MATH 246, Fall 1999

Solution to Final Examination

---

1. (a) (i) For $-\frac{1}{4} \le u < 0$, there are two roots of x given u for the equation: $u = x(x-1)$. Let x_1 and $x_2 (x_1 < x_2)$ be these two roots. It is seen that

$$x_1 = \frac{1-\sqrt{1+4u}}{2} \quad \text{and} \quad x_2 = \frac{1+\sqrt{1+4u}}{2}; \quad x_1 + x_2 = 1.$$

The event $\{u < U \le u + \Delta u\}, \Delta u > 0$, is equivalent to the union of

$$\{x_1 - \Delta x_1 < X \le x_1\} \quad \text{and} \quad \{x_2 < X \le x_2 + \Delta x_2\}, \quad \Delta x_1 > 0, \Delta x_2 > 0,$$

where

$$|\Delta u| \sim |2x_1 - 1| \; |\Delta x_1| \quad \text{and} \quad |\Delta u| \sim |2x_2 - 1| \; |\Delta x_2|.$$

Since $x_1 + x_2 = 1$ and so $|2x_2 - 1| = |2x_1 - 1|$, giving $\Delta x_1 = \Delta x_2$. Now,

$$f_U(u)\Delta u = f_X(x_1)\Delta x_1 + f_X(x_2)\Delta x_2 = \frac{1}{2}\Delta x_1 + \frac{1}{2}\Delta x_2 = \Delta x_2,$$

using the result that $f_X(x_1) = f_X(x_2) = \frac{1}{2}$ as derived from the uniformly distributed property of X over $(0, 2)$. Lastly,

$$f_U(u) = \frac{1}{2}\left[\frac{1}{\left|\frac{du}{dx}\right|_{x=x_1}} + \frac{1}{\left|\frac{du}{dx}\right|_{x=x_2}}\right] = \frac{1}{\sqrt{1+4u}}, \quad -\frac{1}{4} \le u < 0.$$

(ii) For $0 \le u < 2$, there is only one root (call it $\hat{x}$) for the equation: $u = x(x-1)$, where $\hat{x} = \frac{1+\sqrt{1+4u}}{2}$. The event $\{u < U \le u + \Delta u\}$ is equivalent to $\{\hat{x} < X < \hat{x} + \Delta\hat{x}\}$, where $\left|\frac{du}{dx}\right|_{x=\hat{x}} = \frac{1}{2\hat{x}-1} = \frac{1}{\sqrt{1+4u}}$. Now,

$$f_U(u) = \frac{1}{2}\frac{1}{\left|\frac{du}{dx}\right|_{x=\hat{x}}} = \frac{1}{2\sqrt{1+4u}}, \quad 0 \le u < 2.$$

(iii) U cannot assume values outside $\left[-\frac{1}{4}, 2\right)$. Therefore, $f_U(u) = 0$ for $u < -\frac{1}{4}$ or $u \ge 2$.

In summary, the probability density function of U is given by

$$f_U(u) = \begin{cases} \frac{1}{\sqrt{1+4u}}, & -\frac{1}{4} \le u < 0 \\ \frac{1}{2\sqrt{1+4u}}, & 0 \le u < 2 \\ 0, & \text{otherwise} \end{cases}.$$

(b) Let $w_+ = \sqrt{u+1}$ and $w_- = -\sqrt{u+1}$; and observe that when u moves to $u+\Delta u$, w_+ becomes $w_+ + \Delta w_+$ and w_- becomes $w_- - \Delta w_-$. By the symmetry property of the curve: $w^2 = u+1$, it is observed that $\Delta w_+ = \Delta w_-$. Since $P[W < \infty, u < U \leq u + \Delta u] = P[u < U \leq u + \Delta u]$

$$P[W \leq w, u < U \leq u+\Delta u] = \begin{cases} 0, & w < w_- \\ \frac{1}{2}P[u < U \leq u+\Delta u], & w_- \leq w < w_+ \\ P[u < U \leq u + \Delta u], & w \geq w_+ \end{cases}$$

where the factor $1/2$ comes from the symmetry property of $f_{WV}(w,u)$ with respect to w and $\Delta w_+ = \Delta w_-$. Hence,

$$F_W(w|u) = \frac{1}{2}[H(w-w_-) + H(w-w_+)], \quad -\frac{1}{4} \leq u < 2$$

and

$$f_W(w|u) = \frac{1}{2}[\delta(w-w_-) + \delta(w-w_+)], \quad -\frac{1}{4} \leq u < 2.$$

2. (a)
$$\begin{aligned} E[E[Y|X]] &= \int_{-\infty}^{\infty} E[Y|x] f_X(x)\ dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\ f_Y(y|x)\ dy\ f_X(x)\ dx \\ &= \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{XY}(x,y)\ dx\ dy \text{ (interchanging the order of integration)} \\ &= \int_{-\infty}^{\infty} y\ f_Y(y)\ dy = E[Y]. \end{aligned}$$

(b) (i) From the conditional probability formula, we have

$$P[T \leq t, I = i] = P_I(i) P[T \leq t | I = i].$$

The marginal distribution function $P[T \leq t]$ is obtained by summing the joint probability values $P[T \leq t, I = i]$ for all possible values of i. Hence,

$$P[T \leq t] = \sum_{i=1}^{n} P_I(i) P[T \leq t | I = i].$$

Here, $P_I(i) = p_i$ and $P[T \leq t | I = i] = 1 - e^{-\alpha_i t}, t \geq 0$. The probability density function of T is given by

$$f_T(t) = \frac{d}{dt} P[T \leq t] = \begin{cases} \sum_{i=1}^{n} p_i \alpha_i e^{-\alpha_i t}, & t \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

(ii)
$$E[T] = E[E[T|I]] = \sum_{i=1}^{n} P_I(i)E[T|I=i]$$
$$= \sum_{i=1}^{n} p_i \int_0^\infty \alpha_i t e^{-\alpha_i t}\, dt = \sum_{i=1}^{n} p_i/\alpha_i.$$

3. (a) From $\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$, we have $\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}$,
that is $X = \dfrac{U+V}{2}$, $Y = \dfrac{V-U}{2}$; also, $\det A = 2$.
Since X and Y are independent, their joint probability density function $f_{XY}(x,y)$ is given by

$$f_{XY}(x,y) = \begin{cases} \alpha e^{-\alpha x} \beta e^{-\beta y} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Now

$$f_{UV}(u,v) = f_{XY}\left(\frac{u+v}{2}, \frac{v-u}{2}\right)/2$$
$$= \begin{cases} \frac{\alpha\beta}{2} e^{-\alpha\left(\frac{u+v}{2}\right)} e^{-\beta\left(\frac{v-u}{2}\right)} & u+v \geq 0 \text{ and } v-u \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

The density function is non-zero only in the quadrant: $u+v \geq 0$ and $v-u \geq 0$.
Now, $\text{COV}(U,V) = \text{COV}(X-Y, X+Y) = \text{VAR}(X) - \text{VAR}(Y) \neq 0$ since $\alpha \neq \beta$. It is known that when a pair of random variables are independent, they must have zero covariance. Here, the non-zero value of $\text{COV}(U,V)$ would imply dependence of U and V.

(b)

$$f_Z(z) = \int_{-\infty}^{\infty} |y'| f_{XY}(y'z, y')\, dy'$$
$$= \int_0^\infty y' \alpha e^{-\alpha y' z} \beta e^{-\beta y'}\, dy'$$
$$= \alpha\beta \int_0^\infty y' e^{-(\alpha z+\beta)y'}\, dy' = \frac{\alpha\beta}{(\alpha z+\beta)^2}, \quad z \geq 0;$$
$$f_Z(z) = 0 \quad \text{for} \quad z < 0.$$

4. (a) Consider

$$0 \leq E\left[\left(\frac{X-E[X]}{\sigma_X} \pm \frac{Y-E[Y]}{\sigma_Y}\right)^2\right]$$
$$= E\left[\frac{(X-E[X])^2}{\sigma_X^2}\right] \pm 2E\left[\frac{(X-E[X])(Y-E[Y])}{\sigma_X \sigma_Y}\right] + E\left[\frac{(Y-E[Y])^2}{\sigma_Y^2}\right],$$
$$= 1 \pm 2\rho_{XY} + 1 = 2(1 \pm \rho_{XY})$$

and so

$$-1 \le \rho_{XY} \le 1.$$

(b)

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\sigma \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$

$$f_Y(y) = \frac{1}{|a|}\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\left[\frac{y-(am+b)}{2a^2\sigma^2}\right]^2\right);$$

Hence, Y is Gaussian with mean $am+b$ and variance $|a|^2\sigma^2$.
$\text{COV}(X,Y) = \text{COV}(X, aX+b) = a\ \text{COV}(X,X) = a\ \text{VAR}(X)$

$$\rho_{XY} = \frac{\text{COV}(X,Y)}{\sqrt{\text{VAR}(X)\ \text{VAR}(Y)}} = \frac{a\sigma}{\sqrt{|a|^2\sigma^2}} = \frac{a}{|a|} = \begin{cases} 1 & \text{if } a > 0 \\ -1 & \text{if } a < 0 \end{cases}.$$

5. The given problem is a binomial experiment of tossing the die with $n = 120$ and $p = \frac{1}{6}$. The mean and standard deviation are $np = 20$ and $\sqrt{np(1-p)} = \sqrt{120\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)} = 4.08$. The corresponding standard normal variable is given by

$$Z = \frac{N-20}{4.08}.$$

When $N = -0.5$, $Z = \frac{-0.5-20}{4.08} = -5.02$; when $N = 18.5$, $Z = \frac{18.5-20}{4.08} = -0.37$.
By the central limit theorem,

$$\begin{aligned} P[-0.5 \le N \le 18.5] &\approx P[-5.02 \le Z \le -0.37] \\ &= \int_{-5.02}^{-0.37} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}dt = \Phi(-0.37) - \Phi(-5.02). \end{aligned}$$

6. (a) mean $= E[X(t)] = E[A\cos\omega t + B\sin\omega t] = \cos\omega t\ E[A] + \sin\omega t\ E[B] = 0$

$$\begin{aligned} \text{autocovariance } &= E[\{X(t_1) - m_X(t_1)\}\{X(t_2) - m_X(t_2)\}] \\ &= E[X(t_1)X(t_2)] \\ &= E[(A\cos\omega t_1 + B\sin\omega t_1)(A\cos\omega t_2 + B\sin\omega t_2)] \\ &= E[A^2]\cos\omega t_1 \cos\omega t_2 + E[AB](\cos\omega t_1 \sin\omega t_2 + \sin\omega t_1 \cos\omega t_2) \\ &\quad + E[B^2]\sin\omega t_1 \sin\omega t_2 \\ &= (E[A^2] - E[A]^2)\cos\omega t_1 \cos\omega t_2 + (E[B^2] - E[B]^2)\sin\omega t_1 \sin\omega t_2 \\ &= \sigma^2 \cos\omega(t_1 - t_2). \end{aligned}$$

(since A and B are independent, $E[AB] = E[A]E[B]$
and $E[A] = E[B] = 0$)

(b) (i) Independent increments for non-overlapping intervals.

Let $[t_1, t_2]$ and $[t_3, t_4]$ be two non-overlapping time intervals. The independent increments refer that $N[t_2] - N[t_1]$ and $N[t_4] - N[t_3]$ are independent.

(ii) Stationary increments property

Increments in intervals of the same length have the same distribution regardless of when the interval begins.

(c) (i) For $t_1 < t_2$,

$$\begin{aligned}
& P[N(t_1) = i, N(t_2) = j] \\
&= P[N(t_1) = i]P[N(t_2) - N(t_1) = j - i] \quad \text{(independent increments)} \\
&= P[N(t_1) = i]P[N(t_2 - t_1) = j - i] \quad \text{(stationary increments)} \\
&= \frac{(\lambda t_1)^i e^{-\lambda t_1}}{i!} \frac{[\lambda(t_2 - t_1)]^{j-i} e^{-\lambda(t_2 - t_1)}}{(j-i)!}
\end{aligned}$$

(ii) For $t_1 < t_2$,

$$\begin{aligned}
C_N(t_1, t_2) &= E[(N(t_1) - \lambda t_1)(N(t_2) - \lambda t_2)] \\
&= E[N(t_1) - \lambda t_1]E[N(t_2) - N(t_1) - \lambda(t_2 - t_1)] + \text{ VAR } [N(t_1)] \\
&= \text{ VAR } [N(t_1)] = \lambda t_1 = \lambda \min(t_1, t_2);
\end{aligned}$$

similarly, for $t_2 < t_1$,

$$C_N(t_1, t_2) = \lambda t_2 = \lambda \min(t_1, t_2).$$

Hence,

$$C_N(t_1, t_2) = \lambda \min(t_1, t_2).$$