3. Exponential and trigonometric functions

From the first principles, we define the complex exponential function as a complex function $f(z)$ that satisfies the following defining properties:

1. $f(z)$ is entire,
2. $f'(z) = f(z)$,
3. $f(x) = e^x$, x is real.

Let $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$. From property (1), u and v satisfy the Cauchy-Riemann relations. Combining (1) and (2)

$$u_x + iv_x = v_y - iu_y = u + iv.$$

First, we observe that $u_x = u$ and $v_x = v$ and so

$$u = e^x g(y) \quad \text{and} \quad v = e^x h(y),$$

where $g(y)$ and $h(y)$ are arbitrary functions in y.

We also have

$$v_y = u \quad \text{and} \quad u_y = -v,$$

from which we deduce that the arbitrary functions are related by

$$h'(y) = g(y) \quad \text{and} \quad -g'(y) = h(y).$$

By eliminating $g(y)$ in the above relations, we obtain

$$h''(y) = -h(y).$$

The general solution of the above equation is given by

$$h(y) = A \cos y + B \sin y,$$

where A and B are arbitrary constants. Furthermore, using $g(y) = h'(y)$, we have

$$g(y) = -A \sin y + B \cos y.$$

To determine the arbitrary constants A and B, we use property (3) where

$$e^x = u(x, 0) + iv(x, 0) = g(0)e^x + ih(0)e^x = Be^x + iAe^x.$$

We then obtain $B = 1$ and $A = 0$. Putting all the results together, the complex exponential function is found to be

$$f(z) = e^z = e^x \cos y + ie^x \sin y,$$

By setting $x = 0$, we then deduce the Euler formula:

$$e^{iy} = \cos y + i \sin y.$$

It can be verified that

$$e^{z_1 + z_2} = e^{z_1} e^{z_2},$$

another basic property of the exponential function.

The modulus of e^z is non-zero since

$$|e^z| = e^x \neq 0, \qquad \text{for all } z \text{ in } \mathbb{C},$$

and so $e^z \neq 0$ for all z in the complex z-plane. The range of the complex exponential function is the entire complex plane except the zero value.

Periodic property

$$e^{z+2k\pi i} = e^z, \text{ for any } z \text{ and integer } k,$$

that is, e^z is periodic with the fundamental period $2\pi i$. The complex exponential function is periodic while its real counterpart is not.

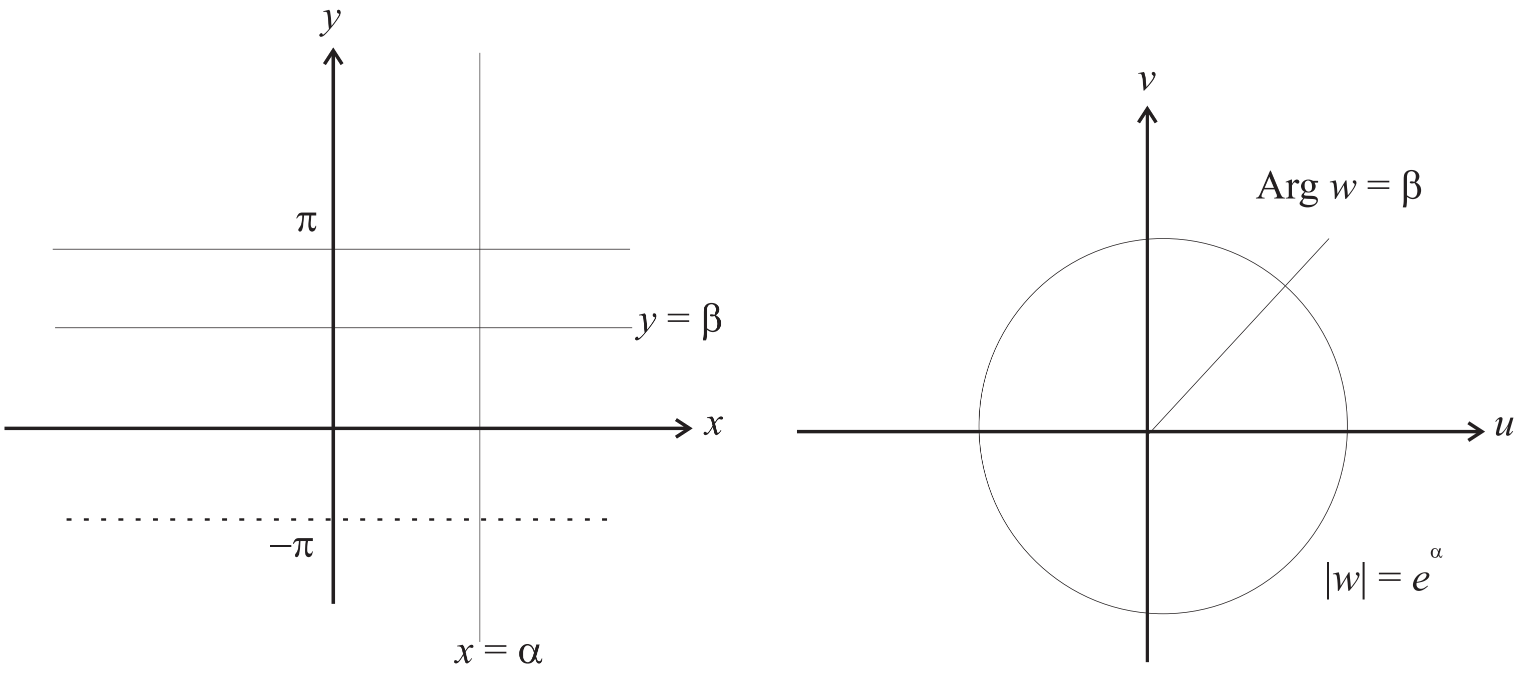
Mapping properties of the complex exponential function

Since the complex exponential function is periodic with fundamental period $2\pi i$, it is a many-to-one function. If we restrict z to lie within the infinite strip $-\pi < \text{Im } z \leq \pi$, then the mapping $w = e^z$ becomes one-to-one.

The vertical line $x = \alpha$ is mapped onto the circle $|w| = e^\alpha$, while the horizontal line $y = \beta$ is mapped onto the ray $\text{Arg } w = \beta$.

When the vertical line $x = \alpha$ moves further to the left, the mapped circle $|w| = e^\alpha$ shrinks to a smaller radius. When the horizontal line in the z-plane moves vertically from $y = -\pi$ to $y = \pi$, the image ray in the w-plane traverses in anticlockwise sense from $\text{Arg } w = -\pi$ to $\text{Arg } w = \pi$.

In particular, the whole x-axis is mapped onto the positive u-axis, and the portion of the y-axis, $-\pi < y \leq \pi$, is mapped onto the unit circle $|w| = 1$.



Example

Consider the following function:

$$f(z) = e^{\frac{\alpha}{z}}, \ \ \alpha \text{ is real.}$$

Show that $|f(z)|$ is constant on the circle $x^2 + y^2 - ax = 0$, a is a real constant.

Solution

Write the equation of the circle as

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$$

which reveals that the circle is centered at $\left(\frac{a}{2}, 0\right)$ and has radius $\frac{a}{2}$. A possible parametric representation of the circle is

$$x = \frac{a}{2}(1 + \cos\theta) \quad \text{and} \quad y = \frac{a}{2}\sin\theta, \qquad -\pi < \theta \leq \pi.$$

The parameter θ is the angle between the positive x-axis and the line joining the center $\left(\frac{a}{2}, 0\right)$ to the point (x, y). The complex representation of the circle can be expressed as

$$z = \frac{a}{2}\,(1 + e^{i\theta}), \qquad -\pi < \theta \leq \pi.$$

The modulus of $f(z)$ when z lies on the circle is found to be

$$|f(z)| = \left| e^{\frac{2\alpha}{a(1+e^{i\theta})}} \right| = \left| e^{\frac{2\alpha}{a}\,\frac{1+\cos\theta - i\sin\theta}{2(1+\cos\theta)}} \right| = e^{\frac{\alpha}{a}}.$$

The modulus value is equal to a constant with no dependence on θ, that is, independent of the choice of the point on the circle.

Trigonometric and hyperbolic functions

Using the Euler formula $e^{iy} = \cos y + i \sin y$, the real sine and cosine functions can be expressed in terms of e^{iy} and e^{-iy} as follows:

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i} \quad \text{and} \quad \cos y = \frac{e^{iy} + e^{-iy}}{2}.$$

We define the complex sine and cosine functions in the same manner

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

The other complex trigonometric functions are defined in terms of the complex sine and cosine functions by the usual formulas:

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}.$$

Let $z = x + iy$, then

$$e^{iz} = e^{-y}(\cos x + i \sin x) \quad \text{and} \quad e^{-iz} = e^{y}(\cos x - i \sin x).$$

The complex sine and cosine functions are seen to be

$$\begin{aligned} \sin z &= \sin x \cosh y + i \cos x \sinh y, \\ \cos z &= \cos x \cosh y - i \sin x \sinh y. \end{aligned}$$

Moreover, their moduli are found to be

$$|\sin z| = \sqrt{\sin^2 x + \sinh^2 y}, \quad |\cos z| = \sqrt{\cos^2 x + \sinh^2 y}.$$

Since $\sinh y$ is unbounded at large values of y, the above modulus values can increase (as y does) without bound.

While the real sine and cosine functions are always bounded between -1 and 1, their complex counterparts are unbounded.

The complex hyperbolic functions are defined by

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}, \quad \tanh z = \frac{\sinh z}{\cosh z}.$$

The other hyperbolic functions cosech z, sech z and coth z are defined as the reciprocal of $\sinh z$, $\cosh z$ and $\tanh z$, respectively.

In fact, the hyperbolic functions are closely related to the trigonometric functions. Suppose z is replaced by iz, we obtain

$$\sinh iz = i \sin z.$$

Similarly, one can show that

$$\sin iz = i \sinh z, \quad \cosh iz = \cos z, \quad \cos iz = \cosh z.$$

The real and imaginary parts of $\sinh z$ and $\cosh z$ are found to be

$$\begin{aligned}\sinh z &= \sinh x \cos y + i \cosh x \sin y,\\ \cosh z &= \cosh x \cos y + i \sinh x \sin y,\end{aligned}$$

and their moduli are given by

$$\begin{aligned}|\sinh z| &= \sqrt{\sinh^2 x + \sin^2 y},\\ |\cosh z| &= \sqrt{\cosh^2 x - \sin^2 y}.\end{aligned}$$

The complex hyperbolic functions $\sinh z$ and $\cosh z$ are periodic with fundamental period $2\pi i$; and $\tanh z$ is periodic with fundamental period πi.

A zero α of a function $f(z)$ satisfies $f(\alpha) = 0$. To find the zeros of $\sinh z$, we observe that

$$\sinh z = 0 \Leftrightarrow |\sinh z| = 0 \Leftrightarrow \sinh^2 x + \sin^2 y = 0.$$

Hence, x and y must satisfy $\sinh x = 0$ and $\sin y = 0$, thus giving $x = 0$ and $y = k\pi$, k is any integer.

The zeros of $\sinh z$ are $z = k\pi i$, k is any integer.

Mapping properties of the complex sine function

Consider the complex sine function

$$w = \sin z = \sin x \cosh y + i \cos x \sinh y, \quad z = x + iy,$$

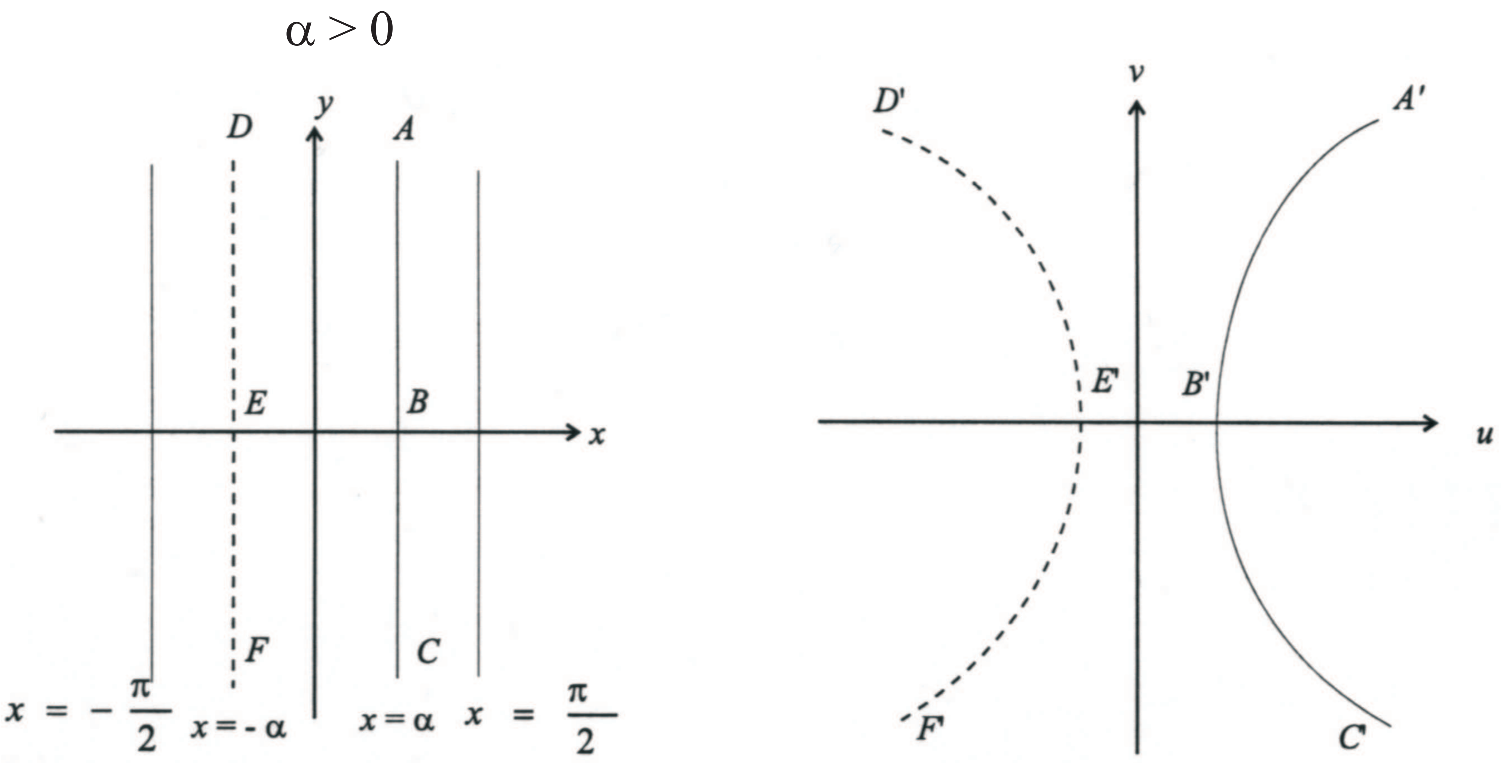
suppose we write $w = u + iv$, then

$$u = \sin x \cosh y \quad \text{and} \quad v = \cos x \sinh y.$$

We find the images of the coordinates lines $x = \alpha$ and $y = \beta$. When $x = \alpha$, $u = \sin\alpha \cosh y$ and $v = \cos\alpha \sinh y$. By eliminating y in the above equations, we obtain

$$\frac{u^2}{\sin^2\alpha} - \frac{v^2}{\cos^2\alpha} = 1,$$

which represents a hyperbola in the w-plane.



Vertical lines are mapped onto hyperbolas under $w = \sin z$.

Take $\alpha > 0$. When $0 < \alpha < \frac{\pi}{2}$, $u = \sin\alpha \cosh y > 0$ for all values of y, so the line $x = \alpha$ is mapped onto the right-hand branch of the hyperbola. Likewise, the line $x = -\alpha$ is mapped onto the left-hand branch of the same hyperbola.

In particular, when $\alpha = \frac{\pi}{2}$, the line $x = \frac{\pi}{2}$ is mapped onto the line segment $v = 0$, $u \geq 1$ (degenerate hyperbola). Also, the y-axis is mapped onto the v-axis.

We conclude that the infinite strip $\left\{0 \leq \text{Re } z \leq \dfrac{\pi}{2}\right\}$ is mapped to the right half-plane $\{u \geq 0\}$. By symmetry, the other infinite strip $\left\{-\dfrac{\pi}{2} \leq \text{Re } z \leq 0\right\}$ is mapped to the left half-plane $\{u \leq 0\}$.

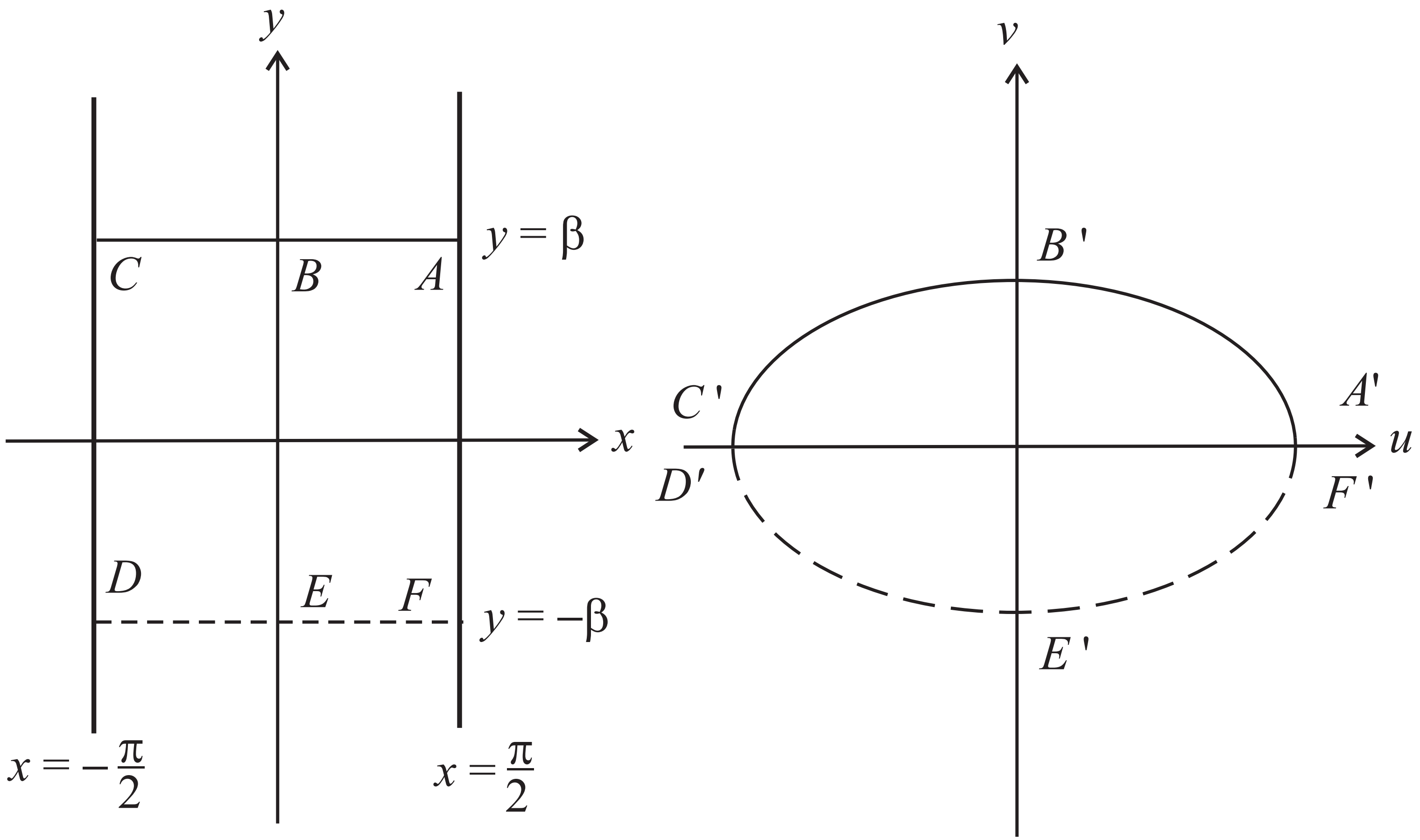
Consider the image of a horizontal line $y = \beta$ $(\beta > 0)$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ under the mapping $w = \sin z$.

When $y = \beta$, $u = \sin x \cosh \beta$ and $v = \cos x \sinh \beta$. By eliminating x in the above equations, we obtain

$$\frac{u^2}{\cosh^2 \beta} + \frac{v^2}{\sinh^2 \beta} = 1,$$

which represents an ellipse in the w-plane.

- The upper line $y = \beta$ (the lower line $y = -\beta$) is mapped onto the upper (lower) portion of the ellipse.
- When $\beta = 0$, the line segment $y = 0, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ is mapped onto the line segment $v = 0, -1 \leq u \leq 1$, which is a degenerate ellipse.



Horizontal lines are mapped onto ellipses under $w = \sin z$.

Example

(a) Find the general solution for $e^z = 1 + i$. Is $|e^z|$ bounded when $\text{Re}\,(z) = \beta$?

(b) Show that $|\sin z|$ is bounded when $\text{Im}\,(z) = \alpha$.

Solution

(a) $e^z = e^x(\cos y + i \sin y) = 1 + i, z = x + iy$.

$e^x = |1 + i| = \sqrt{2}$ so that $x = \frac{1}{2}\ln 2; \tan y = 1$ so that $y = \frac{\pi}{4} + 2k\pi, k$ is integer.

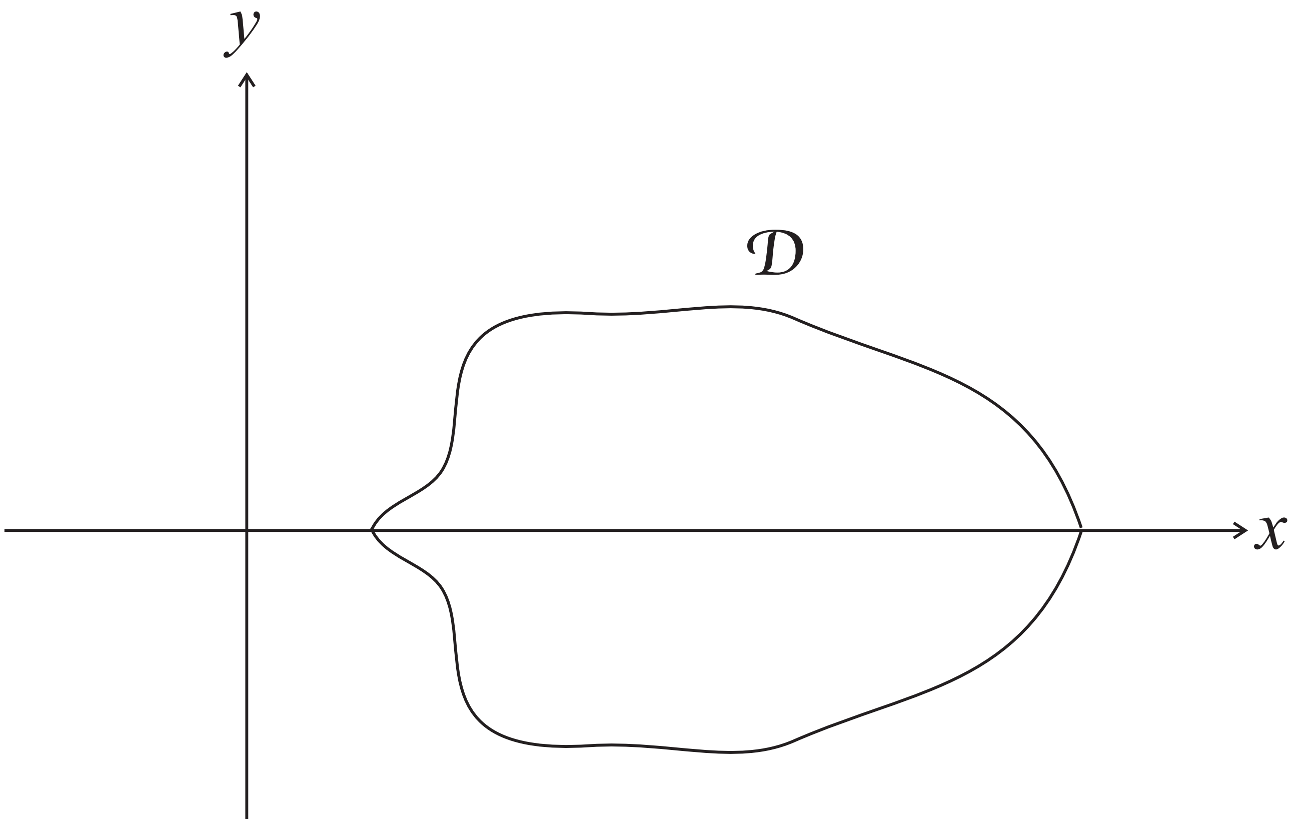
$|e^z| = e^x = e^\beta$ so that $|e^z|$ is bounded when Re $z = \beta$.

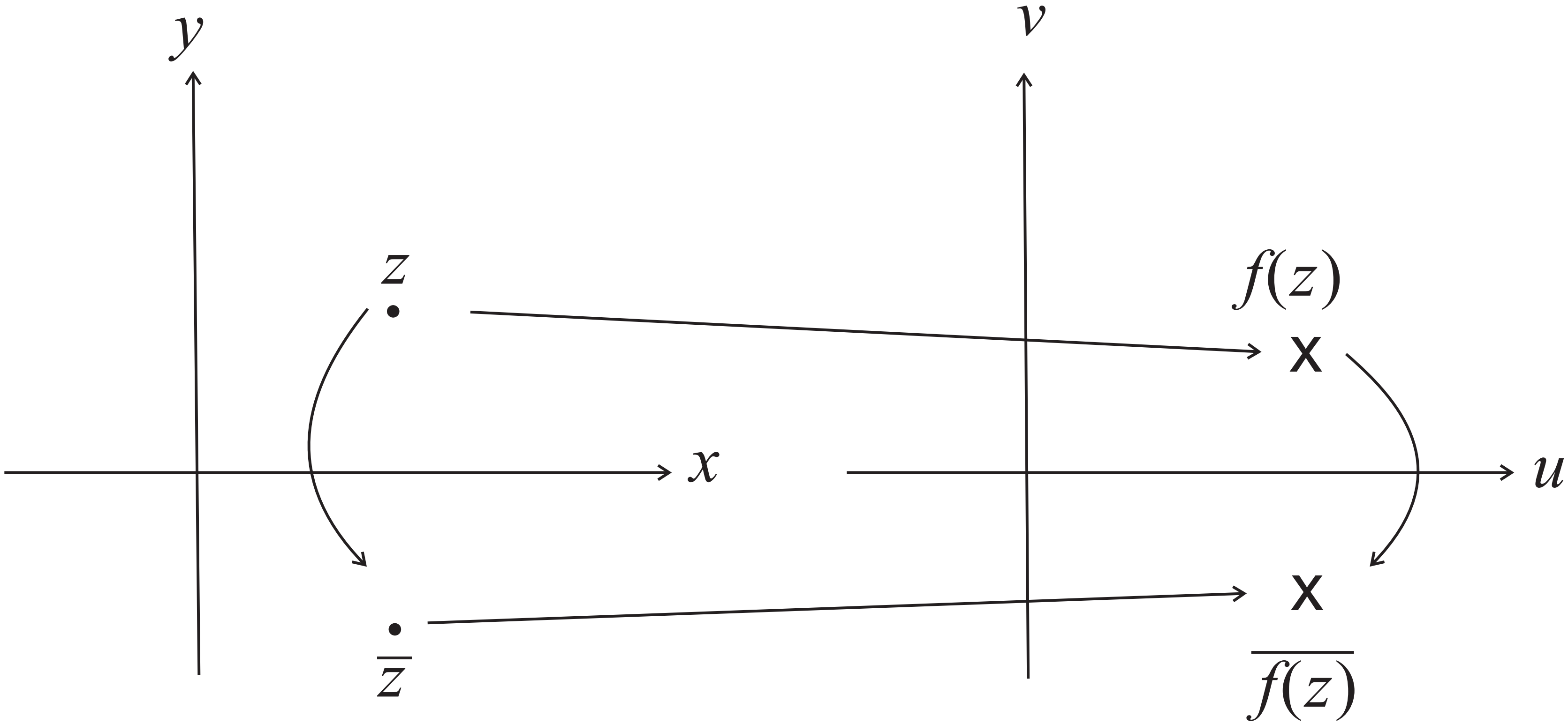
(b) $|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = \sin^2 x + \sinh^2 y$. When Im $z = \alpha, |\sin(x + i\alpha)|^2 = \sin^2 x + \sinh^2 \alpha \leq 1 + \sinh^2 \alpha$. Hence, $|\sin z| \leq \sqrt{1 + \sinh^2 \alpha}$ when Im $z = \alpha$.

Reflection principle

Suppose a function f is analytic in a domain $\mathcal{D}$ which includes part of the real axis and $\mathcal{D}$ is symmetric about the real axis. The reflection principle states that

$f(\overline{z}) = \overline{f(z)}$ if and only if $f(z)$ is real when z is real.




y
v
z
$f(z)$
x
u
$\bar{z}$
$\overline{f(z)}$

An example is the cosine function: $f(z) = \cos z$.

The domain of analyticity is the whole complex plane and it reduces to the real cosine function when z is real.

As a verification, consider

$$\overline{e^{iz}} = \overline{e^{i(x+iy)}} = \overline{e^{-y}e^{ix}} = e^{-y}e^{-ix} = e^{-i(x-iy)} = e^{-i\overline{z}}.$$

Also $\overline{e^{-iz}} = e^{i\overline{z}}$. Now

$$\overline{\cos z} = \overline{\frac{1}{2}(e^{iz} + e^{-iz})} = \frac{1}{2}(e^{-i\overline{z}} + e^{i\overline{z}}) = \cos \overline{z}.$$

Proof of the reflection principle

"$\Longrightarrow$ part"

Write $f(z) = u(x, y) + iv(x, y), z = x + iy$, then

$$\overline{f(\overline{z})} = u(x, -y) - iv(x, -y)$$

which is well defined $\forall z \in \mathcal{D}$ since $\mathcal{D}$ is symmetric with respect to the real axis.

$\overline{f(\overline{z})} = f(z) \Longrightarrow u(x, -y) - iv(x, -y) = u(x, y) + iv(x, y)$

Hence, u is even in y and v is odd in y. Obviously $v(x, 0) = 0$ so that $f(z)$ is real when z is real.

“$\Longleftarrow$ part”

Given that $f(z)$ assumes real value on a line segment of the real axis within $\mathcal{D}, \overline{f(\overline{z})}$ and $f(z)$ have the same value along the segment of real axis.

To prove $\overline{f(\overline{z})} = f(z)$, it suffices to show that $\overline{f(\overline{z})}$ is analytic in $\mathcal{D}$.

[See Theorem 5.5.1 on p. 210. Given that $f_1(z)$ and $f_2(z)$ are analytic in $\mathcal{D}$, if $f_1(z) = f_2(z)$ on an arc inside $\mathcal{D}$, then $f_1(z) = f_2(z)$ throughout $\mathcal{D}$.]

Note that u and v have continuous first order partials in $\mathcal{D}$ and

$$\frac{\partial u}{\partial x}(x,y)=\frac{\partial v}{\partial y}(x,y) \quad \text{and} \quad \frac{\partial u}{\partial y}(x,y)=-\frac{\partial v}{\partial x}(x,y).$$

Let $\widetilde{y}=-y$, we deduce that

$$\frac{\partial u}{\partial x}(x,\widetilde{y})=\frac{\partial(-v)}{\partial \widetilde{y}}(x,\widetilde{y}) \quad \text{and} \quad \frac{\partial u}{\partial \widetilde{y}}(x,\widetilde{y})=-\frac{\partial(-v)}{\partial x}(x,\widetilde{y}),$$

which are the Cauchy-Riemann relations for $\overline{f(\overline{z})}$. Hence, $\overline{f(\overline{z})}$ is analytic in $\mathcal{D}$.

Logarithmic functions

The complex logarithmic function is defined as the inverse of the complex exponential function. For

$$w = \log z,$$

this would imply

$$z = e^{w}.$$

Let $u(x, y)$ and $v(x, y)$ denote the real and imaginary parts of $\log z$, then

$$z = x + iy = e^{u+iv} = e^{u} \cos v + ie^{u} \sin v.$$

Equating the real and imaginary parts on both sides gives

$$x = e^{u} \cos v \quad \text{and} \quad y = e^{u} \sin v.$$

It then follows that

$$e^{2u} = x^2 + y^2 = |z|^2 = r^2 \quad \text{and} \quad v = \tan^{-1}\frac{y}{x}.$$

Using the polar form $z = re^{i\theta}$, we deduce that

$$u = \ln r = \ln|z|, r \neq 0 \quad \text{and} \quad v = \theta = \arg z.$$

Putting the results together, we have

$$w = \log z = \ln|z| + i \arg z, \qquad z \neq 0, \infty.$$

Remark ‘ln’ refers to the real logarithm while ‘log’ refers to the complex logarithm.

Recall that $\arg z$ is multi-valued, so does $\log z$. For a fixed z, there are infinitely many possible values of $\log z$, each differing by a multiple of $2\pi i$.

The principal branch of the complex logarithmic function is denoted by Log z, that is,

$$\text{Log } z = \ln|z| + i\text{Arg } z, \quad -\pi < \text{Arg } z \le \pi.$$

One may write

$$\log z = \text{Log } z + 2k\pi i, \quad k \text{ is any integer.}$$

For example, Log $i = \pi i/2$ and $\log i = \pi i/2 + 2k\pi i, k$ is any integer.

Though $z = e^{\log z}$, it would be incorrect to write $z = \log e^z$ since the logarithmic function is multi-valued.

Given two non-zero complex numbers z_1 and z_2, we have

$$\begin{aligned} \ln|z_1 z_2| &= \ln|z_1| + \ln|z_2|, \\ \arg(z_1 z_2) &= \arg z_1 + \arg z_2, \end{aligned}$$

so

$$\log(z_1 z_2) = \log z_1 + \log z_2.$$

The equality sign actually means that any value of $\log(z_1 z_2)$ equals some value of $\log z_1$ plus some value of $\log z_2$.

Example

Show from the first principles that

$$\frac{d}{dz}\log z = \frac{1}{z}, \quad z \neq 0,\ \infty.$$

Solution

Suppose we perform differentiation along the x-axis, the derivative of $\log z$ becomes

$$\frac{d}{dz}\log z = \frac{\partial}{\partial x}\ln r + i\frac{\partial}{\partial x}(\theta + 2k\pi), \qquad z \neq 0, \infty,$$

where $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\frac{y}{x}$ and k is any integer.

$$\begin{aligned}\frac{d}{dz}\log z &= \frac{1}{r}\frac{\partial r}{\partial x} + i\frac{\partial \theta}{\partial x} \\ &= \frac{1}{\sqrt{x^2+y^2}}\frac{x}{\sqrt{x^2+y^2}} + i\frac{-y}{x^2+y^2} \\ &= \frac{x-iy}{x^2+y^2} = \frac{\overline{z}}{z\overline{z}} = \frac{1}{z}.\end{aligned}$$

Example

Find the solutions of $z^{1+i} = 4$.

Solution

We write this equation as

$$e^{(1+i)\log z} = 4$$

so that $(1+i)\log z = \ln 4 + 2n\pi i, n = 0, \pm 1, \cdots$. Hence,

$$\begin{aligned} \log z &= (1-i)(\ln 2 + n\pi i) \\ &= (\ln 2 + n\pi) + i(n\pi - \ln 2). \end{aligned}$$

Thus,

$$\begin{aligned} z &= 2e^{n\pi}[\cos(n\pi - \ln 2) + i\sin(n\pi - \ln 2)] \\ &= 2e^{n\pi}[(-1)^n \cos(\ln 2) + i(-1)^{n+1}\sin(\ln 2)] \\ &= (-1)^n 2e^{n\pi}[\cos(\ln 2) - i\sin(\ln 2)], \quad n = 0, \pm 1, \cdots . \end{aligned}$$

Inverse trigonometric and hyperbolic functions

We consider the inverse sine function

$$w = \sin^{-1} z,$$

or equivalently,

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}.$$

Considered as a quadratic equation in e^{iw}

$$e^{2iw} - 2ize^{iw} - 1 = 0.$$

Solving the quadratic equation gives

$$e^{iw} = iz + (1 - z^2)^{\frac{1}{2}}.$$

Taking the logarithm on both sides of the above equation

$$\sin^{-1} z = \frac{1}{i} \log\left(iz + (1 - z^2)^{\frac{1}{2}}\right).$$

When $z \neq \pm 1$, the quantity $(1-z^2)^{\frac{1}{2}}$ has two possible values. For each value, the logarithm generates infinitely many values. Therefore, $\sin^{-1} z$ has two sets of infinite values. For example, consider

$$\begin{aligned}
\sin^{-1}\frac{1}{2} &= \frac{1}{i}\ \log\left(\frac{i}{2} \pm \frac{\sqrt{3}}{2}\right) \\
&= \frac{1}{i}\ \left[\ln 1 + i\left(\frac{\pi}{6} + 2k\pi\right)\right] \\
&\qquad \text{or } \frac{1}{i}\ \left[\ln 1 + i\left(\frac{5\pi}{6} + 2k\pi\right)\right] \\
&= \frac{\pi}{6} + 2k\pi \quad \text{or} \quad \frac{5\pi}{6} + 2k\pi,\ \ k \text{ is any integer.}
\end{aligned}$$

Example

If θ is real and non-zero and $\sin\theta\sin\phi = 1$, then

$$\phi = \left(n + \frac{1}{2}\right)\pi \pm i\ln\left|\tan\frac{\theta}{2}\right|,$$

where n is an integer, even or odd, according to $\sin\theta > 0$ or $\sin\theta < 0$.

How to solve $\sin\phi = 2$ using the above result? Let $\sin\theta = \frac{1}{2}$ so that $\theta = \frac{\pi}{6}$, then using the results

$$\phi = \left(n + \frac{1}{2}\right)\pi \pm i\ln\left|\tan\frac{\pi}{12}\right|,$$

where n is even since $\sin\theta > 0$.

Instead, if we solve for $\sin\phi = -2$, then we have the same representation of the solution, except that n is taken to be odd.

In a similar manner, we can derive the following formulas for the other inverse trigonometric and hyperbolic functions:

$$\cos^{-1} z = \frac{1}{i}\ \log(z + (z^2 - 1)^{\frac{1}{2}}),$$

$$\tan^{-1} z = \frac{1}{2i}\ \log\frac{1+iz}{1-iz}, \quad \cot^{-1} z = \frac{1}{2i}\ \log\frac{z+i}{z-i},$$

$$\sinh^{-1} z = \log(z + (1+z^2)^{\frac{1}{2}}), \quad \cosh^{-1} z = \log(z + (z^2-1)^{\frac{1}{2}}),$$

$$\tanh^{-1} z = \frac{1}{2}\ \log\frac{1+z}{1-z}, \quad \coth^{-1} z = \frac{1}{2}\log\frac{z+1}{z-1}, \text{ etc.}$$

The derivative formulas for the inverse trigonometric functions are

$$\frac{d}{dz}\sin^{-1} z = \frac{1}{(1-z^2)^{\frac{1}{2}}}, \quad \frac{d}{dz}\cos^{-1} z = -\frac{1}{(1-z^2)^{\frac{1}{2}}},$$

$$\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}, \quad \text{and so forth.}$$

Example

Consider the inverse tangent function, which is multi-valued:

$$w = \tan^{-1} z \quad \Rightarrow \quad z = \frac{e^{2iw} - 1}{i(e^{2iw} + 1)} \quad \Rightarrow \quad e^{2iw} = \frac{1 + iz}{1 - iz}$$

$$\Rightarrow \quad w = \frac{1}{2i} \log \left(\frac{1 + iz}{1 - iz} \right)$$

Consider the *generalized power function*

$$f(z) = z^a,$$

where a is complex in general, and $z = x+iy = re^{i\theta} = |z|e^{i(\text{Arg } z + 2k\pi)}$ is a complex variable. Consider the following cases:

(i) When $a = n$, n is an integer,

$$z^n = |z|^n \, e^{in\text{Arg } z}.$$

(ii) When a is rational, $a = \frac{m}{n}$ where m, n are irreducible integers, we have

$$\begin{aligned} z^{\frac{m}{n}} &= e^{\frac{m}{n}\log z} \\ &= e^{\frac{m}{n}\ln|z|} \, e^{i\frac{m}{n}\text{ Arg } z} \, e^{2k\frac{m}{n}\pi i}, \qquad k = 0, \pm 1, \pm 2, \ldots. \end{aligned}$$

The factor $e^{2k\frac{m}{n}\pi i}$ takes on n different values for $k = 0, 1, 2, \ldots, n-1$, but repeats itself with period n if k continues to increase through the integer. The power function has n different branches, corresponding to different values of k.

(iii) When $a = \alpha + i\beta$, then

$$\begin{aligned} z^a &= e^{(\alpha\ +\ i\beta)\ [\ln|z|\ +\ i(\text{Arg }z\ +\ 2k\pi)]} \\ &= e^{\alpha \ln|z| - \beta(\text{Arg }z\ +\ 2k\pi)}\ e^{i[\beta \ln|z|\ +\ \alpha(\text{Arg }z\ +\ 2k\pi)]} \\ &= |z|^\alpha e^{-\beta(\text{Arg }z\ +\ 2k\pi)}[\cos(\beta \ln|z| + \alpha(\text{Arg }z + 2k\pi)) \\ &\quad + i \sin(\beta \ln|z| + \alpha(\text{Arg }z + 2k\pi))], \\ &\qquad\qquad\qquad k = 0, \pm 1, \pm 2, \ldots . \end{aligned}$$

In this case, z^a has infinitely many branches.

Example

Find the principal value of each of the following complex quantities:

(a) $(1-i)^{1+i}$; (b) 3^{3-i}; (c) 2^{2i}.

Solution

(a) $$\text{Principal value of } (1-i)^{1+i} = e^{(1+i)\text{Log}(1-i)} = e^{(1+i)(\ln\sqrt{2}-\frac{\pi}{4}i)}$$
$$= e^{\left(\ln\sqrt{2}+\frac{\pi}{4}\right)+i\left(\ln\sqrt{2}-\frac{\pi}{4}\right)}$$
$$=\sqrt{2}e^{\frac{\pi}{4}}\left[\cos\left(\ln\sqrt{2}-\frac{\pi}{4}\right)+i\sin\left(\ln\sqrt{2}-\frac{\pi}{4}\right)\right].$$

(b) $$\text{Principal value of } 3^{3-i} = e^{(3-i)\text{Log }3} = e^{3\ln 3-i\ln 3}$$
$$= 27\,[\cos(\ln 3)-i\sin(\ln 3)].$$

(c) Principal value of $2^{2i} = e^{2i\ln 2} = \cos(\ln 4)+i\sin(\ln 4)$.

Branch points, branch cuts and Riemann surfaces

Why do we need to construct the Riemann surface consisting of overlapping sheets to characterize a multi-valued function?

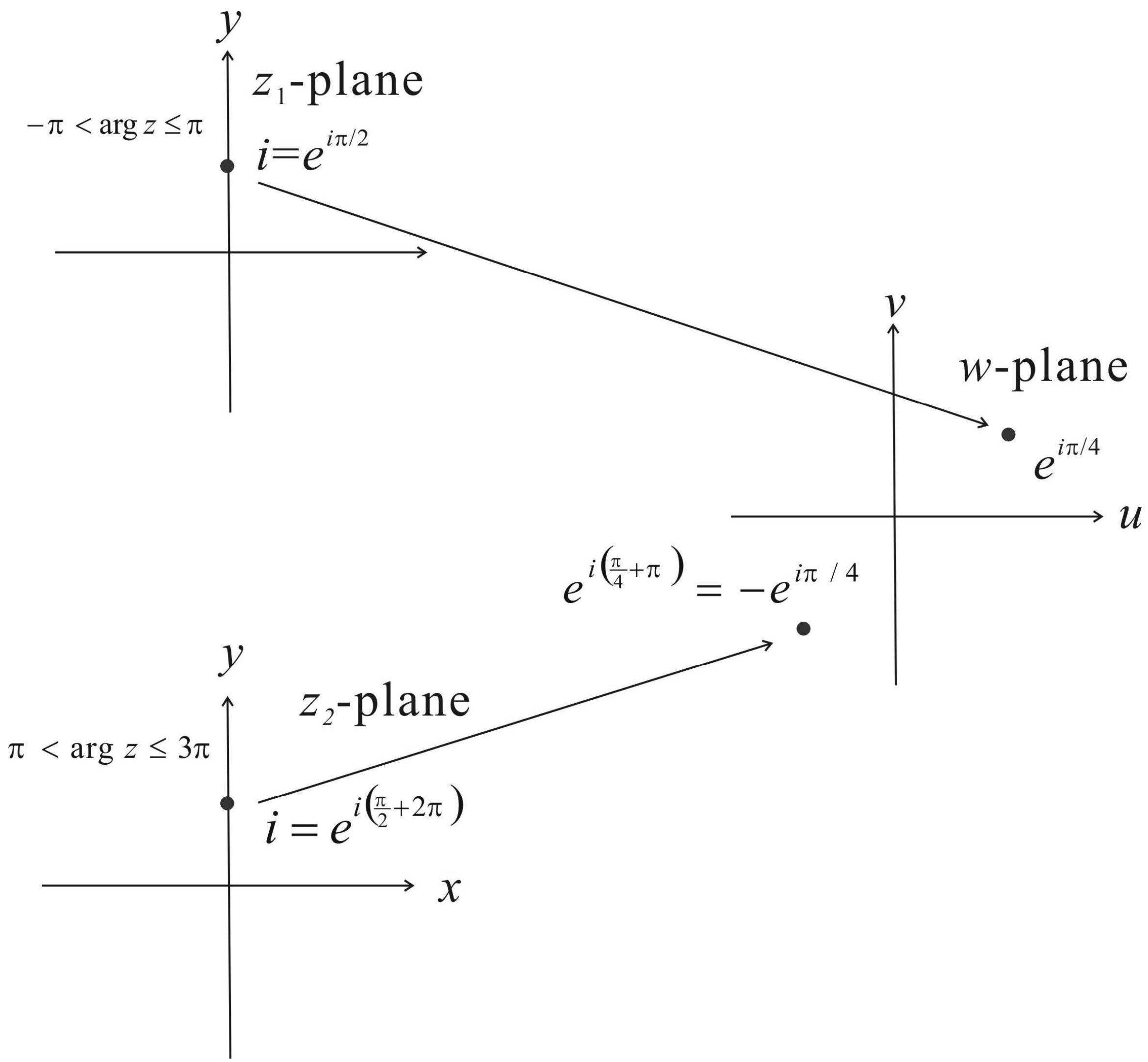
A multi-valued function may be regarded as *single-valued* if we suitably generalize its domain of definition.

Consider the double-valued function

$$w = f(z) = z^{1/2}.$$

There are two possible roots, each root corresponds to a specific branch of the double-valued function. Multi-valued nature stems from the different possible values that can be assumed by arg z.

z_1-plane (z_2-plane) is the domain of definition of the first branch (second branch) of the double-valued function $w = z^{1/2}$.



In general, suppose $z = re^{i \arg z}$, then $z^{1/2} = \sqrt{r}e^{i \arg z/2}$. The two roots correspond to the choice of $\arg z$ which lies in the z_1-plane $(-\pi < \arg z \leq \pi)$ or z_2-plane $(\pi < \arg z \leq 3\pi)$. The two image points lie in different branches of the double-valued function $w = z^{1/2}$.

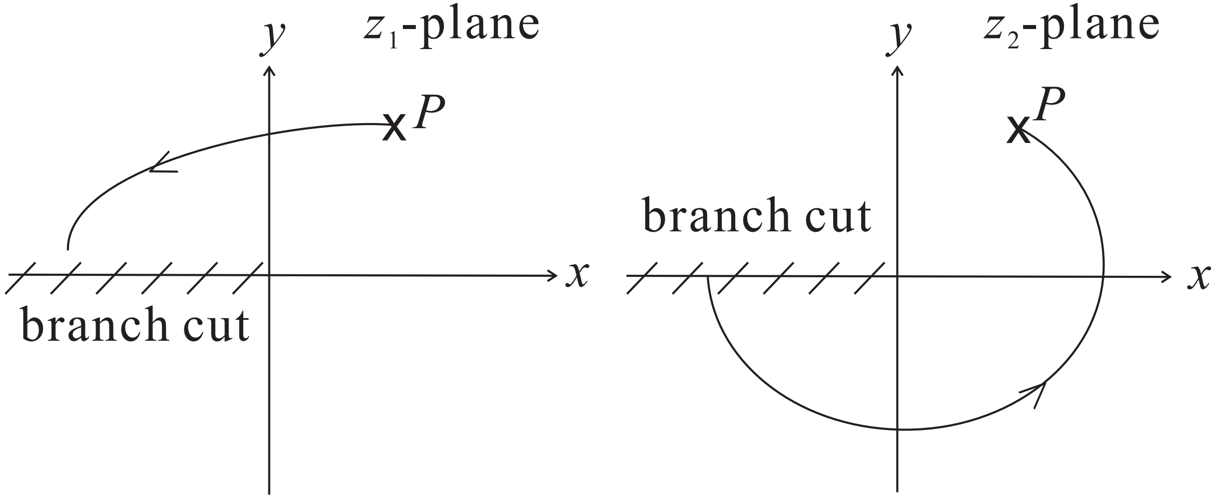
For $w = z^{1/2}$, we take the two copies of the z-plane superimposed upon each other.

Each square root corresponds to a specific branch of the multi-valued function, $w = f(z) = z^{1/2}$. For each branch of the multi-valued function, this becomes a one-to-one function.

For $w = f(z) = z^{1/2}$

$$f(i) = \begin{cases} e^{i(\pi/2)/2} = e^{i\pi/4} & \text{in the first branch} \\ e^{i(5\pi/2)/2} = e^{i5\pi/4} = -e^{i\pi/4} & \text{in the second branch} \end{cases}.$$

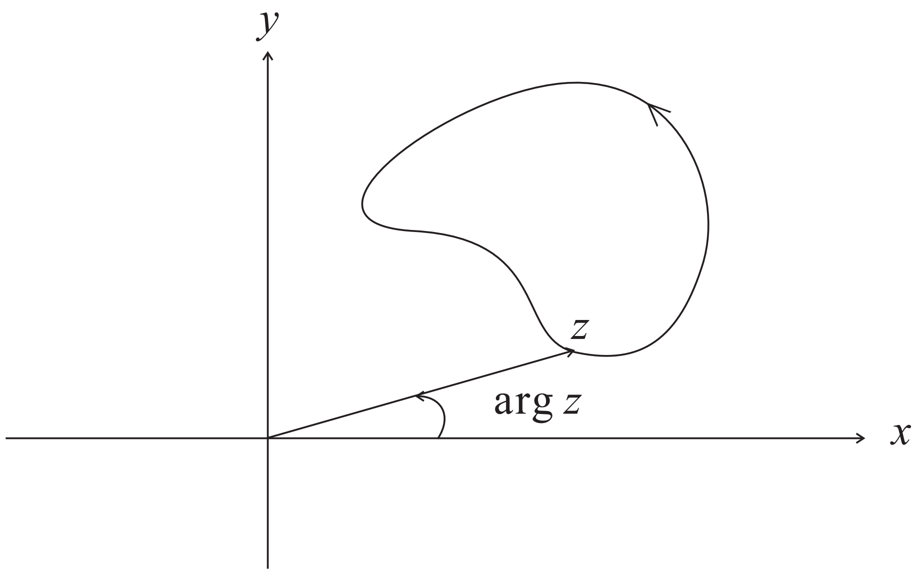
The z_1-plane and z_2-plane are joined together along the negative real axis from $z = 0$ to $z = \infty$. The two ends $z = 0$ and $z = \infty$ are called the *branch points* and the common negative real axis of z_1- and z_2-planes are called the *branch cut*.



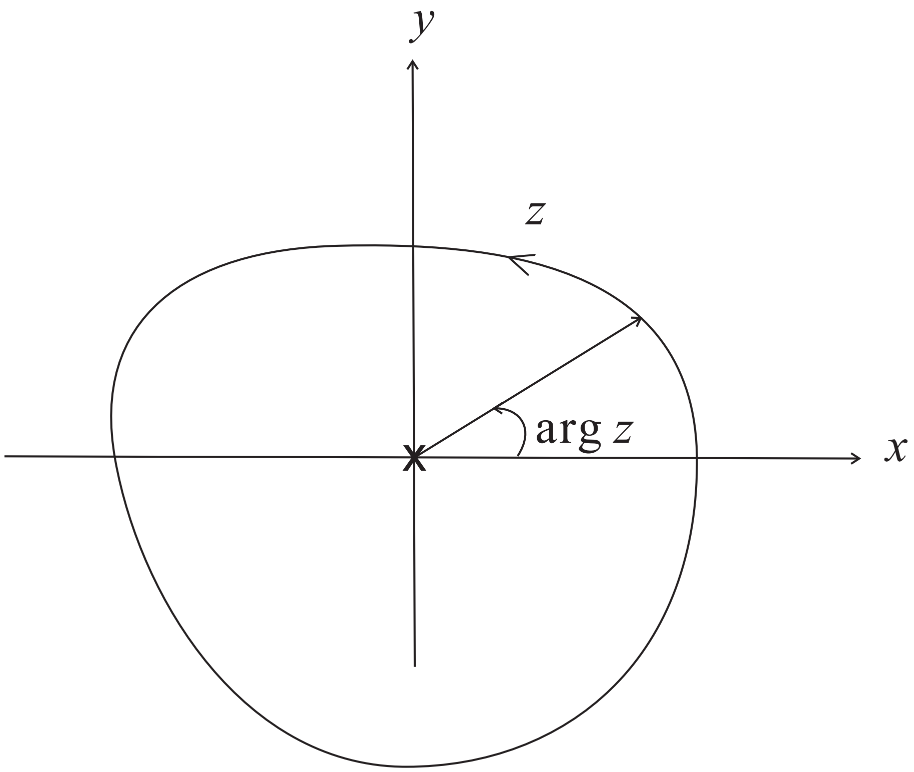
When we transverse a complete loop around the origin (branch point), the loop moves from the z_1-plane to z_2-plane by crossing the *branch cut*. After transversing a closed loop, $\arg z$ increases by 2π.

- A Riemann surface consists of overlapping sheets and these sheets are connected by the branch cuts. Each sheet corresponds to the domain of definition of an individual branch of the multi-valued function. The end points of a branch cut are called the branch points.

Consider a closed loop in the z-plane which does not encircle the origin, arg z increases then decreases, and remains the same value when it moves back to the same point. The closed loop stays on the same branch.



However, if the closed loop encircles the origin, then $\arg z$ increases by an amount 2π when we move in one complete loop and go back to the same point.



- A complete loop around a branch point carries a branch of a given multi-valued function into another branch.

Example

For $w = g(z) = z^{1/4}$, there are 4 branches corresponding to the 4 roots of z. Suppose we take the branch where

$$g(i) = e^{i(\pi/2+4\pi)/4} = -e^{i\pi/8},$$

find $g(1+i)$.

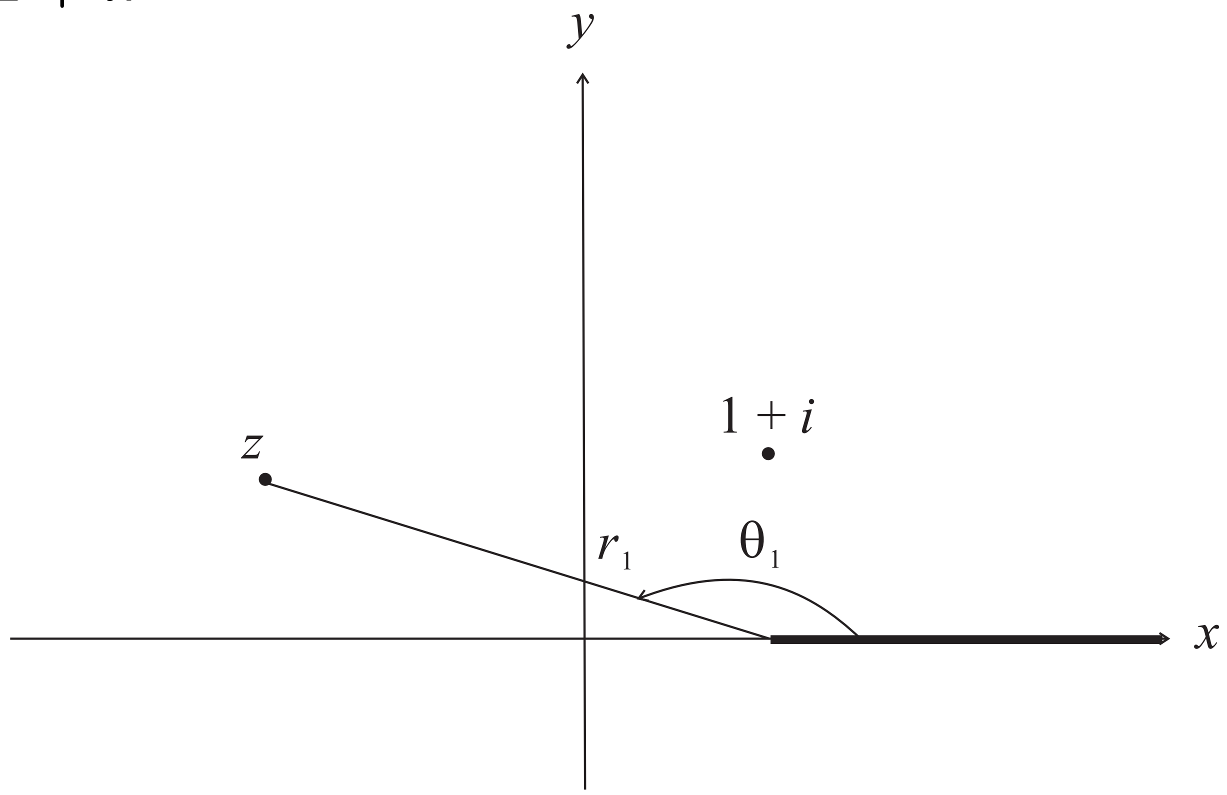
In this branch, we take $3\pi < \arg z \leq 5\pi$. Hence, $1+i = \sqrt{2}e^{i(\pi/4+4\pi)}$ so that

$$g(1+i) = \sqrt[8]{2}e^{i\left(\frac{\pi}{4}+4\pi\right)/4} = -\sqrt[8]{2}e^{i\frac{\pi}{16}}.$$

What are the other 3 values of $g(1+i)$?

Example

For $f(z) = (z-1)^{1/3}$, let a branch cut be constructed along the line $y = 0, x \geq 1$. If we select a branch whose value is a negative real number when $y = 0, x < 1$, what value does this branch assume when $z = 1 + i$.



Solution

Let $r_1 = |z-1|$ and $\theta_1 = \arg(z-1)$, where $0 \le \theta_1 < 2\pi$, then

$$(z-1)^{1/3} = \sqrt[3]{r_1} e^{i(\theta_1/3 + 2k\pi/3)}, \quad k = 0, 1, 2.$$

Taking $\theta_1 = \pi$ on the line $y = 0, x < 1$, we have

$$(z-1)^{1/3} = \sqrt[3]{r_1} e^{i(\pi/3 + 2k\pi/3)},$$

which is a negative real number if we select $k = 1$. On this branch, when $z = 1 + i$, we have $\theta_1 = \pi/2$ and $r_1 = 1$. Hence,

$$f(1+i) = \sqrt[3]{1} e^{i5\pi/6} = -\frac{\sqrt{3}}{2} + \frac{i}{2}.$$

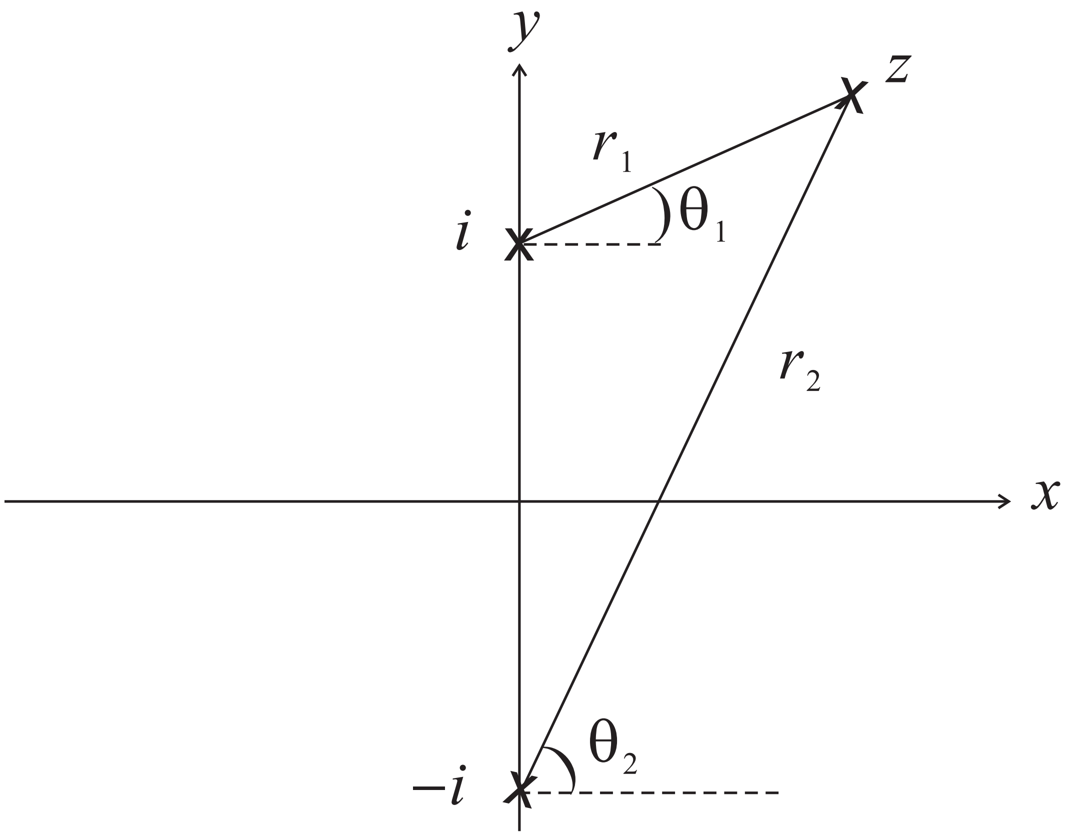
Example

We examine the branch points and branch cut of

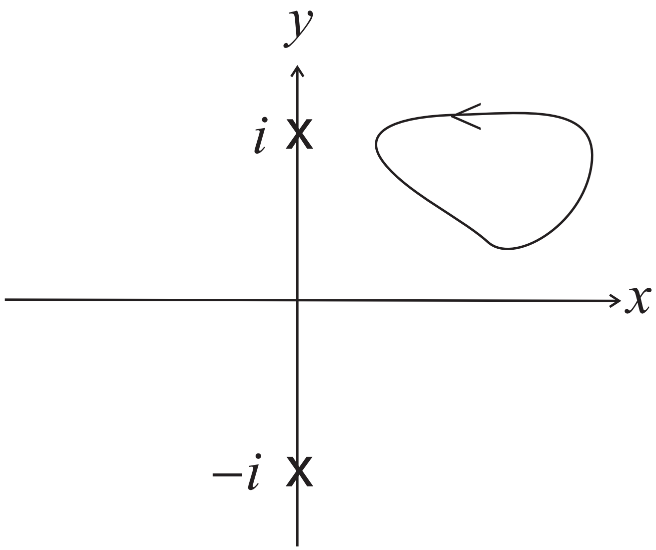
$$w = (z^2 + 1)^{1/2} = [(z+i)(z-i)]^{1/2}.$$

We write $z - i = r_1 e^{i\theta_1}, z + i = r_2 e^{i\theta_2}$ so that

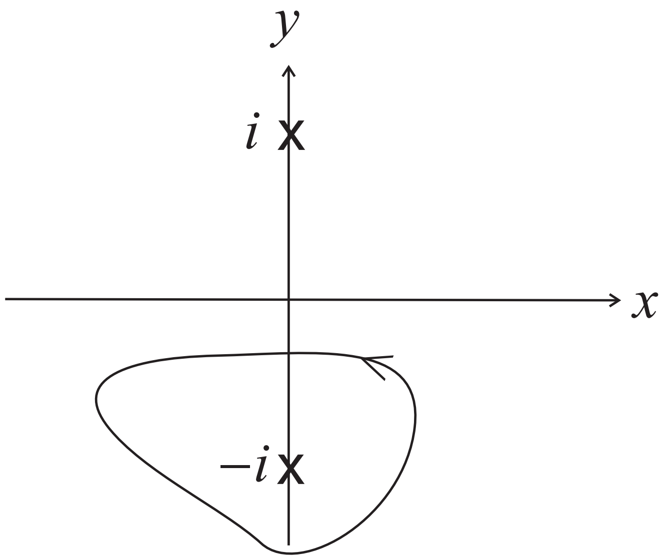
$$w = \sqrt{r_1 r_2} e^{i\theta_1/2} e^{i\theta_2/2}.$$



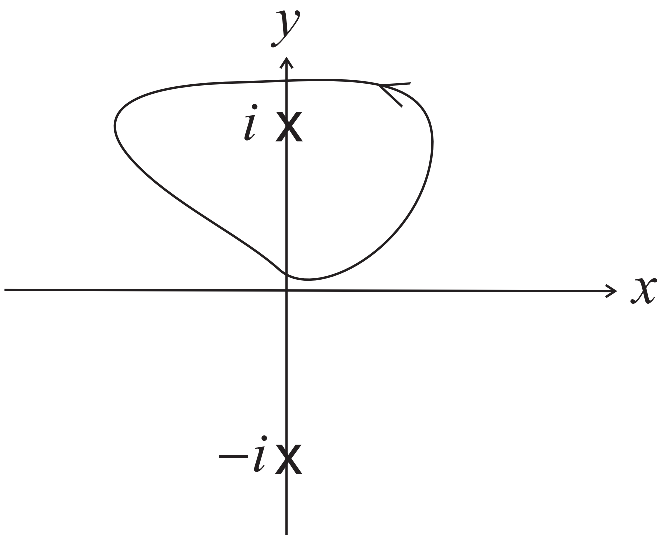
Now, $\theta_1 = \arg(z - i)$ and $\theta_2 = \arg(z + i)$. Consider the following 4 closed loops:



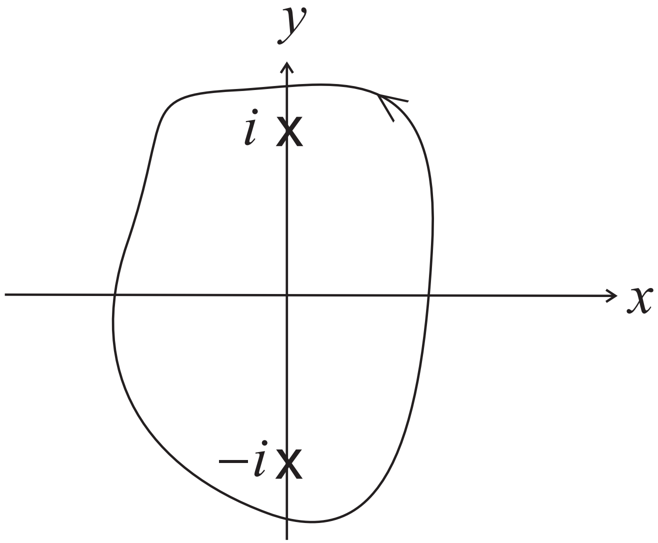
both θ_1 and θ_2 remain the same value



θ_1 remains the same value but θ_2 increases 2π



θ_1 increases by 2π,
θ_2 remains the same value



both θ_1 and θ_2 increase by 2π

case 1: $w = \sqrt{r_1 r_2} e^{i\theta_1/2} e^{i\theta_2/2}$

case 2: $w = \sqrt{r_1 r_2} e^{i\theta_1/2} e^{i(\theta_2+2\pi)/2} = -\sqrt{r_1 r_2} e^{i\theta_1/2} e^{i\theta_2/2}$

case 3: $w = \sqrt{r_1 r_2} e^{i(\theta_1+2\pi)/2} e^{i\theta_2/2} = -\sqrt{r_1 r_2} e^{i\theta_1/2} e^{i\theta_2/2}$

case 4: $w = \sqrt{r_1 r_2} e^{i(\theta_1+2\pi)/2} e^{i(\theta_2+2\pi)/2} = \sqrt{r_1 r_2} e^{i\theta_1/2} e^{i\theta_2/2}$.

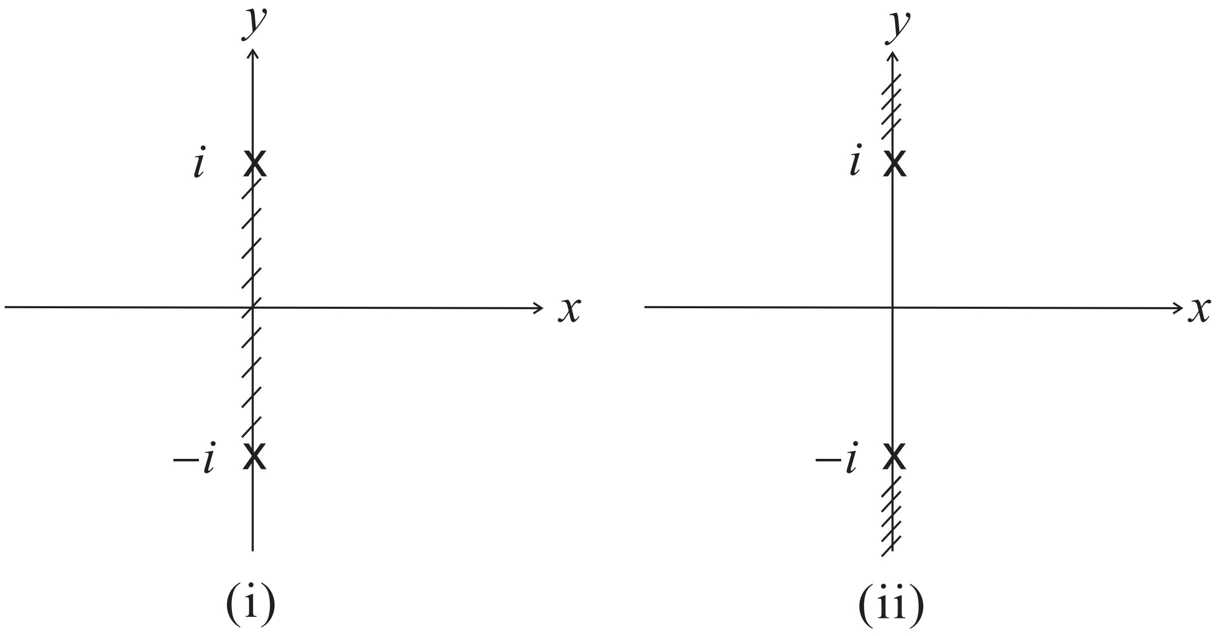
In both cases 1 and 4, the value remains the same and there is no change in branch. For case 4, we expect the closed loop crosses the branch cut twice or does not cross any branch cut at all.

Both cases 2 and 3 signify a change in branch. We expect that the closed loop in case 2 or case 3 crosses a branch cut.

Branch points: $z = \pm i$. The branch cut can be chosen to be either

(i) a cut along the imaginary axis between $z = \pm i$

(ii) two cuts along the imaginary axis, one from i to ∞, the other from $-i$ to ∞.



If the closed loop encircles none or both the branch points, it goes back to the same point on the same plane.

Example

The power function

$$w = f(z) = [z(z-1)(z-2)]^{1/2}$$

has two branches. Show that $f(-1)$ can be either $-\sqrt{6}i$ or $\sqrt{6}i$. Suppose the branch that corresponds to $f(-1) = -\sqrt{6}i$ is chosen, find the value of the function at $z = i$.

Solution

The given power function can be expressed as

$$w = f(z) = \sqrt{|z(z-1)(z-2)|}\, e^{i[\text{Arg } z \; + \; \text{Arg}(z-1) \; + \; \text{Arg}(z-2)]/2}\, e^{ik\pi}, \quad k = 0, 1,$$

where the two possible values of k correspond to the two branches of the double-valued power function.

Note that at $z = -1$,

$\text{Arg}\, z = \text{Arg}(z-1) = \text{Arg}(z-2) = \pi$ and $\sqrt{|z(z-1)(z-2)|} = \sqrt{6}$,

so $f(-1)$ can be either $\sqrt{6}e^{i3\pi/2} = -\sqrt{6}i$ or $\sqrt{6}e^{i3\pi/2}e^{i\pi} = \sqrt{6}i$.

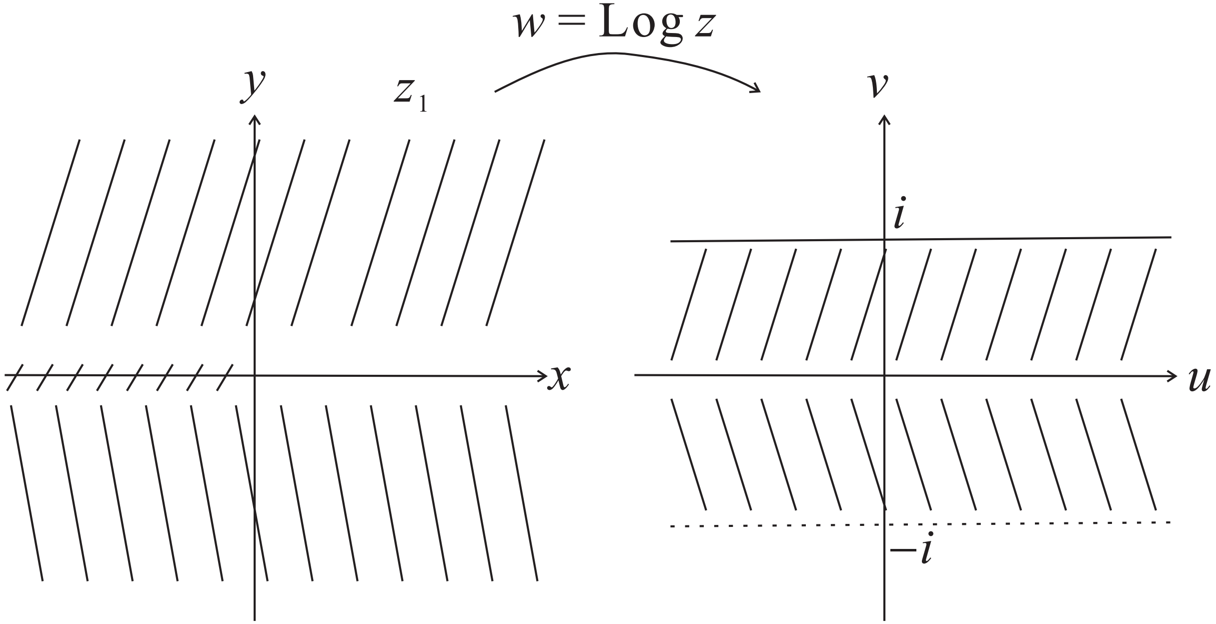
The branch that gives $f(-1) = -\sqrt{6}i$ corresponds to $k = 0$. With the choice of that branch, we have

$$\begin{aligned} f(i) &= \sqrt{|i(i-1)(i-2)|}\; e^{i[\text{Arg}\, i \,+\, \text{Arg}(i-1) \,+\, \text{Arg}(i-2)]/2} \\ &= \sqrt{\sqrt{2}\sqrt{5}}e^{i\left(\frac{\pi}{2} + \frac{3\pi}{4} + \pi - \tan^{-1}\frac{1}{2}\right)/2} \\ &= \sqrt[4]{10}e^{i\pi}e^{i\left(\frac{\pi}{4} - \tan^{-1}\frac{1}{2}\right)/2} \\ &= -\sqrt[4]{10}e^{i\left(\tan^{-1}1 - \tan^{-1}\frac{1}{2}\right)/2} \\ &= -\sqrt[4]{10}e^{i\left[\tan^{-1}\left(\frac{1-\frac{1}{2}}{1+\frac{1}{2}}\right)\right]/2} = -\sqrt[4]{10}e^{i\left(\tan^{-1}\frac{1}{3}\right)/2}. \end{aligned}$$

Note that we have used the idenity: $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\dfrac{x-y}{1+xy}$.

Branches of logarithm function

The logarithmic function has infinitely many branches. The branch points are $z = 0$ and $z = \infty$. The principal branch corresponds to $-\pi < \text{Arg } z \leq \pi$.



All other planes are joined to the adjacent branches along the branch cuts, which are along the negative real axis from $z = 0$ to $z = \infty$.

Example

Find the largest domain of analyticity of

$$f(z) = \text{Log}(z - (3 + 4i)).$$

Solution

The function Log w is analytic in the domain consisting of the entire w-plane minus the semi-infinite line: Im $w = 0$ and Re $w \leq 0$.

For $w = z-(3+4i)$, we ensure analyticity in the z-plane by removing points that satisfy $\text{Im}(z - (3 + 4i)) = 0$ and $\text{Re}(z - (3 + 4i)) \leq 0$, that is, $y = 4$ and $x \leq 3$.