# MATH 246 — Probability and Random Processes

## Solution to Test Two

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1. Let $X$ be the time (in days) between consecutive accidents, and $X$ is exponential with parameter $\lambda$, satisfying $\lambda = 2$. The distribution function of $X$ is

$$F_X(t) = 1 - e^{-2t}, \quad t > 0.$$

Hence, $P[X > 2] = 1 - P[X \leq 2] = 1 - F_X(2) = e^{-4}$.

2. The ratio $P[N = k]/P[N = k-1] = \alpha/k$, which decreases with increasing $k$. For $\alpha < 1, \alpha/k < 1$ for $k \leq 1$ so that $P[N = k]$ attains its maximum value at $k = 0$. Suppose $\alpha \geq 1$ and $\alpha$ is not an integer, then the ratio is greater than 1 at $k = \text{floor}(\alpha)$ and becomes less than 1 at floor $(\alpha) + 1$. When $\alpha$ happens to be an integer, $P[N = \alpha]/P[N = \alpha - 1] = 1$. The maximum value of $P[X = k]$ occurs at both $k - 1$ and $k$.

3. Recall that $P[X = k] = pq^{k-1}$ and $P[X > j] = q^j$

$$P[X = k + j | X > j] = \frac{P[X = k + j, X > j]}{P[X > j]} = \frac{pq^{k+j-1}}{q^j} = pq^{k-1} = P[X = k].$$

If a success has not occurred in the earlier $j$ trials, then the probability of having to perform exactly $k$ more trials to get a success is the same as the probability of initially having to perform exactly $k$ trials to get a success. This is related to the memoryless properties of the geometric random variable.

4. When $-1 < x < 1$, that is, $0 < y < 1, y = x^2$ has two roots, namely, $x_1 = -\sqrt{y}$ and $x_2 = \sqrt{y}$. Therefore,

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left[\frac{2}{9}(1 + \sqrt{y}) + \frac{2}{9}(1 - \sqrt{y})\right] = \frac{2}{9\sqrt{y}}.$$

When $1 \leq x < 2$, that is, $1 \leq y < 4, y = x^2$ is strictly increasing. We have

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{2}{9}(x+1)\frac{1}{2x} = \frac{1}{9}\left(1 + \frac{1}{\sqrt{y}}\right).$$

In summary,

$$f_Y(y) = \begin{cases} \frac{2}{9\sqrt{y}} & 0 < y < 1 \\ \frac{1}{9}\left(1 + \frac{1}{\sqrt{y}}\right) & 1 \leq y < 4 \\ 0 & \text{otherwise} \end{cases}.$$