1. Consider the mapping associated with the complex function
\[ w = \cos z, \quad z = x + iy, \]
find the image curve of \( x = \alpha, \alpha \) is a constant, under the above mapping in the \( w \)-plane. In particular, examine the special cases where \( \cos \alpha = 0 \) and \( \sin \alpha = 0 \).

*Hint:* \( \cos z = \cosh y \cos x - i \sinh y \sin x \).

2. Show that, if \( a \) is a positive real constant, then
\[ \coth^{-1} \frac{z}{a} = \frac{1}{2} \log \frac{z + a}{z - a} = \frac{1}{2} \left[ \ln \left| \frac{z + a}{z - a} \right| + i \arg \left( \frac{z + a}{z - a} \right) \right] \].

*Hint:* \( \sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}, \quad \coth z = \frac{\cosh z}{\sinh z} \).

3. Show that all the values of
\[ \left( \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right)^{\sqrt{2}i} \]
lie on a straight line in the complex plane. Find the equation of this line.

4. Consider the multi-valued function: \( f(z) = (z - 1)^{1/3} \).

(a) Describe the Riemann surface of the function. [Specify the branch cut, branch points and the number of sheets.]

(b) Suppose we choose the branch such that \( f(1 + i) = e^{\frac{5\pi}{6}}i \), compute \( f(-1) \).

5. (a) Evaluate
\[ \int_{C_1} \cosh z \, dz \]
where \( C_1 \) is the line segment joining \( \log 2 \) and \( i\pi/2 \) in the complex plane.

(b) Estimate an upper bound on
\[ \left| \int_{C_2} \frac{1}{\sinh z} \, dz \right| \]
where \( C_2 \) is the line segment joining \( i\pi/4 \) and \( i\pi/2 \) in the complex plane.

*Hint:* \( \sinh iy = i \sin y \).
6. (a) Evaluate
\[ \oint_{x^2+y^2=2x} \frac{\sin \frac{\pi z}{4}}{z^2 - 1} \, dz \]
using Cauchy’s integral formula. \[2\]

(b) Find the maximum value of \[ \left| \frac{1}{z+1} \right| \] on and inside the circle: \( x^2 + y^2 = 2x \).

*Hint:* Use the Maximum Modulus Theorem or other judicious method. \[3\]

7. Let \( f \) be entire and suppose \( \text{Re} f(z) \leq M \) for all \( z \), where \( M \) is a fixed real constant. Prove that \( f \) must be a constant function.

*Hint:* Apply Liouville’s Theorem to the function \( e^f \). It is necessary to show that \( e^f \) is also entire. \[3\]

8. Let \( f \) be an entire function such that 
\[ |f(z)| \leq A|z| \] for all \( z \),
where \( A \) is a fixed positive number.

(a) Let \( f^{(n)}(z) \) denote the \( n \)th order derivative of \( f(z) \). Recall Cauchy’s inequality:
\[ |f^{(n)}(z_0)| \leq n!M_R \frac{R^n}{R^n}, \quad n = 1, 2, \ldots, \]
where \( M_R \) denotes an upper bound of \(|f(z)|\) on \( C_R : |z - z_0| = R \). Use it to show that
\[ |f^{(n)}(z_0)| \leq n!A(R + |z_0|) \frac{R^n}{R^n}, \quad R > 0. \] \[2\]

(b) Hence, show that
\[ f(z) = a_1z, \quad \text{where } a_1 \text{ is a complex constant such that } |a_1| \leq A. \]

*Hint:* Show that \( f^{(n)} \) is zero everywhere in the plane, for \( n \geq 2 \), and \( f(0) = 0. \)

— End —