MATH 246, Fall 1999

Final Examination

Time allowed: 3 hours *Instructor: Dr. Y. K. Kwok*

[**points**]

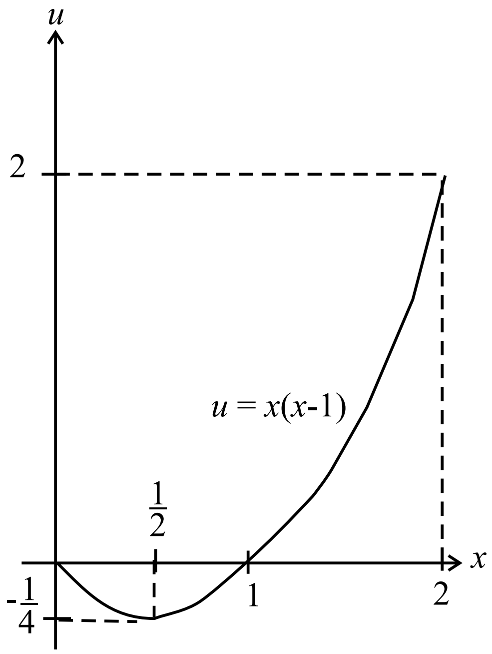
1. Let the pair of random variables U and W be defined by

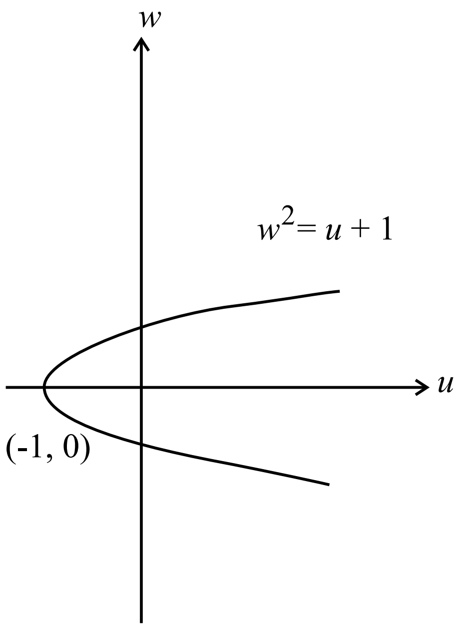
$$\begin{cases} U = X(X-1) \\ W^2 = U+1 \end{cases},$$

where X is a random variable which is uniformly distributed over $(0,2)$. The plots of the functions

$$\begin{cases} u = x(x-1), & 0 < x < 2 \\ w^2 = u+1, & -1 \le u < \infty \end{cases}$$

are shown in the figures below:





(a) Find the probability density function of U, $f_U(u)$. [**6**]

Hint Recall that $f_U(u)\Delta u \approx P[u < U \le u + \Delta u], \Delta u > 0$, where Δu is very small.

Find the events associated with the random variable X which are equivalent to the event $\{u < U \le u + \Delta u\}$. Note the difference between the cases (i) $-\frac{1}{4} \le u < 0$, (ii) $0 \le u < 2$, (iii) $u \notin \left[-\frac{1}{4}, 2\right)$, (correspond to having two roots, one root or no root, respectively, for the equation: $u = x(x-1), 0 < x < 2$).

(b) Find the conditional probability density function of $W, f_W(w|u), -\frac{1}{4} \leq u < 2$, which is defined by $f_W(w|u) = \frac{d}{dw}F_W(w|u)$. The conditional distribution function $F_W(w|u)$ is given by **[6]**

$$\begin{aligned} F_W(w|u) &= \lim_{\Delta u \to 0} P[W \leq w | u < U \leq u + \Delta u] \\ &= \lim_{\Delta u \to 0} \frac{P[W \leq w, u < U \leq u + \Delta u]}{P[u < U \leq u + \Delta u]}, \quad -\frac{1}{4} \leq u < 2. \end{aligned}$$

Hint For $-\frac{1}{4} \leq u < 2$, these are always two roots of w for the equation: $w^2 = u + 1$. Let the two roots be denoted by

$$w_+ = \sqrt{u+1} \quad \text{and} \quad w_- = -\sqrt{u+1}.$$

Distinguish the cases: (i) $w < w_-$, (ii) $w_- < w < w_+$, and (iii) $w > w_+$.

2. (a) Let X and Y be a pair of continuous random variables. Suppose the conditional expectation of $E[Y|x]$ can be viewed as defining a function of x, and so $E[Y|X]$ is a function of the random variable X. Show that the expectation of $E[Y|X]$ is equal to the expectation of Y, that is,

$$E[E[Y|X]] = E[Y].$$

Hint

$$E[E[Y|X]] = \int_{-\infty}^{\infty} E[Y|x] f_X(x)\, dx,$$

where $f_X(x)$ is the marginal density function of X. **[4]**

(b) A customer entering a service station is served by serviceman i with probability $p_i, i = 1, 2, \cdots, n$. The time taken by serviceman i to service a customer is an exponentially distributed random variable with parameter α_i. Let I be the discrete random variable which assumes the value i if the customer is serviced by the i^{th} serviceman, and let $P_I(i)$ denote the probability mass function of I. Let T denote the time taken to service a customer.

(i) Explain the meaning of the following formula

$$P[T \leq t] = \sum_{i=1}^{n} P_I(i) P[T \leq t | I = i]$$

and use it to find the probability density function of T. **[3]**

(ii) Use the conditional expectation to compute $E[T]$. **[3]**

3. Let X and Y be a pair of independent exponentially distributed random variables with parameters α and β, respectively, $\alpha \neq \beta$. Define another pair of random variables U and V by

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

(a) Find the joint probability density function of U and V, and determine whether U and V are independent. **[5]**

(b) Let $Z = X/Y$, find the probability density function of Z. **[5]**

Hint Compute $\mathrm{COV}(U, V)$ and show that it is non-zero. How to relate independence with non-zero covariance?

4. (a) Show that the correlation coefficient ρ_{XY} between a pair of random variables X and Y must satisfy

$$-1 \leq \rho_{XY} \leq 1.$$

[5]

(b) Let X be a Gaussian random variable with mean m and variance σ^2. Define $Y = aX + b$, where a and b are constants.

(i) Show that Y is also Gaussian, and find the probability density function of Y. **[3]**

(ii) Show that $\rho_{XY} = \dfrac{a}{|a|}$. **[2]**

5. Suppose a fair die is tossed 120 times. Use the central limit theorem to find approximately the probability that the face '4' will turn 18 times or less. Express your answer in terms of the standard normal cumulative function, $\Phi(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-t^2/2}\, dt$. **[6]**

Hint Let N be the number of times that the face '4' turns up. Taking the distribution to be continuous as an approximation, the problem is to find $P[-0.5 \leq N \leq 18.5]$.

6. (a) Let $X(t) = A\cos\omega t + B\sin\omega t$, where A and B are independent identically distributed Gaussian random variables with zero mean and variance σ^2. Find the mean and autocovariance of $X(t)$. **[4]**

(b) State the stationary increments and independent increments properties of a Poisson process. **[2]**

(c) Let $N(t), t \geq 0$, be a Poisson process with parameter $\lambda > 0$.

(i) If $t_2 > t_1$, compute the joint probability mass function

$$P[N(t_1) = i, N(t_2) = j]$$

[3]

(ii) Show that the autocovariance of $N(t)$ is given by **[3]**

$$C_N(t_1, t_2) = \lambda \min(t_1, t_2).$$

List of useful formulae

Binomial Random Variable

$S_X = \{0, 1, \ldots, n\}$ $\qquad p_k = C_k^n p^k (1-p)^{n-k} \quad k = 0, 1, \ldots, n$

$E[X] = np \qquad \text{VAR}[X] = np(1-p)$

Poisson Random Variable

$S_X = \{0, 1, 2, \ldots\}$ $\qquad p_k = \dfrac{\alpha^k}{k!} e^{-\alpha} \qquad k = 0, 1, \ldots$ and $\alpha > 0$

$E[X] = \alpha \qquad \text{VAR}[X] = \alpha$

Uniform Random Variable

$S_X = [a, b]$ $\qquad f_X(x) = \dfrac{1}{b-a} \quad a \le x \le b$

$E[X] = \dfrac{a+b}{2} \qquad \text{VAR}[X] = \dfrac{(b-a)^2}{12}$

Exponential Random Variable

$S_X = [0, \infty)$ $\qquad f_X(x) = \lambda e^{-\lambda x} \quad x \ge 0$ and $\lambda > 0$

$E[X] = \dfrac{1}{\lambda} \qquad \text{VAR}[X] = \dfrac{1}{\lambda^2}$

Gaussian (Normal) Random Variable

$S_X = (-\infty, \infty)$ $\qquad f_X(x) = \dfrac{e^{-(x-m)^2/2\sigma^2}}{\sqrt{2\pi}\sigma} \quad -\infty < x < \infty$ and $\sigma > 0$

$E[X] = m \qquad \text{VAR}[X] = \sigma^2$

Relations between pdf's when $Y = g(X)$

(i) $Y = aX + b$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

(ii) a non-linear function $Y = g(X)$

$$f_Y(y) = \sum_k \left.\frac{f_X(x)}{\left|\frac{dy}{dx}\right|}\right|_{x=x_k}$$

Marginal pdf's

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y')\, dy' \quad \text{and} \quad f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x', y)\, dx'$$

Independence of X and Y

X and Y are independent if and only if $f_{XY}(x,y) = f_X(x)f_Y(y)$, for all x, y

Conditional pdf of Y given $X = x$

$$f_Y(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

Conditional expectation of Y given $X = x$

Continuous $\quad E[Y|x] = \int_{-\infty}^{\infty} y f_Y(y|x)\ dy$

discrete $\quad F[Y|x] = \sum_{y_j} y_j P_Y(y_j|x)$

Functions of several random variables

(i) $Z = X + Y$, $\quad F_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x'} f_{XY}(x', y')\ dy' dx'$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x', z - x')\ dx'$$

If X and Y are independent, then $f_Z(z) = \int_{-\infty}^{\infty} f_X(x') f_Y(z - x')\ dx'$

(ii) $Z = X/Y$, $\quad f_Z(z|y) = |y| f_X(yz|y)$

$$f_Z(z) = \int_{-\infty}^{\infty} f_Z(z|y') f_Y(y')\ dy'$$

(iii) $\mathbf{Z} = A\mathbf{X}$

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{f_{\mathbf{X}}(A^{-1}\mathbf{z})}{|\det\ A|}$$

Correlation and covariance of two random variables

$\mathrm{COV}(X, Y) = E[(X - m_X)(Y - m_Y)]$, where m_X and m_Y are $E[X]$ and $E[Y]$, resp.

$\rho_{XY} = \frac{\mathrm{COV}(X,Y)}{\sigma_X \sigma_Y} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y}$

autocovariance $C_X(t_1, t_2)$ of a random process $X(t)$

$$C_X(t_1, t_2) = E[\{X(t_1) - m_X(t_1)\}\{X(t_2) - m_X(t_2)\}]$$

— *End* —