MATH 304 — Complex Analysis

Solution to Homework 6

1. Note that z_n is a double pole of $1/\cos^2 z$ for any integer n. Since $f(z)$ is analytic on the whole real axis, so $f(z) = \sum_{k=0}^{\infty} a_k(z - z_n)^k$, where $a_k = \dfrac{f^{(k)}(z_n)}{k!}$.

   Also, $\dfrac{1}{\cos^2 z} = \dfrac{b_2}{(z - z_n)^2} + \dfrac{b_1}{(z - z_n)} + \sum_{k=0}^{\infty} c_k(z - z_n)^k$ (since z_n is a double pole). As shown in later calculations, it is necessary to find the value of b_2. The trick is to observe that $\dfrac{z - z_n}{\cos^2 z}$ has a simple pole at z_n and $\text{Res}\left(\dfrac{z - z_n}{\cos^2 z}, z_n\right) = b_2$. Note that

$$b_2 = \lim_{z \to z_n} \frac{(z - z_n)^2}{\cos^2 z} = \lim_{z \to z_n} \frac{2(z - z_n)}{-2 \sin z \cos z} = \lim_{z \to z_n} \frac{2}{-2 \cos^2 z + 2 \sin^2 z} = 1.$$

   Furthermore, since the Taylor series of $\cos^2 z$ at $z = z_n$ has even power terms only (see below), the even power terms in the Laurent expansion of $\dfrac{1}{\cos^2 z}$ at $z = z_n$ are zero. In particular, we have $b_1 = 0$.

   Consider the following series expansion

$$\frac{f(z)}{\cos^2 z} = \left(\sum_{k=0}^{\infty} a_k(z - z_n)^k\right)\left(\frac{b_2}{(z - z_n)^2} + \sum_{k=0}^{\infty} c_{2k}(z - z_n)^{2k}\right)$$

   so that the coefficient of $\dfrac{1}{z - z_n}$ is $a_1 b_2$. Hence, we obtain

$$\text{Res}\left(\frac{f(z)}{\cos^2 z}, z_n\right) = a_1 b_2 = a_1 = f'(z_n).$$

   Alternative method (if $f(z_n) \neq 0$)

$$\begin{aligned} \cos^2 z &= 2(z - z_n)^2 - 8(z - z_n)^4 + 32(z - z_n)^6 - \cdots \\ &= (z - z_n)^2\{2 - 8(z - z_n)^2 + 32(z - z_n)^4 - \cdots\} \end{aligned}$$

   so z_n is a double pole of $\dfrac{1}{\cos^2 z}$. Using the formula in Example 6.2.2 (p. 230), we have

$$\text{Res}\left(\frac{f(z)}{\cos^2 z}, z_n\right) = \left.\frac{6f'(z)(\cos^2 z)'' - 2f(z)(\cos^2 z)'''}{3[(\cos^2 z)'']^2}\right|_{z=z_n} = f'(z_n).$$

2. (a) Note that $z = \dfrac{\pi}{2} + k\pi$, k is any integer, are simple pole of $\tan z$ since

$$\sin\left(\frac{\pi}{2} + k\pi\right) = (-1)^k \neq 0.$$

   Now, $\text{Res}\left(\tan z, \dfrac{\pi}{2} + k\pi\right) = \dfrac{\sin\left(\frac{\pi}{2} + k\pi\right)}{\left.\frac{d}{dz}\cos z\right|_{z=\frac{\pi}{2}+k\pi}} = -1.$

(b) Obviously, $z = 1$ is a pole of order n. We have

$$\begin{aligned}\text{Res }(f,1) &= \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-1)^n\frac{z^{2n}}{(z-1)^n}\right] \\ &= \frac{(2n)!}{(n-1)!(n+1)!}.\end{aligned}$$

3. The point $z = 0$ is a removable singularity of $f(z)$ since the Laurent expansion of $f(z)$ valid in the region $|z| > 0$ is given by

$$f(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \quad |z| > 0.$$

The function f is simply defined "incorrectly" at $z = 0$.

$$\text{Res }(f,0) = \text{ coefficient of } \frac{1}{z} \text{ in the above Laurent series} = 0.$$

4. First, consider the Taylor series expansion of $2\cos z - 2 + z^2$ at $z = 0$:

$$\begin{aligned}2\cos z - 2 + z^2 &= 2\left[1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right] - 2 + z^2 \\ &= \frac{z^4}{12}\left(1 - \frac{z^2}{30} + \cdots\right).\end{aligned}$$

Consider

$$\lim_{z\to 0} z^8 f = \lim_{z\to 0}\frac{12^2}{1 - \frac{z^2}{30} + \cdots} = 144,$$

so that f has a pole of order 8 at $z = 0$. Since f is even so that $\text{Res}\,(f,0) = -\text{Res}\,(f,0)$; hence,

$$\text{Res }(f,0) = 0.$$

5. (a) Note that $z = 1$ is a double pole of the integrand and it is the only pole included inside $|z| = 2$. We then have

$$\begin{aligned}\oint_{|z|=2}\frac{z^4+z}{(z-1)^2}\,dz &= 2\pi i\text{Res}\left(\frac{z^4+z}{(z-1)^2}, 1\right) \\ &= 2\pi i\lim_{z\to 1}(z^4+z)' = 2\pi i\lim_{z\to 1}(4z^3+1) = 10\pi i.\end{aligned}$$

(b) Note that $z = 0$ is a double pole of the integrand and it is the only pole included inside $|z| = 2$. We then have

$$\begin{aligned}\oint_{|z|=2}\frac{z^3+3z+1}{z^4-5z^2}\,dz &= 2\pi i\text{Res}\left(\frac{z^3+3z+1}{z^4-5z^2}, 0\right) \\ &= 2\pi i\lim_{z\to 0}\left(\frac{z^3+3z+1}{z^2-5}\right)' \\ &= 2\pi i\lim_{z\to 0}\frac{(z^2-5)(3z^2+3)-(z^2+3z+1)(2z)}{(z^2-5)^2} = \frac{-6\pi i}{5}.\end{aligned}$$

(c) Note that $z = 0$ is a double pole of $\sinh^2 z/z^4$. Hence

$$\begin{aligned}
\oint_{|z|=2} \frac{\sinh^2 z}{z^4}\,dz &= 2\pi i \text{Res}\left(\frac{\sinh^2 z}{z^4}, 0\right) \\
&= 2\pi i \lim_{z\to 0} \frac{d}{dz}\left[\frac{\sinh^2 z}{z^2}\right] \\
&= 2\pi i \lim_{z\to 0} \frac{d}{dz}\left[\frac{\left(z + \frac{z^3}{3!} + \cdots\right)^2}{z^2}\right] = 0.
\end{aligned}$$

(d) Consider

$$\begin{aligned}
\oint_{|z-i|=2} \frac{e^z + z}{(z-1)^4}\,dz &= 2\pi i \text{Res}\left(\frac{e^z + z}{(z-1)^4}, 1\right) \\
&= 2\pi i \lim_{z\to 1} \frac{(e^z + z)'''}{3!} = 2\pi i \left(\frac{e}{6}\right) = \frac{\pi e i}{3}.
\end{aligned}$$

6. Recall the following Taylor series:

$$\begin{aligned}
e^z - 1 &= z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\
\sin^3 z &= z^3 - \frac{z^5}{2} + \cdots \\
\frac{e^z - 1}{\sin^3 z} &= \frac{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}{z^3 - \frac{z^5}{2} + \cdots}
\end{aligned}$$

By performing long division

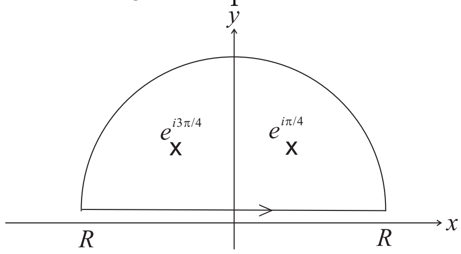
$$\begin{array}{r|l} & \frac{1}{z^2} \quad \frac{1}{2z} \\ \hline z^3 \quad \frac{z^5}{2} & z \quad \frac{z^2}{2!} \quad \frac{z^3}{3!} \end{array}$$

the coefficient of $\dfrac{1}{z}$ is seen to be $\dfrac{1}{2}$ so that

$$\text{Res}\left(\frac{e^z - 1}{\sin^3 z}, 0\right) = \frac{1}{2}.$$

7. (a)
$$\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta + a^2} = \oint_{|z|=1} \frac{dz}{iz\left(1-2a\left(\frac{z+z^{-1}}{2}\right)+a^2\right)} = \oint_{|z|=1} \frac{dz}{-ai(z-a)\left(z-\frac{1}{a}\right)}$$
$$= \begin{cases} 2\pi i \text{Res}\left(\frac{1}{-ai(z-a)\left(z-\frac{1}{a}\right)}, a\right), & |a|>1 \\ 2\pi i \text{Res}\left(\frac{1}{-ai(z-a)\left(z-\frac{1}{a}\right)}, \frac{1}{a}\right), & |a|<1 \end{cases}$$
$$= \begin{cases} \frac{2\pi}{a}\lim_{z\to a}\left(\frac{1}{z-\frac{1}{a}}\right) = \frac{2\pi}{a^2-1}, & |a|>1 \\ \frac{2\pi}{a}\lim_{z\to \frac{1}{a}}\left(\frac{1}{z-a}\right) = \frac{2\pi}{1-a^2}, & |a|<1 \end{cases}.$$

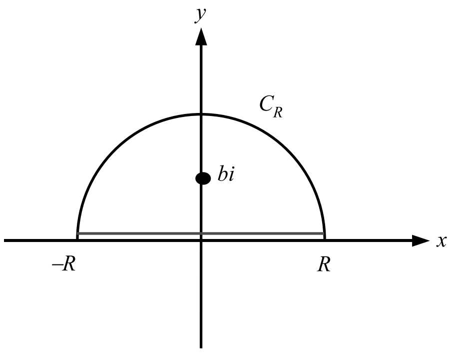
(b) Consider the closed contour C as depicted in the following figure



$$\oint_C \frac{z^2}{z^4+1}\,dz = \int_{C_R} \frac{z^2}{z^4+1}\,dz + \int_{-R}^{R} \frac{x^2}{x^4+1}\,dx.$$

By Residue calculus
$$\oint_C \frac{z^2}{z^4+1}\,dz = 2\pi i\left[\text{Res}\left(\frac{z^2}{z^4+1}, e^{i\pi/4}\right) + \text{Res}\left(\frac{z^2}{z^4+1}, e^{i3\pi/4}\right)\right],$$
where the isolated singularities of $\dfrac{z^2}{z^4+1}$ enclosed inside C are $z = e^{i\pi/4}$ and $z = e^{i3\pi/4}$. Both are simple poles so that
$$\text{Res}\left(\frac{z^2}{z^4+1}, e^{i\pi/4}\right) = \left.\frac{z^2}{4z^3}\right|_{z=e^{i\pi/4}} = \frac{1}{4}e^{-i\pi/4}$$
$$\text{Res}\left(\frac{z^2}{z^4+1}, e^{i3\pi/4}\right) = \frac{1}{4}e^{-i3\pi/4}.$$
$$\int_{C_R} \frac{z^2}{z^4+1}\,dz = O\left(\frac{R^2}{R^4}\right) R \to 0 \text{ as } R\to\infty.$$
By taking the limit $R\to\infty$, we obtain
$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx = 2\pi i\left[\text{Res}\left(\frac{z^2}{z^4+1}, e^{i\pi/4}\right) + \text{Res}\left(\frac{z^2}{z^4+1}, e^{i3\pi/4}\right)\right]$$
$$= \frac{2\pi i}{4}\left[e^{-i\pi/4} + e^{-i3\pi/4}\right] = \frac{\pi}{\sqrt{2}}.$$

(c) Consider
$$\oint_C \frac{ze^{iaz}}{z^2+b^2}\,dz = \int_{-R}^{R} \frac{xe^{iax}}{x^2+b^2}\,dx + \int_{C_R} \frac{ze^{iaz}}{z^2+b^2}\,dz.$$



Letting $R \to \infty$, then the integral over C_R vanishes by Jordan's lemma. This is because $\left|\dfrac{z}{z^2+b^2}\right| \to 0$ as $R \to \infty$. The integrand has a singularity at $z = bi$ which is enclosed inside the closed contour. Since $b > 0$, we have

$$2\pi i \text{Res}\left(\frac{ze^{iaz}}{z^2+b^2}, bi\right) = 2\pi i\left(\frac{-ze^{iaz}}{z+ib}\right)\bigg|_{z=bi} = \frac{\pi}{b}bie^{-ab}$$

so that

$$\text{Im} \oint_C \frac{ze^{iaz}}{z^2+b^2}\,dz = \int_{-\infty}^{\infty} \frac{x\sin ax}{x^2+b^2}\,dx = \frac{\pi}{e^{ab}}.$$

8. One may be tempted to say that the given integral equals the imaginary part of

$$\text{PV}\int_{-\infty}^{\infty} \frac{e^{ix}}{x+i}dx.$$

This is wrong! (Why?) Moreover, we cannot use $(\sin z)/(z+i)$ either, because it is unbounded in both the upper and lower half-planes. We try the substitution

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i},$$

which lead to the representation

$$I = \frac{1}{2i}\left(\text{PV}\int_{-\infty}^{\infty}\frac{e^{ix}}{x+i}dx - \text{PV}\int_{-\infty}^{\infty}\frac{e^{-ix}}{x+i}dx\right).$$

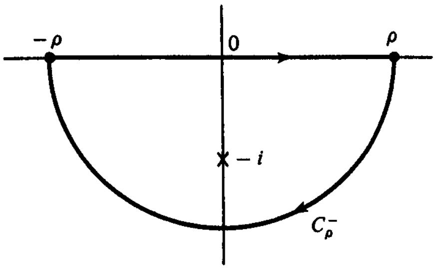
Now we deal with each integral separately. For

$$I_1 := \text{PV}\int_{-\infty}^{\infty}\frac{e^{ie}}{x+i}dx$$

we close the contour $[-\rho,\rho]$ with the half-circle C_ρ^+ in the upper half-plane. Then, by Jordan's lemma

$$\lim_{\rho\to\infty}\int_{C_\rho^+}\frac{e^{iz}}{z+i}dz = 0,$$

and since the only singularity of the integrand is in the *lower* half-plane at $z = -i$, we deduce that $I_1 = 0$.



Now the second integral

$$I_2 := \text{PV} \int_{-\infty}^{\infty} \frac{e^{-ix}}{x+i} dx$$

involves the function e^{-iz}, which is unbounded in the upper half-plane, so we close the contour $[-\rho, \rho]$ in the lower half-plane with the semicircle $C_\rho^- : z = \rho e^{-it}, 0 \le t \le \pi$ (see the above figure). Then by the analogue of Jordan's lemma for the case when $m < 0$, we deduce that

$$\lim_{\rho\to\infty} \int_{C_\rho^-} \frac{e^{-iz}}{z+i} dz = 0.$$

Observing that the closed contour in the figure is negatively oriented, we obtain

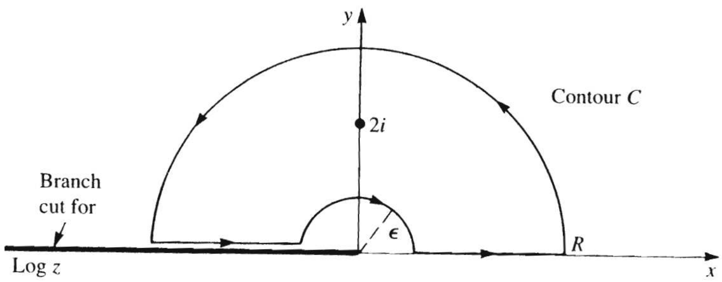
$$\begin{aligned} I_2 &= -2\pi i \text{Res}\left(\frac{e^{-iz}}{z+i}; -i\right) \\ &= -2\pi i \lim_{z\to -i} e^{-iz} = -2\pi i e^{-1}. \end{aligned}$$

Consequently,

$$I = \frac{1}{2i}(I_1 - I_2) = \frac{1}{2i}(0 + 2\pi i e^{-1}) = \frac{\pi}{e}.$$

9. Consider

$$\begin{aligned} \oint_C \frac{\text{Log } z}{z^2+4} dz &= \int_{-R}^{-\varepsilon} \frac{\text{Log } z}{z^2+4} dx + \int_{C_\varepsilon} \frac{\text{Log } z}{z^2+4} dz \\ &\quad + \int_{\varepsilon}^{R} \frac{\ln x}{x^2+4} dx + \int_{C_R} \frac{\text{Log } z}{z^2+4} dz, \end{aligned}$$



It is necessary to show that the second integral vanishes as $\varepsilon \to 0$ and the fourth integral vanishes as $R \to \infty$. The integrand has a simple pole at $z = 2i$.

Now

$$\int_{C_\varepsilon} \frac{\text{Log } z}{z^2+4} dz = O(\varepsilon \ln \varepsilon) \to 0 \text{ as } \varepsilon \to 0$$
$$\int_{C_R} \frac{\text{Log } z}{z^2+4} dz = O\left(\frac{\ln R}{R^2} \cdot R\right) \to 0 \text{ as } R \to \infty.$$

The sum of the first and third integrals is

$$\begin{aligned} & i\pi \int_{-\infty}^{0} \frac{1}{x^2+4} dx + 2\int_0^\infty \frac{\ln x}{x^2+4} dx \\ = & \ 2\pi i \text{Res}\left(\frac{\text{Log } z}{z^2+4}, 2i\right) \\ = & \ 2\pi i \frac{\text{Log } 2i}{4i} = \frac{\pi}{2}\left(\ln 2 + i\frac{\pi}{2}\right). \end{aligned}$$

By equating the real parts, we obtain

$$\int_0^\infty \frac{\ln x}{x^2+4}\,dx = \frac{\pi}{4}\ln 2.$$

10. Along ℓ_1, we have

$$I_1 = \int_{\ell_1} \frac{ze^z}{e^{4z}+1}\,dz = \int_{-R}^{R} \frac{xe^x}{e^{4x}+1}\,dx.$$

Along ℓ_3, we have $z = x + i\dfrac{\pi}{2}$ so that

$$\begin{aligned} I_3 &= \int_R^{-R} \frac{\left(x+i\frac{\pi}{2}\right) e^x e^{i\pi/2}}{e^{4x}+1}\,dx \\ &= -i\int_{-R}^{R} \frac{xe^x}{e^{4x}+1}\,dx + \frac{\pi}{2}\int_{-R}^{R} \frac{e^x}{e^{4x}+1}\,dx. \end{aligned}$$

For $z = R+iy, 0 \le y \le \dfrac{\pi}{2}$ on ℓ_2, we have $|z| \le R + y \le R + \dfrac{\pi}{2}$ so that

$$\left|\frac{ze^z}{e^{4z}+1}\right| = |z|\left|\frac{e^z}{e^{4z}+1}\right| \le \left(R+\frac{\pi}{2}\right)\frac{e^R}{e^{4R}-1} = \frac{R+\frac{\pi}{2}}{e^{3R}-e^{-R}}$$

which tends to 0 as $R \to \infty$. Hence,

$$|I_2| = \left|\int_{\ell_2} \frac{ze^z}{e^{4z}+1}\,dz\right| \le \frac{R+\frac{\pi}{2}}{e^{3R}-e^{-R}}\frac{\pi}{2} \to 0 \text{ as } R \to \infty.$$

In a similar manner, $|I_4| = \left|\displaystyle\int_{\ell_4} \frac{ze^z}{e^{4z}+1}\,dz\right| \to 0$ as $R \to \infty$.

The integrand function has a simple pole at $z = i\dfrac{\pi}{4}$ inside the closed rectangular contour. We have

$$\begin{aligned} \oint_C \frac{ze^z}{e^{4z}+1}\,dz &= 2\pi i \operatorname{Res}\left(\frac{ze^z}{e^{4z}+1}; i\frac{\pi}{4}\right) \\ &= (2\pi i)\frac{i\frac{\pi}{4}e^{i\pi/4}}{4e^{i\pi}} = \frac{(2\pi i)(-i\pi)(1+i)}{16\sqrt{2}} = \frac{\pi^2(1+i)}{8\sqrt{2}}. \end{aligned}$$

Lastly,

$$\begin{aligned} \lim_{R\to\infty} \oint_C \frac{ze^z}{e^{4z}+1}\,dz &= \lim_{R\to\infty} [I_1 + I_2 + I_3 + I_4] \\ &= (1-i)\int_{-\infty}^{\infty} \frac{xe^x}{e^{4x}+1}\,dx + \frac{\pi}{2}\int_{-\infty}^{\infty} \frac{e^x}{e^{4x}+1}\,dx \\ &= \frac{\pi^2(1+i)}{8\sqrt{2}}. \end{aligned}$$

Taking the imaginary parts of both sides, we obtain

$$\int_{-\infty}^{\infty} \frac{xe^x}{e^{4x}+1}\,dx = -\frac{\pi^2}{8\sqrt{2}}.$$