MATH304

Homework 5

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1. Intuitively,
$$\frac{n}{1+in} = \frac{1}{\frac{1}{n}+i} \longrightarrow -i \text{ as } n \to \infty.$$
Prove that
$$\lim_{n\to\infty} \frac{n}{1+in} = -i$$
using the definition of limit of a sequence.

2. Consider the following two complex series
$$\sum_{n=1}^{\infty} \alpha_n \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_n|.$$
Suppose $|\text{Arg } \alpha_n| \leq \frac{\pi}{2} - \delta, \delta > 0$, show that when $\sum_{n=1}^{\infty} |\alpha_n|$ diverges, $\sum_{n=1}^{\infty} \alpha_n$ also diverges.

3. Consider the following series of complex functions
$$\sum_{k=0}^{\infty} \frac{z^2}{(1+|z|^2)^k}.$$
Define the partial sum $S_n(z)$, where
$$S_n(z) = \sum_{k=0}^{n} \frac{z^2}{(1+|z|^2)^k}.$$
Compute $S(z)$ where
$$S(z) = \lim_{n\to\infty} S_n(z).$$
Show that the convergence of $S_n(z)$ to $S(z)$ inside $|z| < 1$ as $n \to \infty$ is *not* of uniform convergence.

4. Using the Weierstrass M-test, establish the uniform convergence of
$$\sum_{n=1}^{\infty} \frac{1}{(1-z)^n}, \quad 1.01 < |1-z|.$$

5. Find the region of convergence of each of the following Taylor series

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$,

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3 4^n}$,

(c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\sin(n+1)\frac{\pi}{4}}{2^{\frac{n+1}{2}}} (z-1)^n$.

Remark The last Taylor series converges to $\dfrac{1}{1+z^2}$ inside its region of convergence.

6. Show that

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = -\text{Log}\,(1-z) \quad \text{for } z \in \mathcal{D} = \{z : |z| < 1\}.$$

Explain why the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n}$, $\theta \neq 0$, is conditionally convergent. What happens when $\theta = 0$? Use the above series to show that

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\theta}{n} = -\ln\left(2 \sin \frac{\theta}{2}\right)$;

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = \frac{\pi - \theta}{2}, \quad 0 < \theta < 2\pi$.

7. Expand each of the following functions in a Taylor series about the indicated point and determine the region of convergence in each case.

   (a) $\cos z$; $z = \pi/2$,

   (b) $1/(1+z)$; $z = 1$.

8. (a) Expand $y = e^x \cos x$ in Taylor series at $x = 0$.

   (b) Find the Taylor expansion of $f(z) = \dfrac{1}{1+z^2}$ at $z = 1$.

9. If each of the following functions is expanded into a Taylor series about the indicated point, what would be the region of convergence? Do not perform the expansion.

   (a) $\dfrac{\sin z}{(z^2+4)}$; $z = 0$

   (b) $\dfrac{(z+3)}{(z-1)(z-4)}$; $z = 2$

   (c) $\dfrac{e^z}{z(z-1)}$; $z = 4i$.

10. Expand $f(z) = \dfrac{z}{(z-1)(2-z)}$ in a Laurent series valid for:

   (a) $|z| < 1$,

   (b) $1 < |z| < 2$,

   (c) $|z| > 2$,

(d) $|z-1|>1$.

11. Locate each of the isolated singularities of the given function and determine whether it is a removable singularity, a pole or an essential singularity. If the singularity is removable, give the value of the function at the point; if the singularity is a pole, give the order of the pole.

    (i) $\dfrac{e^z-1}{z}$,

    (ii) $\dfrac{e^z-1}{e^{2z}-1}$.