MATH304, Spring 2007

Solution to Final Examination

1. (a) Note that $a_{2n-1} = 3, a_{2n} = 1, n = 1, 2, \cdots$. By the Root Test,

$$R = \frac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}} = 1.$$

(b) For $|z| \leq 1$, $\left|\dfrac{z^n}{n^2(n+1)}\right| \leq \dfrac{1}{n^2(n+1)}$. The infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$ is convergent. Hence, we have uniform convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2(n+1)}$ for $|z| \leq 1$.

(c) Let $S_n(z) = \displaystyle\sum_{k=1}^{n} \frac{z^4}{(1+z^4)^{k-1}}$, then

$$|S_n(z) - (1+z^4)| = \left|\frac{1}{(1+z^4)^{n-1}}\right|.$$

For $|1+z^4| > 1$, we take $\epsilon > 0$, then $\left|\dfrac{1}{(1+z^4)^{n-1}}\right| < \epsilon$ when

$$(n-1)\ln\frac{1}{|1+z^4|} < \ln\epsilon$$

or

$$n > \frac{\ln\epsilon}{\ln\frac{1}{|1+z^4|}} + 1.$$

Hence, $\displaystyle\sum_{n=1}^{\infty} \frac{z^4}{(1+z^4)^n} = 1 + z^4$ for $|1+z^4| > 1$.

To show the property of uniform convergence for $|1+z^4| \geq r, r > 1$, we take N to be $\dfrac{\ln\epsilon}{\ln\frac{1}{r}} + 1$, which is independent of z.

Whenever $n > \dfrac{\ln\epsilon}{\ln\frac{1}{r}} + 1$, we would have

$$|R_n(z)| < \epsilon, \quad \text{for } |1+z^4| > 1,$$

where $R_n(z) = 1 + z^4 - S_n(z)$.

2. (a) $f(z) = \dfrac{\alpha}{(z-\alpha)^2} + \dfrac{1}{z-\alpha}$, valid for $|z-\alpha| > 0$.

(b) (i) Inside the domain $|z| < |\alpha|$, the Laurent series reduces to the Taylor series as $f(z)$ is analytic in the region. Hence,

$$f(z) = \frac{z}{(z-\alpha)^2} = \frac{z}{\alpha^2}\left(1 - \frac{z}{\alpha}\right)^{-2} = \sum_{n=1}^{\infty} \frac{nz^n}{\alpha^{n+1}} = \frac{z}{\alpha^2} + \frac{2z^2}{\alpha^3} + \frac{3z^3}{\alpha^4} + \cdots.$$

(ii) For $|\alpha| < |z| < \infty$, we have

$$f(z) = \frac{1}{z\left(1 - \frac{\alpha}{z}\right)^2} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{(n+1)\alpha^n}{z^n} = \frac{1}{z} + \frac{2\alpha}{z^2} + \frac{3\alpha^2}{z^3} + \cdots .$$

(c) $\text{Res}(f, \alpha) = 1$; $\text{Res}(f, 0)$ is not defined since $z = 0$ is not an isolated singularity of f.

(d) $\displaystyle\oint_{|z|=1} \frac{z}{(z-\alpha)^2}\, dz.$

If $|\alpha| > 1$, then $f(z) = \dfrac{z}{(z-\alpha)^2}$ is analytic inside $|z| = 1$, thus

$$\oint_{|z|=1} \frac{z}{(z-\alpha)^2}\, dz = 0.$$

If $|\alpha| < 1$, then

$$\oint_{|z|=1} \frac{z}{(z-\alpha)^2}\, dz = 2\pi i \text{Res}(f, \alpha) = 2\pi i.$$

3. (a) $f(z) = \dfrac{p'(z_0)(z-z_0) + p''(z_0)(z-z_0)^2/2 + \cdots}{q'''(z_0)(z-z_0)^3/3! + q''''(z_0)(z-z_0)^4/4! + \cdots}.$

Consider $\displaystyle\lim_{z\to z_0} (z-z_0)^2 f(z) = \frac{6p'(z_0)}{q'''(z_0)}$, which is finite and non-zero.

Hence, $z = z_0$ is a double pole of $f(z)$.

(b) Consider

$$\begin{aligned}
\text{Res}\,(f, z_0) &= \lim_{z\to z_0} \frac{d}{dz}[(z-z_0)^2 f] \\
&= \lim_{z\to z_0} \frac{d}{dz}\left[\frac{p'(z_0) + p''(z_0)(z-z_0)/2! + \cdots}{q'''(z_0)/3! + q''''(z_0)(z-z_0)/4! + \cdots}\right] \\
&= \lim_{z\to z_0} \left\{\frac{[p''(z_0)/2! + \cdots][q'''(z_0)/3! + \cdots] - [q''''(z_0)/4! + \cdots][p'(z_0) + \cdots]}{[q'''(z_0)/3! + q''''(z_0)(z-z_0)/4!]^2}\right\} \\
&= \frac{3p''(z_0)}{q'''(z_0)} - \frac{3}{2}\frac{p'(z_0)q''''(z_0)}{q'''(z_0)^2}.
\end{aligned}$$

4. The singularities of $f(z)$ are at $z = 0$ and $\sin \pi/z = 0$, that is, $z = 0, \pm 1, \pm\dfrac{1}{2}, \cdots$.

For $z = \pm\dfrac{1}{n}$, n is any integer, they are all isolated singularities. Consider

$$\begin{aligned}
\lim_{z\to\frac{1}{n}} \left(z - \frac{1}{n}\right)\pi \cot\frac{\pi}{z} &= \lim_{z\to\frac{1}{n}} \frac{z - \frac{1}{n}}{\sin\frac{\pi}{z}} \lim_{z\to\frac{1}{n}} \pi\cos\frac{\pi}{z} \\
&= \lim_{z\to\frac{1}{n}} \frac{1}{-\frac{\pi}{z^2}\cos\frac{\pi}{z}} \lim_{z\to\frac{1}{n}} \pi\cos\frac{\pi}{z} = -\frac{1}{n^2}.
\end{aligned}$$

Hence, $z = \dfrac{1}{n}$, n is any integer, $\cdots$ represents a pole of order 1 of $f(z) = \pi\cot\frac{\pi}{z}$. Also,

$$\text{res}\left(f, \frac{1}{n}\right) = -\frac{1}{n^2}.$$

The singular point $z = 0$ is not an isolated singularity since any ϵ-neighborhood of $z = 0$ contains points of the form $z = \dfrac{1}{n}$, n is some integer, and these points are singularities of $f(z)$.

5. The principal value of the integral is given by

$$I = \lim_{\substack{\rho\to\infty \\ \varepsilon,\delta\to 0^+}} \left(\int_{\varepsilon}^{4-\delta} + \int_{4+\delta}^{\rho} \right) \frac{dx}{x^{\lambda}(x-4)}.$$

Choosing the branch

$$f(z) = \frac{1}{e^{\lambda(\operatorname{Log} r + i\theta)}(re^{i\theta}-4)}, \quad \text{for } z = re^{i\theta}, \quad 0 < \theta < 2\pi,$$

we form the contour as given in the question. Since $f(z)$ has no singularities “inside” the closed contour, the integral over the latter must be zero. Utilizing different definitions for f on the upper and lower sides of the branch cut, we can write this as

$$(1 - e^{-2\pi i\lambda}) \left(\int_{\varepsilon}^{4-\delta} + \int_{4+\delta}^{\rho} \right) \frac{dx}{x^{\lambda}(x-4)} \\ + \left(\int_{\Gamma_\varepsilon} + \int_{S_\delta^+} + \int_{S_\delta^-} + \int_{C_\rho} \right) f(z)\, dz = 0.$$

Now for $0 < \lambda < 1$, on the circle of radius ρ, we have

$$|f(z)| = \frac{1}{|\sqrt{z}|\,|z+4|} \le \frac{1}{\sqrt{\rho}(\rho-4)} \quad (\rho > 4),$$

which yields the estimate

$$\left| \int_{C_\rho} f(z)\, dz \right| \le \frac{2\pi\rho}{\sqrt{\rho}(\rho-4)}.$$

Consequently, the integral over C_ρ tends to zero as $\rho \to \infty$. Similarly, on the inner circle of radius ε we have

$$|f(z)| \le \frac{1}{\sqrt{\varepsilon}(4-\varepsilon)} \quad (\varepsilon < 4),$$

which implies that

$$\left| \int_{\Gamma_\varepsilon} f(z)\, dz \right| \le \frac{2\pi\varepsilon}{\sqrt{\varepsilon}(4-\varepsilon)} = \frac{2\pi\sqrt{\varepsilon}}{4-\varepsilon}.$$

As $\varepsilon \to 0^+$ this also goes to zero. Hence, we obtain

$$\lim_{\varepsilon\to 0^+} \int_{\Gamma_\varepsilon} f(z)\, dz = 0 \quad \text{and} \quad \lim_{\rho\to\infty} \int_{C_\rho} f(z)\, dz = 0.$$

To compute the limits as $\delta \to 0^+$ of the integrals over S_δ^+ and S_δ^-, we apply the results concerning the behavior of integrals near simple poles. On the upper half-circle around $z = 4$, the function f agrees with the principal branch

$$f_1(z) := \frac{1}{e^{\lambda \operatorname{Log} z}(z-4)},$$

which is *analytic* on the positive real axis except for its simple pole at $z = 4$. Hence

$$\lim_{\delta\to 0^+}\int_{S_\delta^+} f(z)\,dz = -i\pi \text{Res}\,(f_1, 4) = -i\pi \lim_{z\to 4} e^{-\lambda \text{Log}\, z} = -i\pi 4^{-\lambda}.$$

However, on the lower half-circle, $f(z)$ equals $e^{-2\pi i\lambda} \times f_1(z)$, and so

$$\lim_{\delta\to 0^+}\int_{S_\delta^-} f(z)\,dz = -i\pi 4^{-\lambda} e^{-2\pi i\lambda}.$$

Finally, on taking the limit as $\rho \to \infty, \varepsilon \to 0^+$ and $\delta \to 0^+$, we deduce that

$$(1 - e^{-2\pi i\lambda})I + 0 - i\pi 4^{-\lambda} - i\pi 4^{-\lambda} e^{-2\pi i\lambda} + 0 = 0,$$

or, equivalently,

$$I = i\pi 4^{-\lambda}\frac{1+e^{-2\pi i\lambda}}{1-e^{-2\pi i\lambda}} = i\pi 4^{-\lambda}\frac{e^{i\pi\lambda}+e^{-i\pi\lambda}}{e^{i\pi\lambda}-e^{-i\pi\lambda}} = \frac{\pi\cot\lambda\pi}{4^\lambda}.$$

6. (a) First, we find the pole of the integrand function. Consider

$$e^z + 1 = 0 \quad \text{so that} \quad z = \pi i.$$

By the Residue Theorem, we have $\displaystyle\oint_C \frac{e^{pz}}{1+e^z}\,dz = 2\pi i \text{Res}\left(\frac{e^{pz}}{1+e^z}, \pi i\right)$.
Take $\Gamma = \{z = x + 2\pi i, \quad x \in [A, -A]\}$, we have

$$\begin{aligned}
\oint_C \frac{e^{pz}}{1+e^z}\,dz &= \int_{-A}^{A}\frac{e^{px}}{1+e^x}\,dx + \int_{S_1} f(z)\,dz + \int_{S_2} f(z)\,dz + \int_\Gamma \frac{e^{pz}}{1+e^z}\,dz \\
&= \int_{-A}^{A}\frac{e^{px}}{1+e^x}\,dx + \int_{S_1} f(z)\,dz + \int_{S_2} f(z)\,dz + \int_A^{-A}\frac{e^{p(x+2\pi i)}}{1+e^{(x+2\pi i)}}\,dx \\
&= \int_{-A}^{A}\frac{e^{px}}{1+e^x}\,dx + \int_{S_1} f(z)\,dz + \int_{S_2} f(z)\,dz - e^{2p\pi i}\int_{-A}^{A}\frac{e^{px}}{1+e^x}\,dx.
\end{aligned}$$

(b) Consider

$$\int_{S_1} f(z)\,dz = i\int_0^{2\pi}\frac{e^{p(A+y_i)}}{1+e^{(A+yi)}}\,dy.$$

Since $0 < p < 1$, so for large value A, $\displaystyle\int_{S_1} f(z)\,dz$ tends to zero.

Similar argument can be applied to $\displaystyle\int_{S_2} f(z)\,dz$. Now,

$$\begin{aligned}
\text{Res}\left(\frac{e^{pz}}{1+e^z}, \pi i\right) &= \lim_{z\to\pi i}\frac{(z-\pi i)e^{pz}}{1+e^z} \\
&= \lim_{z-\pi i}\frac{e^{pz}+p(z-\pi i)e^{pz}}{e^z} = \frac{e^{p\pi i}}{e^{\pi i}} = -e^{p\pi i}.
\end{aligned}$$

We then have

$$\begin{aligned}
(1-e^{2p\pi i})\int_{-\infty}^{\infty}\frac{e^{px}}{1+e^x}\,dx &= 2\pi i(-e^{p\pi i}) \\
\int_{-\infty}^{\infty}\frac{e^{px}}{1+e^x}\,dx &= -\frac{2\pi i e^{p\pi i}}{1-e^{2p\pi i}} = \frac{2\pi i}{e^{p\pi i}-e^{-p\pi i}} = \frac{\pi}{\sin p\pi}.
\end{aligned}$$

(c) It holds if p is a complex number where $0 < \operatorname{Re} p < 1$ since

$$\int_0^{2\pi} \frac{e^{p(A+yi)}}{1+e^{(A+yi)}}\,dy \approx \int_0^{2\pi} \frac{e^{(\operatorname{Re} p)A}}{e^A}\,dy \to 0 \text{ as } A \to \infty.$$

Take $p = \dfrac{1}{2} + \dfrac{i}{2}$, we have

$$\int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x}\,dx \quad = \quad \int_{-\infty}^{\infty} \frac{\cos\frac{x}{2} + i\sin\frac{x}{2}}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}\,dx = \int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{\cosh x}\,dx.$$

By comparing the real parts, we have

$$\int_{-\infty}^{\infty} \frac{\cos x}{\cosh x}\,dx = \operatorname{Re}\frac{\pi}{\sin\left(\frac{1+i}{2}\right)\pi} = \frac{2\pi}{e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}}.$$