MATH304

Homework 2

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1. If $f(z) = 1/z = u + iv$, construct several members of the families: $u(x,y) = \alpha, v(x,y) = \beta$ where α and β are non-zero constants, showing that they are families of circle.

2. For each of the following functions, examine whether the function is continuous at $z = 0$:

$$\text{(a)} \quad f(z) = \begin{cases} 0 & z = 0 \\ \frac{\text{Re } z}{|z|} & z \neq 0 \end{cases}; \qquad \text{(b)} \quad f(z) = \begin{cases} 0 & z = 0 \\ \frac{(\text{Re } z)^2}{|z|} & z \neq 0 \end{cases}.$$

3. A particle moves along a curve $z = e^{-t}(2 \sin t + i \cos t)$.
   (a) Find a unit tangent vector to the curve at the point where $t = \pi/4$.
   (b) Determine the magnitudes of velocity and acceleration of the particle at $t = 0$ and $\pi/2$.

4. Consider the function $f(z) = xy^2 + ix^2y, z = x + iy$. Find the point set where
   (a) the Cauchy-Riemann relations are satisfied;
   (b) the function is differentiable;
   (c) the function is analytic.

5. Let $f(z)$ be analytic in a domain $\mathcal{D}$. Suppose $\text{Re}f(z) = [\text{Im}f(z)]^2$ inside $\mathcal{D}$, show that $f(z)$ is constant inside $\mathcal{D}$.

6. Find an analytic function $f(z)$ whose real part $u(x,y)$ is
   (a) $u(x,y) = y^3 - 3x^2y, \quad f(i) = 1 + i;$
   (b) $u(x,y) = \dfrac{y}{x^2 + y^2}, \quad f(1) = 0;$
   (c) $u(x,y) = (x - y)(x^2 + 4xy + y^2).$

7. Find the orthogonal trajectories of the following families of curves:
   (a) $x^3y - xy^3 = \alpha;$
   (b) $2e^{-x} \sin y + x^2 - y^2 = \alpha;$
   (c) $(r^2 + 1) \cos \theta = \alpha r.$

8. Let $\theta = \angle APB$, which is the angle included between the line segments PA and PA. Here, A and B are the fixed points $(-a, 0)$ and $(a, 0)$, respectively, and P is the variable point $z = x+iy$. Show that $\theta(x,y)$ is a harmonic function. Find the corresponding harmonic conjugate v such that $\theta + iv$ is an analytic function.

9. If u and v are harmonic in a region $\mathcal{R}$, prove that

$$\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

is analytic in $\mathcal{R}$.

10. Suppose the isothermal lines of a steady state temperature field are the family of curves

$$x^2 + y^2 = \alpha, \quad \alpha > 0.$$

Find the general solution of the temperature function, and the equation of the family of flux lines.