Expected value of functions of random variables

$$Z = g(X, Y), \text{ expected value of } Z \text{ is given by}$$

$$E[Z] = \begin{cases} 
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{XY}(x, y) \, dx \, dy & X, Y \text{ jointly continuous} \\
\sum_{i} \sum_{n} g(x_i, y_n) P_{XY}(x_i, y_n) & X, Y \text{ discrete} 
\end{cases}.$$

Sum of random variables

$$E[X + Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x' + y') f_{XY}(x', y') \, dx' \, dy'$$

$$= \int_{-\infty}^{\infty} x' f_{X}(x') \, dx' + \int_{-\infty}^{\infty} y' f_{Y}(y') \, dy' = E[X] + E[Y].$$

In general, $$E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots E[X_n].$$

The random variables do not have to be independent.
Independent random variables

Suppose $X$ and $Y$ are independent random variables, and let $g(X, Y) = g_1(X)g_2(Y)$, then

$$
E[g_1(X)g_2(Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(x')g_2(y')f_X(x')f_Y(y') \ dx'\ dy'
$$

$$
= \left[\int_{-\infty}^{\infty} g_1(x')f_X(x') \ dx'\right]\left[\int_{-\infty}^{\infty} g_2(y')f_Y(y') \ dy'\right]
$$

$$
= E[g_1(X)]E[g_2(Y)].
$$

In general, if $X_1, \ldots, X_n$ are independent random variables, then

$$
E[g_1(X_1)g_2(X_2) \cdots g_n(X_n)] = E[g_1(X_1)]E[g_2(X_2)] \cdots E[g_n(X_n)].
$$
Correlation and covariance of a pair of random variables

The $jk$th joint moment of $X$ and $Y$ is defined by

$$E[X^jY^k] = \begin{cases} 
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^jy^k f_{XY}(x, y) \, dx \, dy, & \text{X and Y jointly continuous} \\
\sum \sum x_i^j y_n^k P_{XY}(x_i, y_n) & \text{X, Y discrete}
\end{cases}$$

When $E[XY] = 0$, then $X$ and $Y$ are orthogonal.

The $jk$th central moment of $X$ and $Y$ is defined as

$$E[(X - E[X])^j(Y - E[Y])^k].$$

When $j = 2, k = 0$, it gives $\text{VAR}(X)$; and when $j = 0, k = 2$, it gives $\text{VAR}(Y)$. When $j = k = 1$, it gives $\text{COV}(X, Y) = E[(X - E[X])(Y - E[Y])]$. 
Covariance

\[
\]
\[
\]
\[
= E[XY] - E[X]E[Y].
\]

Note that \( \text{COV}[X, Y] = E[XY] \) if either of the random variables has mean zero.

Suppose \( X \) and \( Y \) are continuous random variables

\[
E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf_{XY}(x, y) \, dy \, dx
\]
\[
E[X] = \int_{-\infty}^{\infty} xf_X(x) \, dx \quad \text{and} \quad E[Y] = \int_{-\infty}^{\infty} yf_Y(y) \, dy.
\]
Example Let $X$ and $Y$ be independent random variables, then

$$\text{COV}(X, Y) = E[(X - E[X])(Y - E[Y])]$$

$$= E[X - E[X]]E[Y - E[Y]] = 0.$$ 

Hence, a pair of independent random variables have zero covariance.
Correlation coefficient of $X$ and $Y$

$$\rho_{XY} = \frac{\text{COV}(X, Y)}{\sigma_X \sigma_Y} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y},$$

where $\sigma_X$ and $\sigma_Y$ are standard deviations of $X$ and $Y$, respectively.

Properties of $\rho_{XY}$

Consider

$$0 \leq E \left\{ \left( \frac{X - E[X]}{\sigma_X} \pm \frac{Y - E[Y]}{\sigma_Y} \right)^2 \right\} = 1 \pm 2\rho_{XY} + 1 = 2(1 \pm \rho_{XY}),$$

and so $-1 \leq \rho_{XY} \leq 1$. 
positive correlation  
negative correlation
The extreme values of $\rho_{XY}$ are achieved when $X$ and $Y$ are related linearly.

When $Y = aX + b$, $\rho_{XY} = \begin{cases} 1 & \text{if } a > 0 \\ -1 & \text{if } a < 0 \end{cases}$.

$X$ and $Y$ are said to be uncorrelated if $\rho_{XY} = 0$.

If $X$ and $Y$ are independent, then $\text{COV}(X,Y) = 0$, so $\rho_{XY} = 0$. We have independent $\Rightarrow$ uncorrelated.

How about uncorrelated $\Rightarrow$ independent?

As a special case, when $X$ and $Y$ are jointly Gaussian and $\rho_{XY} = 0$, then $X$ and $Y$ are independent random variables. However, it is possible for $X$ and $Y$ to be uncorrelated but not independent.
Example Let $\theta$ be uniformly distributed in $[0, 2\pi]$. Let $X = \cos \theta$ and $Y = \sin \theta$. The marginal pdf's of $X$ and $Y$ are arcsine pdf's, where
\[
 f_X(x) = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1; \quad f_Y(y) = \frac{1}{\pi} \frac{1}{\sqrt{1-y^2}}, \quad -1 < y < 1.
\]
The product of the marginals is non-zero in the square $\{(x, y): -1 < x < 1, -1 < y < 1\}$. If $X$ and $Y$ were independent,
\[
 f_{XY}(x, y) = \frac{1}{\pi^2} \frac{1}{\sqrt{1-x^2}} \frac{1}{\sqrt{1-y^2}} > 0, \quad -1 < x < 1 \text{ and } -1 < y < 1,
\]
so that the point $(X, Y)$ would assume all values in this square. However, $(X, Y)$ must lie on the unit circle. Hence, $(X, Y)$ cannot assume all values in the square, so $X$ and $Y$ are dependent. On the other hand, consider $E[XY] = E(\sin \theta \cos \theta) = \frac{1}{2\pi} \int_0^{2\pi} \sin \phi \cos \phi \, d\phi = 0$. Since $E[X] = E[Y] = 0$, we have
\[
 \rho_{XY} = \frac{E[XY]}{\sigma_X \sigma_Y} = 0,
\]
so $X$ and $Y$ are uncorrelated.
Example Suppose $X$ and $Y$ have the joint pdf:

$$f_{XY}(x, y) = \begin{cases} 2e^{-x}e^{-y} & 0 \leq y \leq x < \infty \\ 0 & \text{otherwise} \end{cases}.$$ 

The marginal pdf's are found to be

$$f_X(x) = 2e^{-x}(1 - e^{-x}), \quad 0 \leq x < \infty,$$

$$f_Y(y) = 2e^{-2y}, \quad 0 \leq y < \infty.$$


$$E[XY] = \int_0^\infty \int_0^x 2xye^{-x}e^{-y} \, dy \, dx$$

$$= \int_0^\infty 2xe^{-x}(1 - e^{-x} - xe^{-x}) \, dx = 1.$$

Correlation coefficient, $\rho_{XY} = \frac{1 - \frac{3}{2} \cdot \frac{1}{2}}{\sqrt{\frac{5}{4}} \sqrt{\frac{1}{4}}} = \frac{1}{\sqrt{5}}$.
Function of several random variables

Suppose $Z = g(X_1, X_2, \cdots, X_n)$, where $X_1, \cdots, X_n$ are random variables. The cdf of $Z$ is found by first finding the equivalent event of $\{Z \leq z\}$, i.e. the set $R_Z = \{x = (x_1, \cdots, x_n) \text{ such that } g(x) \leq z\}$, then

$$F_Z(z) = P[x \in R_Z] = \int_{x \in R_Z} \cdots \int f_{X_1X_2\cdots X_n}(x'_1, \cdots, x'_n) \, dx'_1, \cdots dx'_n.$$

The pdf of $Z$ is $f_Z(z) = \frac{d}{dz} F_Z(z)$. 
Example Sum of two random variables

Let \( Z = X + Y \)

\[
F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x'} f_{XY}(x', y') \, dy' \, dx'.
\]

\[
f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x', z-x') \, dx'.
\]

If \( X \) and \( Y \) are independent random variables, then

\[
f_Z(z) = \int_{-\infty}^{\infty} f_X(x') f_Y(z-x') \, dx'.
\]

This is the convolution integral of the marginal pdf of \( X \) and \( Y \).
For example, suppose $X$ and $Y$ are two zero-mean unit variance Gaussian random variables with correlation coefficient $\rho$, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x', z - x') \, dx'$$

$$= \frac{1}{2\pi\sqrt{1 - \rho^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{[x'2 - 2\rho x'(z - x') + (z - x')^2]}{2(1 - \rho^2)}\right) \, dx'$$

$$= \frac{e^{-z^2/2}}{\sqrt{2\pi}}.$$ 

Therefore, the sum of two non-independent Gaussian random variables remains to be Gaussian.
Example (System with standby redundancy)

The system has a single key component in operation and a duplicate of that component in standby mode. When the first component fails, the second component is put into operation. Find the pdf of the lifetime of the standby system if the components have independent exponentially distributed lifetimes with the same mean.

Solution Let $T_1$ and $T_2$ be the lifetimes of the two components, so system lifetime $T = T_1 + T_2$.

$$f_{T_1}(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad f_{T_2}(z-x) = \begin{cases} \lambda e^{-\lambda(z-x)} & z-x \geq 0 \\ 0 & x > z \end{cases}$$

$$f_T(z) = \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} dx$$

$$= \lambda^2 e^{-\lambda z} \int_0^z dx = \lambda^2 ze^{-\lambda z}, \quad \text{for } z \geq 0;$$

$$f_T(z) = 0, \quad \text{for } z < 0.$$
The conditional pdf can be used to find the pdf of a function of several random variables.

Let \( Z = g(X, Y) \), we first find the pdf of \( Z \) given \( Y = y \), that is, \( f_{Z}(z|Y = y) \). Then, the pdf is given by

\[
f_{Z}(z) = \int_{-\infty}^{\infty} f_{Z}(z|y') f_{Y}(y') \, dy'.
\]
Example Let $Z = X/Y$, find the pdf of $Z$ if $X$ and $Y$ are independent and both are exponentially distributed with mean one.

Hint Recall that for $Y = \alpha X + \beta$, then $f_Y(y) = \frac{1}{|\alpha|} f_X\left(\frac{y - \beta}{\alpha}\right)$.

Solution

Suppose $Y = y, Z = \frac{1}{y}X$. Taking $\alpha = \frac{1}{y}$, we have

$$f_Z(z|y) = |y| f_X(yz|y);$$

pdf of $Z = f_Z(z) = \int_{-\infty}^{\infty} |y'| f_X(y'z|y') f_Y(y') \, dy'$$

$$= \int_{-\infty}^{\infty} |y'| f_{XY}(y'z, y') \, dy'.$$

If $X$ and $Y$ are independent and exponentially distributed with mean one, then

$$f_Z(z) = \int_{0}^{\infty} y' f_X(y'z) f_Y(y') \, dy', \quad z > 0$$

$$= \int_{0}^{\infty} y' e^{-y'z} e^{-y'} \, dy' = \frac{1}{(1 + z)^2}, \quad z > 0.$$