n = 0] = e^{-1}/0! = 0.368, P[D_n = 1] = e^{-1}/1! = 0.368,$

$P[D_n = 2] = e^{-1}/2! = 0.184, P[D_n = 3] = e^{-1}/3! = 0.061,$

$P[D_n > 3] = 1 - \{P[D_n = 0] + P[D_n = 1] + P[D_n = 2] + P[D_n = 3]\} = 0.019.$

When $S_n = 3$, then

$$
\begin{aligned}
P[S_{n+1} = 1 | S_n = 3] &= P[D_n = 2] = 0.184, \\
P[S_{n+1} = 2 | S_n = 3] &= P[D_n = 1] = 0.368, \\
P[S_{n+1} = 3 | S_n = 3] &= 1 - 0.184 - 0.368 = 0.448.
\end{aligned}
$$

Similarly, we can find the other entries in the transition probability matrix.

The transition probability matrix is given by

$$P = \begin{pmatrix} 0.368 & 0 & 0.632 \\ 0.368 & 0.368 & 0.264 \\ 0.184 & 0.368 & 0.448 \end{pmatrix}.$$

It can be shown that

$$\boldsymbol{\pi}_\infty = (0.285 \quad 0.263 \quad 0.452).$$

When $n \to \infty, P[S_n = 1] = 0.285, P[S_n = 2] = 0.263, P[S_n = 3] = 0.452.$

$$P[D_n > 1|S_n = 1] = P[D_n = 2] + P[D_n = 3] + P[D_n > 3] = 0.264$$
$$P[D_n > 2|S_n = 2] = P[D_n = 3] + P[D_n > 3] = 0.08$$
$$P[D_n > 3|S_n = 3] = P[D_n > 3] = 0.019.$$

Lastly, $P[D_n > S_n]$

$$= \sum_{i=1}^{3} P[D_n > S_n | S_n = i] \; P[S_n = i]$$

$$= 0.264 \times 0.285 + 0.080 \times 0.263 + 0.019 \times 0.452 = 0.105.$$

In the long run, the demand will exceed the supply about 10% of the time.