MATH304, Spring 2007

Solution to Test One

1. Note that $|z^n+\alpha| \le |z|^n + |\alpha| \le 1+|\alpha|$ for $|z|\le 1$. The upper bound is attained when z is chosen such that $|z|=1$ and $z^n = k\alpha$, where k is real positive. Hence, the maximum value of $|z^n+\alpha| = 1+|\alpha|$ for $|z|\le 1$.

2. (a) Consider

$$|g(z_2)-g(z_1)| = \left|\frac{f(z_2)-f(z_1)}{f(1)-f(0)}\right| = \frac{|z_2-z_1|}{|1-0|} = |z_2-z_1|$$

so $g(z)$ is an isometry.

(b) (i) From $|g(z)|^2 = |z|^2$ and $|g(z)-1|^2 = |z-1|^2$ for any z in $\mathbb{C}$, we have

$$[\mathrm{Re}\, g(z)]^2 + [\mathrm{Im}\, g(z)]^2 = (\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2$$
$$[\mathrm{Re}\, g(z)]^2 - 2\mathrm{Re}\, g(z) + 1 + [\mathrm{Im}\, g(z)]^2 = (\mathrm{Re}\, z)^2 - 2\mathrm{Re}\, z + 1 + (\mathrm{Im}\, z)^2.$$

Subtracting the two equations, we obtain

$$\mathrm{Re}\, g(z) = \mathrm{Re}\, z.$$

(ii) Since $|g(i)|^2 = 1$ and $\mathrm{Re}\, g(i) = \mathrm{Re}\, i = 0$; hence $g(i) = \pm i$.

3. See lecture note.

4. $f(z) = (x-y)^2 + 2i(x+y)$.

(a) $u_x = 2(x-y), \quad u_y = -2(x-y), \quad v_x = 2, \quad v_y = 2.$

$$u_x = v_y \quad \Leftrightarrow \quad 2(x-y)=2 \quad \Leftrightarrow \quad x-y=1$$
$$u_y = -v_x \quad \Leftrightarrow \quad -2(x-y) = -2 \quad \Leftrightarrow \quad x-y=1.$$

Hence, the Cauchy-Riemann relations are satisfied along $x-y=1$.

(b) Along $x-y=1$, $f'(z)$ exists and it is equal to $u_x + iv_x = 2(1+i)$.
But f' exists only along the line $x-y=1$, and every neighborhood of any point on the line contains points not on the line. Hence, f is *nowhere* analytic.

5. Consider $f(z) = \dfrac{1}{z} = \dfrac{1}{x+iy} = \dfrac{x-iy}{x^2+y^2} = \dfrac{x}{x^2+y^2} - i\dfrac{y}{x^2+y^2}$.

We then have

$$u(x,y) = \frac{x}{x^2+y^2}, \quad v(x,y) = \frac{-y}{x^2+y^2}.$$

For $u(x,y)=1$, we then have $x = x^2+y^2$

$$x^2 - x + y^2 = 0$$
$$\left(x-\frac{1}{2}\right)^2 + y^2 = \frac{1}{4}.$$

Therefore, the preimage of of $u(x,y)=1$ is a circle in the x-y plane with center at $\left(\dfrac{1}{2}, 0\right)$ and radius $= \dfrac{1}{2}$.

6. It is known that z and z^2 are entire and

$$f'(z) = \begin{cases} 2z & \text{for } |z| < 1 \\ 1 & \text{for } |z| > 1 \end{cases}.$$

Hence, f is not differentiable on the circle: $|z| = 1$. Define $D = \{z : |z| \neq 1\}$, which is an open set since its complement $D' = \{z : |z| = 1\}$ is closed. Since every point inside D is an interior point and f is differentiable throughout D so f is analytic in D.

7. Consider

$$\begin{aligned}
(uv)_{xx} &= u_{xx}v + 2u_x v_x + uv_{xx} \\
(uv)_{yy} &= u_{yy}v + 2u_y v_y + uv_{yy} \\
(uv)_{xx} + (uv)_{yy} &= (u_{xx} + u_{yy})v + u(v_{xx} + v_{yy}) + 2(u_x v_x + u_y v_y) \\
&= 2[u_x(-u_y) + u_y u_x] = 0.
\end{aligned}$$

hence, uv is harmonic.

8. (a) Let $t = y/x$ and let $T(x, y)$ denote the temperature field where

$$T(x, y) = f(t).$$

We have

$$\begin{aligned}
T_x &= \frac{-y}{x^2} f', & T_y &= \frac{f'}{x} \\
T_{xx} &= \frac{2y}{x^3} f' + \frac{y^2}{x^4} f'', & T_{yy} &= \frac{f''}{x^2}.
\end{aligned}$$

Since T is harmonic

$$0 = T_{xx} + T_{yy} = \frac{1}{x^2}\left(\frac{2y}{x} f' + \frac{y^2}{x^2} f'' + f''\right)$$

so that

$$(1 + t^2)f'' + 2tf' = 0.$$

To solve for f, consider

$$\frac{f''}{f'} = (\ln f')' = -\frac{2t}{1 + t^2}$$

so that

$$\ln f' = -\ln(1 + t^2) + C'$$

giving

$$f' = \frac{A}{1 + t^2}, \quad A > 0.$$

Lastly, we obtain

$$f = A\tan^{-1} t + B = A\tan^{-1}\frac{y}{x} + B.$$

(b) Let $F(x, y)$ be the flux function so that

$$\begin{aligned} F_y &= T_x = -\frac{Ay}{x^2+y^2} \quad \text{so} \quad F = -\frac{A}{2}\ln(x^2+y^2) + g(x) \\ -F_x &= T_y = \frac{Ax}{x^2+y^2}. \end{aligned}$$

To determine $g(x)$, we equate

$$-F_x = \frac{Ax}{x^2+y^2} = \frac{Ax}{x^2+y^2} - g'(x)$$

giving $g(x) = C$. Set $F(x, y) = \text{constant}$, the flux lines are family of circles: $x^2 + y^2 = \alpha, \alpha > 0$.