MATH362 – Fundamentals of Mathematical Finance

Topic 1 — Mean variance portfolio theory

1.1 Mean and variance of portfolio return

1.2 Markowitz mean-variance formulation

1.3 Two-fund Theorem

1.4 Inclusion of the risk free asset: One-fund Theorem

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1.1 Mean and variance of portfolio return

Asset return

Suppose that you purchase an asset at time zero, and 1 year later you sell the asset. The **total return** on your investment is defined to be

$$\text{total return} \; = \frac{\text{amount received}}{\text{amount invested}}.$$

If X_0 and X_1 are, respectively, the amounts of money invested and received and R is the total return, then

$$R = \frac{X_1}{X_0}.$$

rate of return is

$$r = \frac{\text{amount received} - \text{amount invested}}{\text{amount invested}} = \frac{X_1 - X_0}{X_0}.$$

It is clear that

$$R = 1 + r \quad \text{and} \quad X_1 = (1 + r)X_0.$$

Short sales

- It is possible to sell an asset that you do not own through the process of **short selling**, or **shorting**, the asset. You then sell the borrowed asset to someone else, receiving an amount X_0. At a later date, you repay your loan by purchasing the asset for, say, X_1 and return the asset to your lender. Short selling is profitable if the asset price declines.

- When short selling a stock, you are essentially duplicating the role of the issuing corporation. You sell the stock to raise immediate capital. If the stock pays dividends during the period that you have borrowed it, you too must pay that same dividend to the person from whom you borrowed the stock.

Return associated with short selling

We *receive* X_0 initially and *pay* X_1 later, so the outlay (支出) is $-X_0$ and the final receipt (進款，收入) is $-X_1$, and hence the total return is

$$R = \frac{-X_1}{-X_0} = \frac{X_1}{X_0}.$$

The minus signs cancel out, so we obtain the same expression as that for purchasing the asset. The return value R applies algebraically to both purchases and short sales.

We can write

$$-X_1 = -X_0 R = -X_0(1 + r)$$

to show that final receipt is related to initial outlay.

Example

Suppose I short 100 shares of stock in company CBA. This stock is currently selling for \$10 per share. I borrow 100 shares from my broker and sell these in the stock market, receiving $\$1,000$. At the end of 1 year the price of CBA has dropped to \$9 per share. I buy back 100 shares for \$900 and give these shares to my broker to repay the original loan. Because the stock price fell, this has been a favorable transaction for me. I made a profit of \$100.

The rate of return is clearly negative as $r = -10\%$.

Shorting converts a negative rate of return into a profit because the original investment is also negative.

Portfolio return

Suppose now that n different assets are available. We form a **portfolio** of these n assets. Suppose that this is done by apportioning an amount X_0 among the n assets. We then select amounts $X_{0i}, i = 1, 2, \cdots, n$, such that $\sum_{i=1}^{n} X_{0i} = X_0$, where X_{0i} represents the amount invested in the i^{th} asset. If we are allowed to sell an asset short, then some of the X_{0i}'s can be negative.

We write

$$X_{0i} = w_i X_0, \quad i = 1, 2, \cdots, n$$

where w_i is the **weight** of asset i in the portfolio. Clearly,

$$\sum_{i=1}^{n} w_i = 1$$

and some w_i's may be negative if short selling is allowed.

Let R_i denote the total return of asset i. Then the amount of money generated at the end of the period by the i^{th} asset is $R_i X_{0i} = R_i w_i X_0$.

The total amount received by this portfolio at the end of the period is therefore $\sum_{i=1}^{n} R_i w_i X_0$.

The overall total return of the portfolio is

$$R = \frac{\sum_{i=1}^{n} R_i w_i X_0}{X_0} = \sum_{i=1}^{n} w_i R_i.$$

Since $\sum_{i=1}^{n} w_i = 1$, we have

$$r = \sum_{i=1}^{n} w_i r_i.$$

Covariance

When considering two or more random variables, their mutual dependence can be summarized by their **covariance**.

Let x_1 and x_2 be two random variables with expected values $\overline{x}_1$ and $\overline{x}_2$. The covariance of these variables is defined to be the expectation of the product of deviations from the respective mean of x_1 and x_2:

$$\text{cov}(x_1, x_2) = E[(x_1 - \overline{x}_1)(x_2 - \overline{x}_2)].$$

The covariance of two random variables x and y is denoted by σ_{xy}. We write $\text{cov}(x_1, x_2) = \sigma_{12}$. Note that, by symmetry, $\sigma_{12} = \sigma_{21}$, and

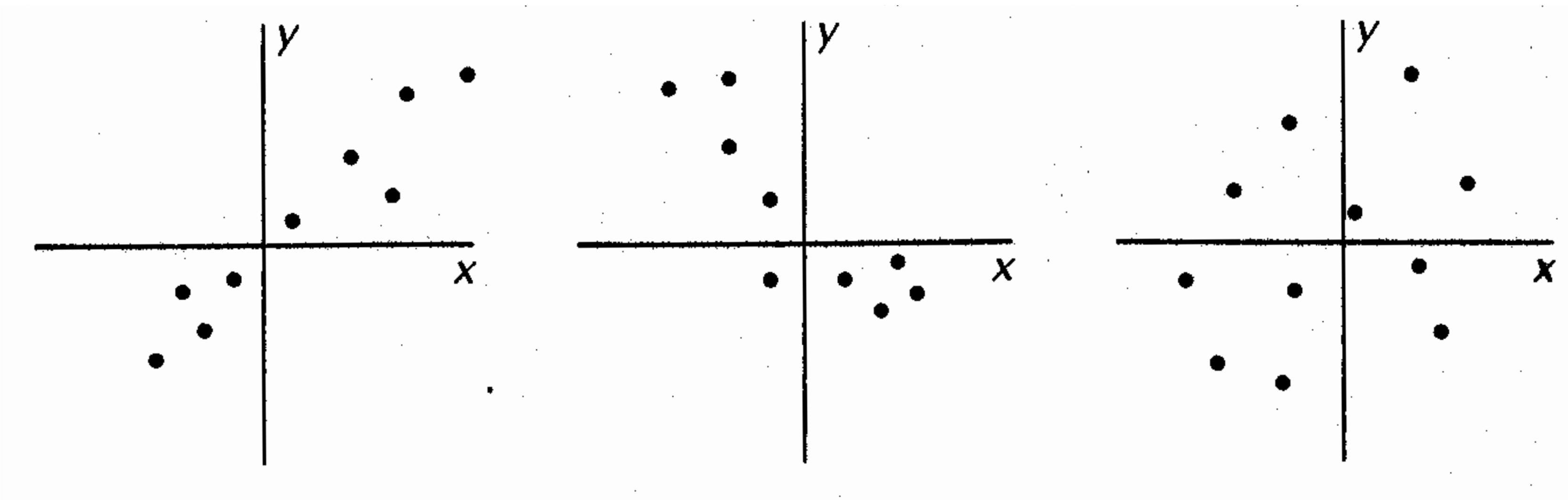
$$\sigma_{12} = E[x_1 x_2 - \overline{x}_1 x_2 - x_1 \overline{x}_2 + \overline{x}_1 \overline{x}_2] = E[x_1 x_2] - \overline{x}_1 \overline{x}_2.$$

- If two random variables x_1 and x_2 have the property that $\sigma_{12} = 0$, then they are said to be **uncorrelated**. This is the situation (roughly) where knowledge of the value of one variable gives no information about the other.

- If two random variables are independent, then they are uncorrelated. When x_1 and x_2 are independent, $E[x_1 x_2] = \overline{x}_1 \overline{x}_2$ so that $\text{cov}(x_1, x_2) = 0$.

- If $\sigma_{12} > 0$, the two variables are said to be **positively correlated**. In this case, if one variable is above its mean, the other is likely to be above its mean as well.

- On the other hand, if $\sigma_{12} < 0$, the two variables are said to be **negatively correlated**.



(a) Positively correlated (b) Negatively correlated (c) Uncorrelated

When x_1 and x_2 are positively correlated, positive deviation from mean of one random variable has a higher tendency to have positive deviation from mean of the other random variable.

The covariance of two random variables satisfies

$$|\sigma_{12}| \leq \sigma_1\sigma_2.$$

If $\sigma_{12} = \sigma_1\sigma_2$, the variables are **perfectly correlated**. In this situation, the covariance is as large as possible for the given variance. If one random variable were a fixed positive multiple of the other, the two would be perfectly correlated.

Conversely, if $\sigma_{12} = -\sigma_1\sigma_2$, the two variables exhibit **perfect negative correlation**.

The **correlation coefficient** of two random variables is defined as

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1\sigma_2}.$$

It can be shown that $|\rho_{12}| \leq 1$.

Mean return of a portfolio

Suppose that there are n assets with (random) rates of return $r_1, r_2, \cdots, r_n$, and their expected values $E(r_1) = \overline{r}_1, E(r_2) = \overline{r}_2, \cdots$, $E(r_n) = \overline{r}_n$. The rate of return of the portfolio in terms of the return of the individual returns is

$$r = w_1 r_1 + w_2 r_2 + \cdots + w_n r_n,$$

so that

$$E(r) = w_1 E(r_1) + w_2 E(r_2) + \cdots + w_n E(r_n).$$

We denote the variance of the return of asset i by σ_i^2, the variance of the return of the portfolio by σ^2, and the covariance of the return of asset i with asset j by σ_{ij}.

Portfolio variance is given by

$$\begin{aligned}
\sigma^2 &= E[(r-\overline{r})^2] \\
&= E\left[\left(\sum_{i=1}^n w_i r_i - \sum_{i=1}^n w_i \overline{r}_i\right)^2\right] \\
&= E\left[\left(\sum_{i=1}^n w_i (r_i - \overline{r}_i)\right)\left(\sum_{j=1}^n w_j (r_j - \overline{r}_j)\right)\right] \\
&= E\left[\sum_{i=1}^n \sum_{j=1}^n w_i w_j (r_i - \overline{r}_i)(r_j - \overline{r}_j)\right] \\
&= \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij}.
\end{aligned}$$

Zero correlation

Suppose that a portfolio is constructed by taking equal portions of n of these assets; that is, $w_i = 1/n$ for each i. The overall rate of return of this portfolio is

$$r = \frac{1}{n} \sum_{i=1}^{n} r_i.$$

When the returns are uncorrelated, the corresponding variance is

$$\mathrm{var}(r) = \frac{1}{n^2} \sum_{i=1}^{n} \sigma^2 = \frac{\sigma^2}{n}.$$

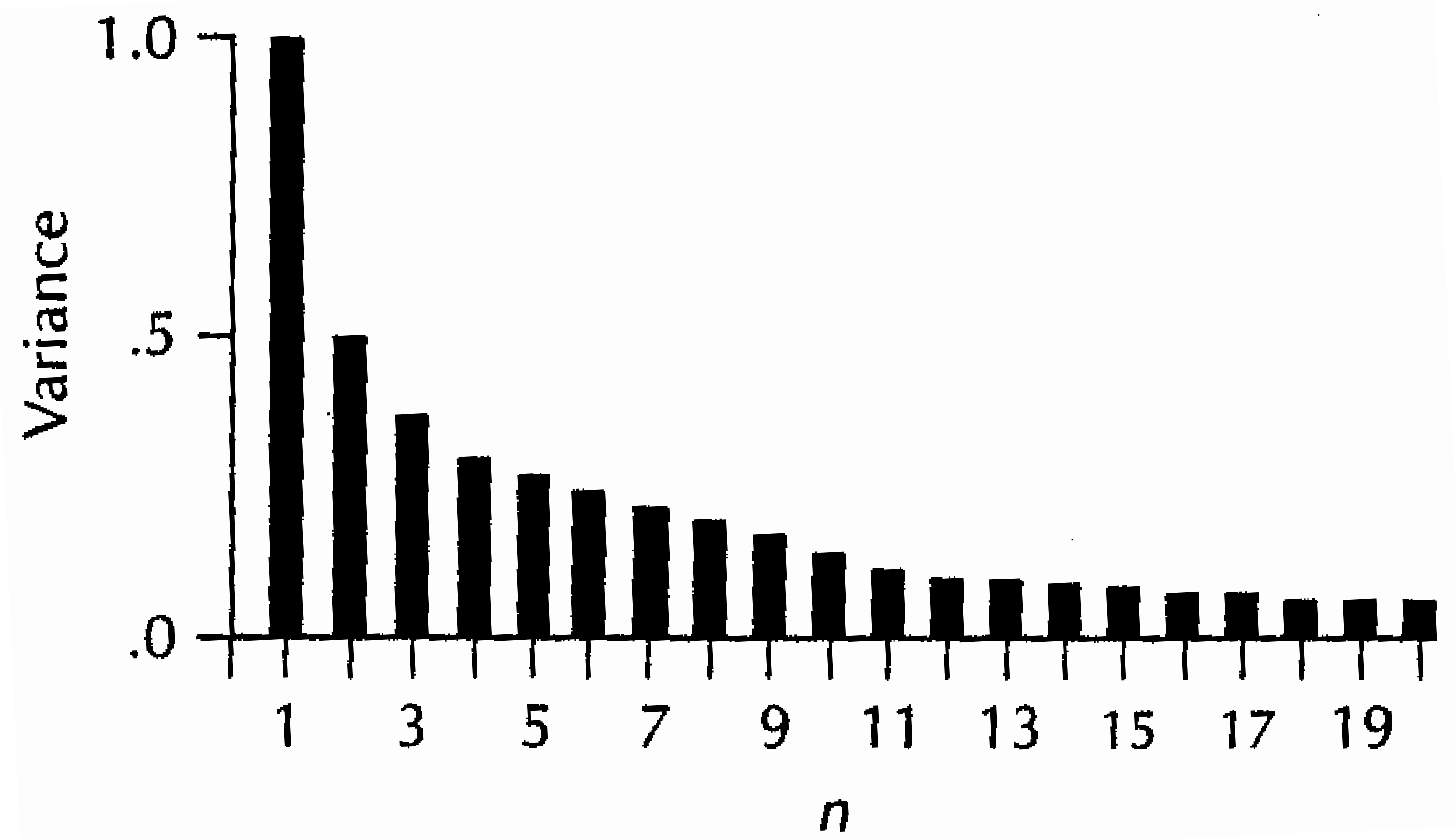
The variance decreases rapidly as n increases.

Non-zero correlation

Each asset has a rate of return with mean m and variance σ^2, but now each return pair has a covariance of $\text{cov}(r_i, r_j) = 0.3\sigma^2$ for $i \neq j$. We form a portfolio by taking equal portions of n of these assets. In this case,

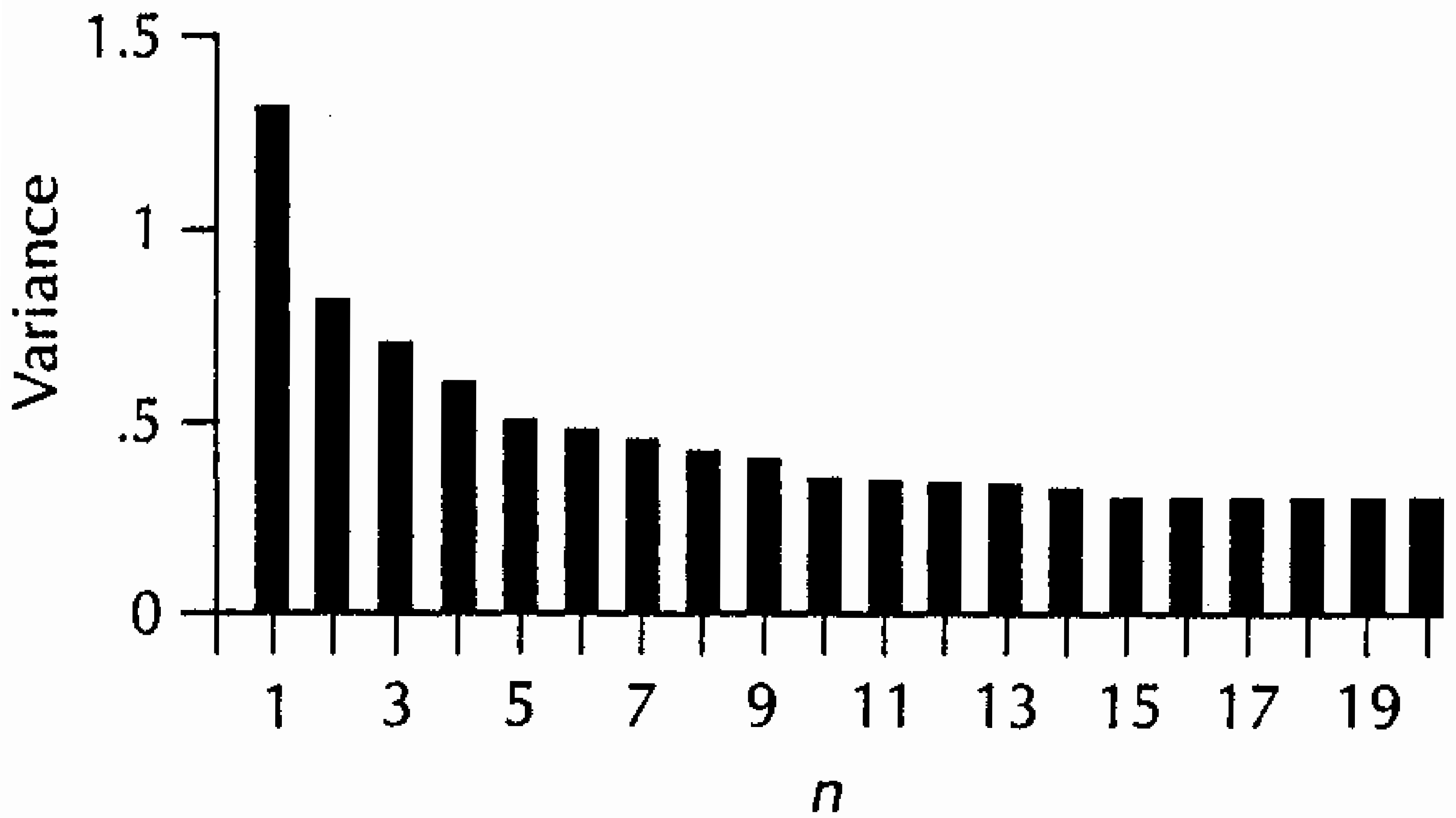
$$\begin{aligned}
\text{var}(r) &= E\left[\sum_{i=1}^{n} \frac{1}{n}(r_i - \overline{r})\right]^2 \\
&= \frac{1}{n^2} E\left\{\left[\sum_{i=1}^{n} (r_i - \overline{r})\right]\left[\sum_{j=1}^{n} (r_j - \overline{r})\right]\right\} \\
&= \frac{1}{n^2} \sum_{i,j} \sigma_{ij} = \frac{1}{n^2}\left\{\sum_{i=j} \sigma_{ij} + \sum_{i \neq j} \sigma_{ij}\right\} \\
&= \frac{1}{n^2}\{n\sigma^2 + 0.3(n^2 - n)\sigma^2\} \\
&= \frac{\sigma^2}{n} + 0.3\sigma^2\left(1 - \frac{1}{n}\right) \\
&= \frac{0.7\sigma^2}{n} + 0.3\sigma^2.
\end{aligned}$$

Uncorrelated assets



If assets are uncorrelated, the variance of a portfolio can be made very small.

Correlated assets



If asset are positively correlated, there is likely to be a lower limit to the variance that can be achieved.

1.2 Markowitz mean-variance formulation

We consider a single-period investment model. Suppose there are N risky assets, whose rates of returns are given by the random variables $r_1, \cdots, r_N$, where

$$r_n = \frac{S_n(1) - S_n(0)}{S_n(0)}, \quad n = 1, 2, \cdots, N.$$

Here $S_n(0)$ is known while $S_n(1)$ is random, $n = 1, 2, \cdots, N$. Let $\boldsymbol{w} = (w_1 \cdots w_N)^T, w_n$ denotes the proportion of wealth invested in asset n, with $\sum_{n=1}^{N} w_n = 1$. The rate of return of the portfolio r_P is

$$r_P = \sum_{n=1}^{N} w_n r_n.$$

Assumptions

1. There does not exist any asset that is replicable by a combination of other assets in the portfolio. That is, no *redundant* asset.

2. The two vectors $\boldsymbol{\mu} = (\overline{r}_1 \quad \overline{r}_2 \cdots \overline{r}_N)$ and $\mathbf{1} = (1 \quad 1 \cdots 1)$ are linearly independent. Avoidance of the degenerate case.

The first two moments of r_P are

$$\mu_P = E[r_P] = \sum_{n=1}^{N} E[w_n r_n] = \sum_{n=1}^{N} w_n \mu_n, \text{ where } \mu_n = \overline{r}_n,$$

and

$$\sigma_P^2 = \text{var}(r_P) = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \text{cov}(r_i, r_j) = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}.$$

Let Ω denote the covariance matrix so that

$$\sigma_P^2 = \boldsymbol{w}^T \Omega \boldsymbol{w},$$

where Ω is symmetric and $(\Omega)_{ij} = \sigma_{ij} = \text{cov}(r_i, r_j)$. For example, when $n = 2$, we have

$$(w_1 \quad w_2) \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = w_1^2 \sigma_1^2 + w_1 w_2 (\sigma_{12} + \sigma_{21}) + w_2^2 \sigma_2^2.$$

Also, note that

$$\begin{aligned} \frac{\partial \sigma_P^2}{\partial w_k} &= \sum_{j=1}^N \sum_{i=1}^N \frac{\partial w_i}{\partial w_k} w_j \sigma_{ij} + \sum_{i=1}^N \sum_{j=1}^N w_i \frac{\partial w_j}{\partial w_k} \sigma_{ij} \\ &= \sum_{j=1}^N w_j \sigma_{kj} + \sum_{i=1}^N w_i \sigma_{ik}. \end{aligned}$$

Since $\sigma_{kj} = \sigma_{jk}$, we obtain

$$\frac{\partial \sigma_P^2}{\partial w_k} = 2 \sum_{j=1}^N w_j \sigma_{kj} = 2(\Omega \boldsymbol{w})_k,$$

where $(\Omega \boldsymbol{w})_k$ is the k^{th} component of the vector $\Omega \boldsymbol{w}$.

Remark

1. The portfolio risk of return is quantified by σ_P^2. In the mean-variance analysis, only the first two moments are considered in the portfolio investment model. Earlier investment theory prior to Markowitz only considered the maximization of μ_P without σ_P.

2. The measure of risk by variance would place equal weight on the upside and downside deviations.

3. The assets are characterized by their random rates of return, $r_i, i = 1, \cdots, N$. In the mean-variance model, it is assumed that their first and second order moments: μ_i, σ_i and σ_{ij} are all known. We would like to determine the choice variables: $w_1, \cdots, w_N$ such that σ_P^2 is minimized for a given preset value of μ_P.

Two-asset portfolio

Consider a portfolio of two assets with known means $\overline{r}_1$ and $\overline{r}_2$, variances σ_1^2 and σ_2^2, of the rates of return r_1 and r_2, together with the correlation coefficient ρ, where cov$(r_1, r_2) = \rho\sigma_1\sigma_2$.

Let $1 - \alpha$ and α be the weights of assets 1 and 2 in this two-asset portfolio.

Portfolio mean: $\overline{r}_P = (1 - \alpha)\overline{r}_1 + \alpha\overline{r}_2$,

Portfolio variance: $\sigma_P^2 = (1 - \alpha)^2\sigma_1^2 + 2\rho\alpha(1 - \alpha)\sigma_1\sigma_2 + \alpha^2\sigma_2^2$.

assets' mean and variance

	Asset A	**Asset B**
Mean return (%)	10	20
Variance (%)	10	15

Portfolio mean[a] and variance[b] for weights and asset correlations

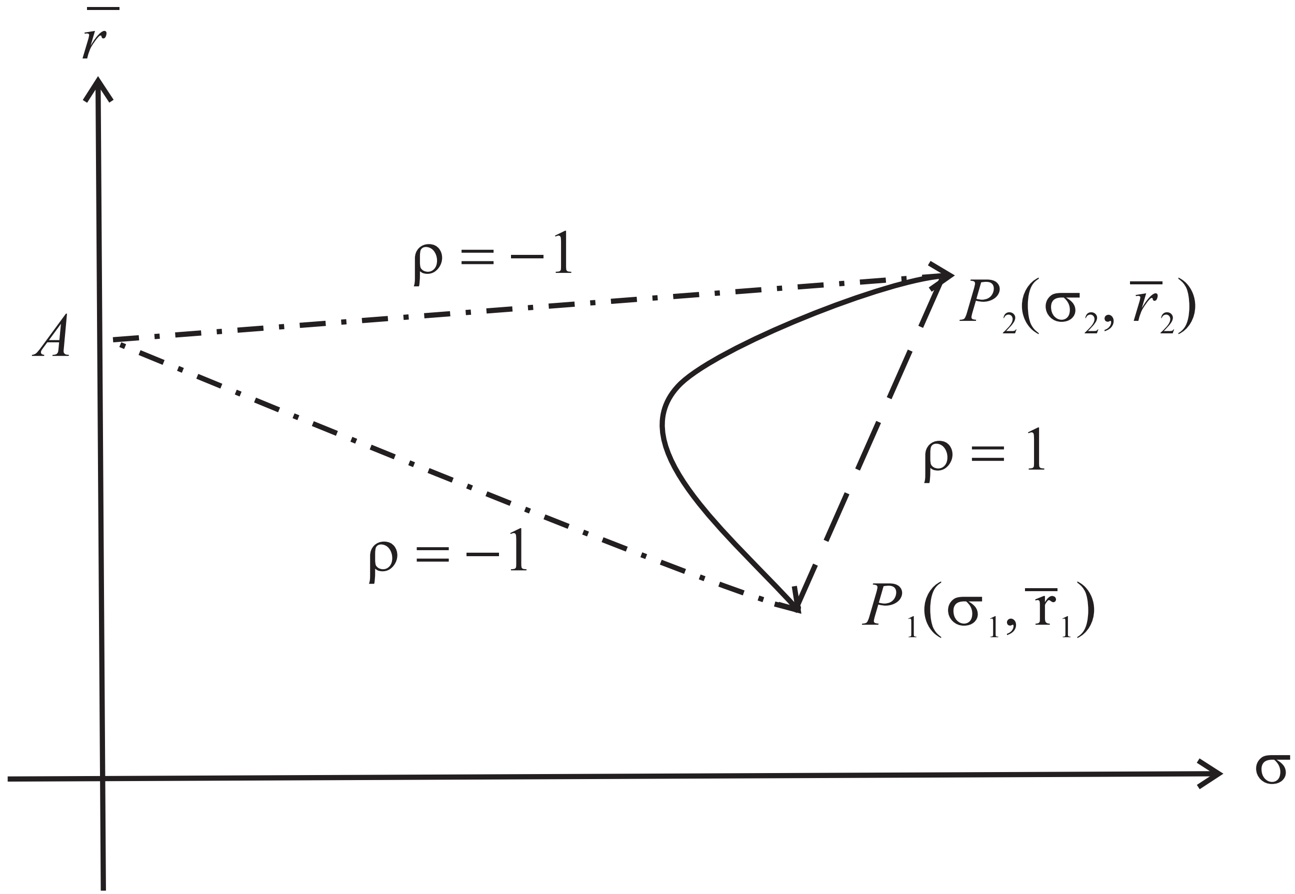
weight		$\rho = -1$		$\rho = -0.5$		$\rho = 0.5$		$\rho = 1$	
w_A	$w_B = 1 - w_A$	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance
1.0	0.0	10.0	10.00	10.0	10.00	10.0	10.00	10.0	10.00
0.8	0.2	12.0	3.08	12.0	5.04	12.0	8.96	12.0	10.92
0.5	0.5	15.0	0.13	15.0	3.19	15.0	9.31	15.0	12.37
0.2	0.8	18.0	6.08	18.0	8.04	18.0	11.96	18.00	13.92
0.0	1.0	20.0	15.00	20.0	15.00	20.0	15.00	20.0	15.00

[a] The mean is calculated as $E(R) = w_A 10 + (1 - w_A)20$.

[b] The variance is calculated as $\sigma_P^2 = w_A^2 10 + (1 - w_A)^2 15 + 2 w_A (1 - w_A) \rho \sqrt{10}\sqrt{5}$ where ρ is the assumed correlation and $\sqrt{10}$ and $\sqrt{5}$ are standard deviations of the two assets, respectively.

Observation: Apparently, lower variance is achieved for a given mean when the correlation becomes more negative.

We represent the two assets in a mean-standard deviation diagram (recall: standard deviation = $\sqrt{\text{variance}}$)



As α varies, $(\sigma_P, \overline{r}_P)$ traces out a conic curve in the σ-$\overline{r}$ plane. With $\rho = -1$, it is possible to have $\sigma_P = 0$ for some suitable choice of weight α.

Consider the special case where $\rho = 1$,

$$\begin{aligned}\sigma_P(\alpha; \rho = 1) &= \sqrt{(1-\alpha)^2\sigma_1^2 + 2\alpha(1-\alpha)\sigma_1\sigma_2 + \alpha^2\sigma_2^2} \\ &= (1-\alpha)\sigma_1 + \alpha\sigma_2.\end{aligned}$$

Since $\overline{r}_P$ and σ_P are linear in α, and if we choose $0 \leq \alpha \leq 1$, then the portfolios are represented by the straight line joining $P_1(\sigma_1, \overline{r}_1)$ and $P_2(\sigma_2, \overline{r}_2)$.

When $\rho = -1$, we have

$$\sigma_P(\alpha; \rho = -1) = \sqrt{[(1-\alpha)\sigma_1 - \alpha\sigma_2]^2} = |(1-\alpha)\sigma_1 - \alpha\sigma_2|.$$

When α is small (close to zero), the corresponding point is close to $P_1(\sigma_1, \overline{r}_1)$. The line AP_1 corresponds to

$$\sigma_P(\alpha; \rho = -1) = (1-\alpha)\sigma_1 - \alpha\sigma_2.$$

The point A corresponds to $\alpha = \dfrac{\sigma_1}{\sigma_1 + \sigma_2}$. It is a point on the vertical axis which has zero value of σ_P.

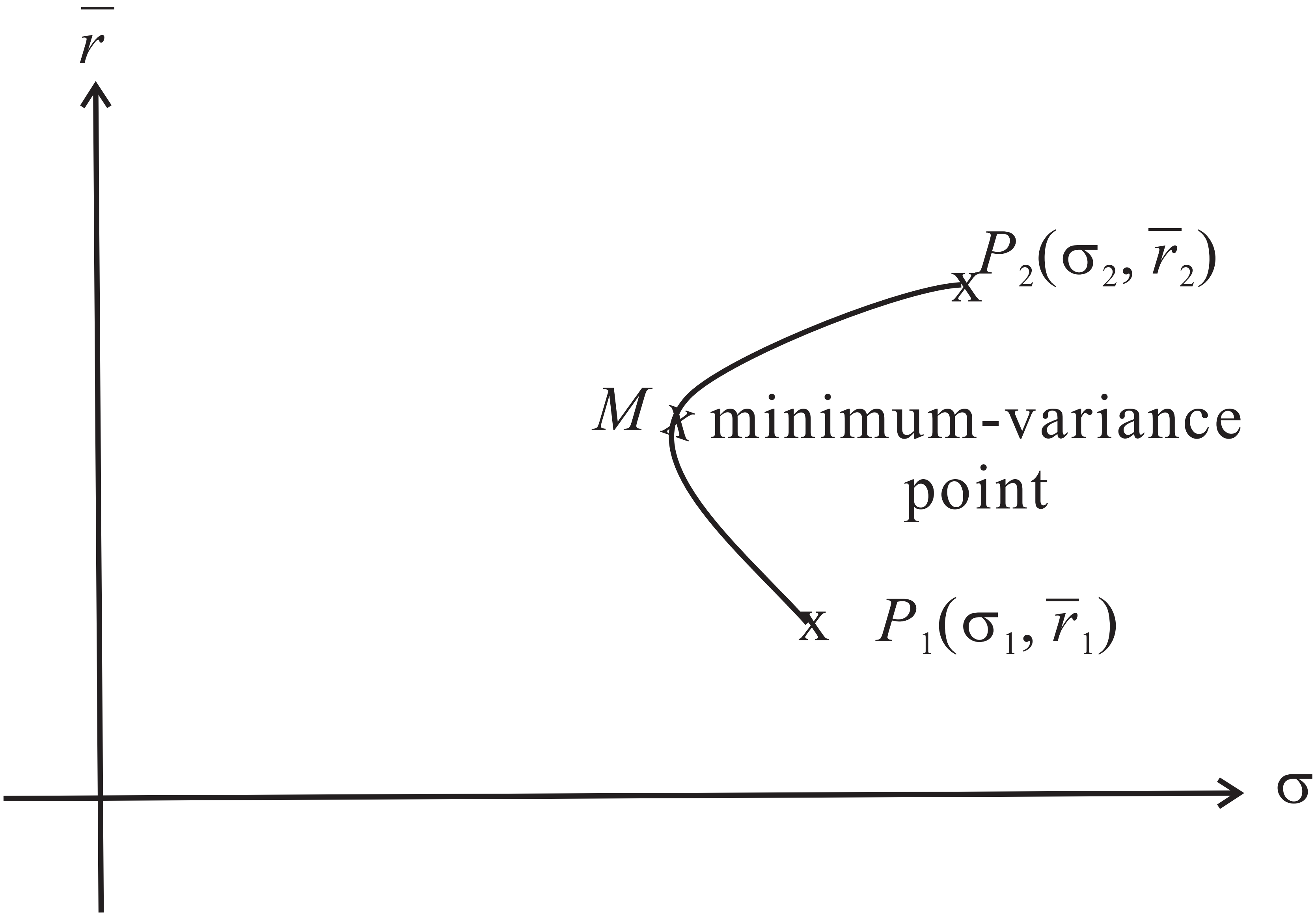
The quantity $(1-\alpha)\sigma_1 - \alpha\sigma_2$ remains positive until $\alpha = \dfrac{\sigma_1}{\sigma_1+\sigma_2}$. When $\alpha > \dfrac{\sigma_1}{\sigma_1+\sigma_2}$, the locus traces out the upper line AP_2.

Suppose $-1 < \rho < 1$, the minimum variance point on the curve that represents various portfolio combinations is determined by

$$\frac{\partial \sigma_P^2}{\partial \alpha} = -2(1-\alpha)\sigma_1^2 + 2\alpha\sigma_2^2 + 2(1-2\alpha)\rho\sigma_1\sigma_2 \underset{\substack{\uparrow \\ \text{set}}}{=} 0$$

giving

$$\alpha = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}.$$


$\bar{r}$
$P_2(\sigma_2, \bar{r}_2)$
M
minimum-variance point
$P_1(\sigma_1, \bar{r}_1)$
σ

Mathematical formulation of Markowitz's mean-variance analysis

$$\text{minimize } \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}$$

subject to $\displaystyle\sum_{i=1}^{N} w_i \overline{r}_i = \mu_P$ and $\displaystyle\sum_{i=1}^{N} w_i = 1$. Given the target expected rate of return of portfolio μ_P, we find the optimal portfolio strategy that minimizes σ_P^2.

Solution

We form the Lagrangian

$$L = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij} - \lambda_1\left(\sum_{i=1}^{N} w_i - 1\right) - \lambda_2\left(\sum_{i=1}^{N} w_i \overline{r}_i - \mu_P\right)$$

where λ_1 and λ_2 are the Lagrangian multipliers.

We then differentiate L with respect to w_i and the Lagrangian multipliers, and set all the derivatives be zero.

$$\frac{\partial L}{\partial w_i} = \sum_{j=1}^{N} \sigma_{ij} w_j - \lambda_1 - \lambda_2 \overline{r}_i = 0, \quad i = 1, 2, \cdots, N. \tag{1}$$

$$\frac{\partial L}{\partial \lambda_1} = \sum_{i=1}^{N} w_i - 1 = 0; \tag{2}$$

$$\frac{\partial L}{\partial \lambda_2} = \sum_{i=1}^{N} w_i \overline{r}_i - \mu_P = 0. \tag{3}$$

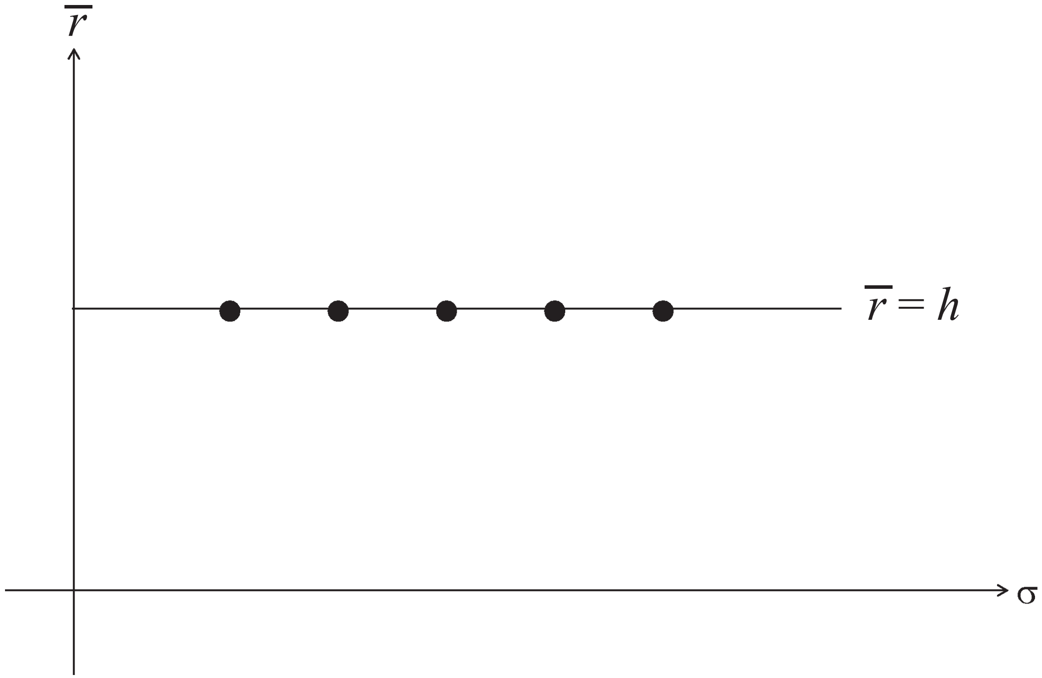
From Eq. (1), the optimal portfolio vector weight $\boldsymbol{w}^*$ admits solution of the form

$$\boldsymbol{w}^* = \Omega^{-1}(\lambda_1 \mathbf{1} + \lambda_2 \boldsymbol{\mu})$$

where $\mathbf{1} = (1 \quad 1 \cdots 1)^T$ and $\boldsymbol{\mu} = (\overline{r}_1 \quad \overline{r}_2 \cdots \overline{r}_N)^T$.

Remark

Consider the case where all assets have the same expected rate of return, that is, $\boldsymbol{\mu} = h\mathbf{1}$ for some constant h. In this case, the solution to Eqs. (2) and (3) gives $\mu_P = h$. The assets are represented by points that all lie on the horizontal line: $\overline{r} = h$.



In this case, the expected portfolio return cannot be arbitrarily prescribed. We must observe $\mu_P = h$, so the constraint on expected portfolio return is relaxed.

To determine λ_1 and λ_2, we apply the two constraints:

$$\begin{aligned} 1 &= \mathbf{1}^T \Omega^{-1} \Omega \boldsymbol{w}^* = \lambda_1 \mathbf{1}^T \Omega^{-1} \mathbf{1} + \lambda_2 \mathbf{1}^T \Omega^{-1} \boldsymbol{\mu} \\ \mu_P &= \boldsymbol{\mu}^T \Omega^{-1} \Omega \boldsymbol{w}^* = \lambda_1 \boldsymbol{\mu}^T \Omega^{-1} \mathbf{1} + \lambda_2 \boldsymbol{\mu}^T \Omega^{-1} \boldsymbol{\mu}. \end{aligned}$$

Writing $a = \mathbf{1}^T \Omega^{-1} \mathbf{1}, b = \mathbf{1}^T \Omega^{-1} \boldsymbol{\mu}$ and $c = \boldsymbol{\mu}^T \Omega^{-1} \boldsymbol{\mu}$, we have

$$1 = \lambda_1 a + \lambda_2 b \text{ and } \mu_P = \lambda_1 b + \lambda_2 c.$$

Solving for λ_1 and λ_2:

$$\lambda_1 = \frac{c - b\mu_P}{\Delta} \quad \text{and} \quad \lambda_2 = \frac{a\mu_P - b}{\Delta},$$

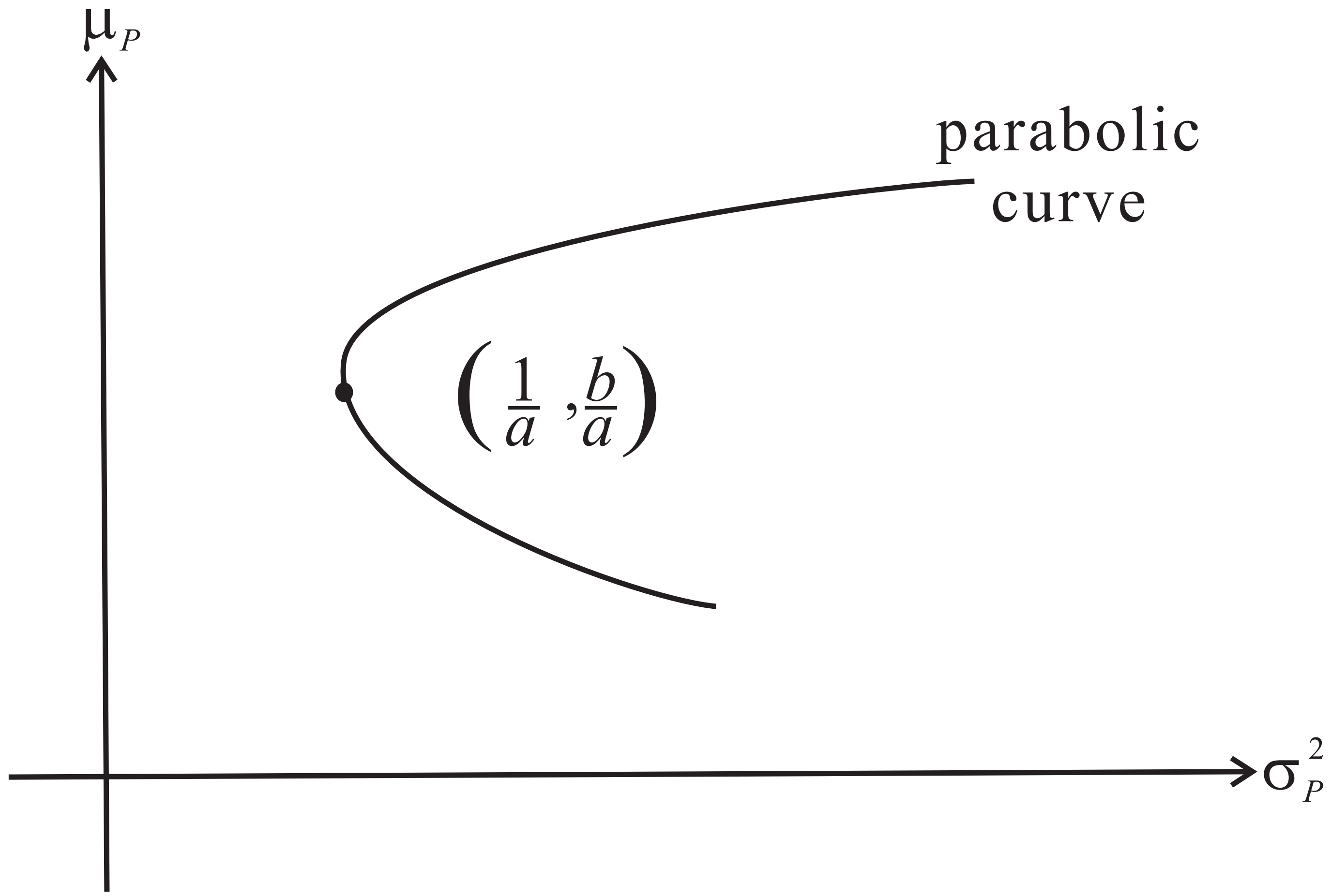
where $\Delta = ac - b^2$. Provided that $\boldsymbol{\mu} \neq h\mathbf{1}$ for some scalar h, we then have $\Delta \neq 0$.

Note that λ_1 and λ_2 have dependence on μ_P, where μ_P is the target mean prescribed in the variance minimization problem.

Note that $\sigma_P^2 = \boldsymbol{w}^T \Omega \boldsymbol{w} \geq 0$, for all $\boldsymbol{w}$, so Ω is guaranteed to be semi-positive definite. In our subsequent analysis, we assume Ω to be positive definite. In this case, Ω^{-1} exists and $a > 0, c > 0$. By virtue of the Cauchy-Schwarz inequality, $\Delta > 0$. The minimum portfolio variance for a given value of μ_P is given by

$$\begin{aligned}
\sigma_P^2 &= \boldsymbol{w}^{*T} \Omega \boldsymbol{w}^* = \boldsymbol{w}^{*T} \Omega (\lambda_1 \Omega^{-1} \mathbf{1} + \lambda_2 \Omega^{-1} \boldsymbol{\mu}) \\
&= \lambda_1 + \lambda_2 \mu_P = \frac{a\mu_P^2 - 2b\mu_P + c}{\Delta}.
\end{aligned}$$

The set of minimum variance portfolios is represented by a parabolic curve in the $\sigma_P^2 - \mu_P$ plane. The parabolic curve is generated by varying the value of the parameter μ_P.



Non-optimal portfolios are represented by points which must fall on the right side of the parabolic curve.

Alternatively, when μ_P is plotted against σ_P, the set of minimum variance portfolio is a hyperbolic curve.

What are the asymptotic values of $\lim_{\mu_P \to \pm\infty} \frac{d\mu_P}{d\sigma_P}$?

$$\begin{aligned}
\frac{d\mu_P}{d\sigma_P} &= \frac{d\mu_P}{d\sigma_P^2}\frac{d\sigma_P^2}{d\sigma_P} \\
&= \frac{\Delta}{2a\mu_P - 2b} 2\sigma_P \\
&= \frac{\sqrt{\Delta}}{a\mu_P - b}\sqrt{a\mu_P^2 - 2b\mu_P + c}
\end{aligned}$$

so that

$$\lim_{\mu_P \to \pm\infty} \frac{d\mu_P}{d\sigma_P} = \pm\sqrt{\frac{\Delta}{a}}.$$

Given μ_P, we obtain $\lambda_1 = \frac{c - b\mu_P}{\Delta}$ and $\lambda_2 = \frac{a\mu_P - b}{\Delta}$, and the optimal weight $\boldsymbol{w}^* = \Omega^{-1}(\lambda_1\mathbf{1} + \lambda_2\boldsymbol{\mu})$.

To find the global minimum variance portfolio, we set

$$\frac{d\sigma_P^2}{d\mu_P} = \frac{2a\mu_P - 2b}{\Delta} = 0$$

so that $\mu_P = b/a$ and $\sigma_P^2 = 1/a$. Correspondingly, $\lambda_1 = 1/a$ and $\lambda_2 = 0$. The weight vector that gives the global minimum variance portfolio is found to be

$$\boldsymbol{w}_g = \lambda_1\Omega^{-1}\mathbf{1} = \frac{\Omega^{-1}\mathbf{1}}{a} = \frac{\Omega^{-1}\mathbf{1}}{\mathbf{1}^T\Omega^{-1}\mathbf{1}}.$$

It is not surprising to see that $\lambda_2 = 0$ corresponds to $\boldsymbol{w}_g^*$ since the constraint on the targeted mean vanishes when λ_2 is taken to be zero. In this case, we minimize risk paying no regard to targeted mean, thus the global minimum variance portfolio is resulted.

The other portfolio that corresponds to $\lambda_1 = 0$ is obtained when μ_P is taken to be $\frac{c}{b}$. The value of the other Lagrangian multiplier λ_2 is

$$\lambda_2 = \frac{a\left(\frac{c}{b}\right) - b}{\Delta} = \frac{1}{b}.$$

The weight vector of this particular portfolio is

$$\boldsymbol{w}_d = \frac{\Omega^{-1}\boldsymbol{\mu}}{b} = \frac{\Omega^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}}.$$

The corresponding variance is $\sigma_d^2 = \dfrac{a\left(\frac{c}{b}\right)^2 - 2b\left(\frac{c}{b}\right) + c}{\Delta} = \dfrac{c}{b^2}$. Even with $\lambda_1 = 0$, we still have $\boldsymbol{w}_d^T\mathbf{1} = 1$.

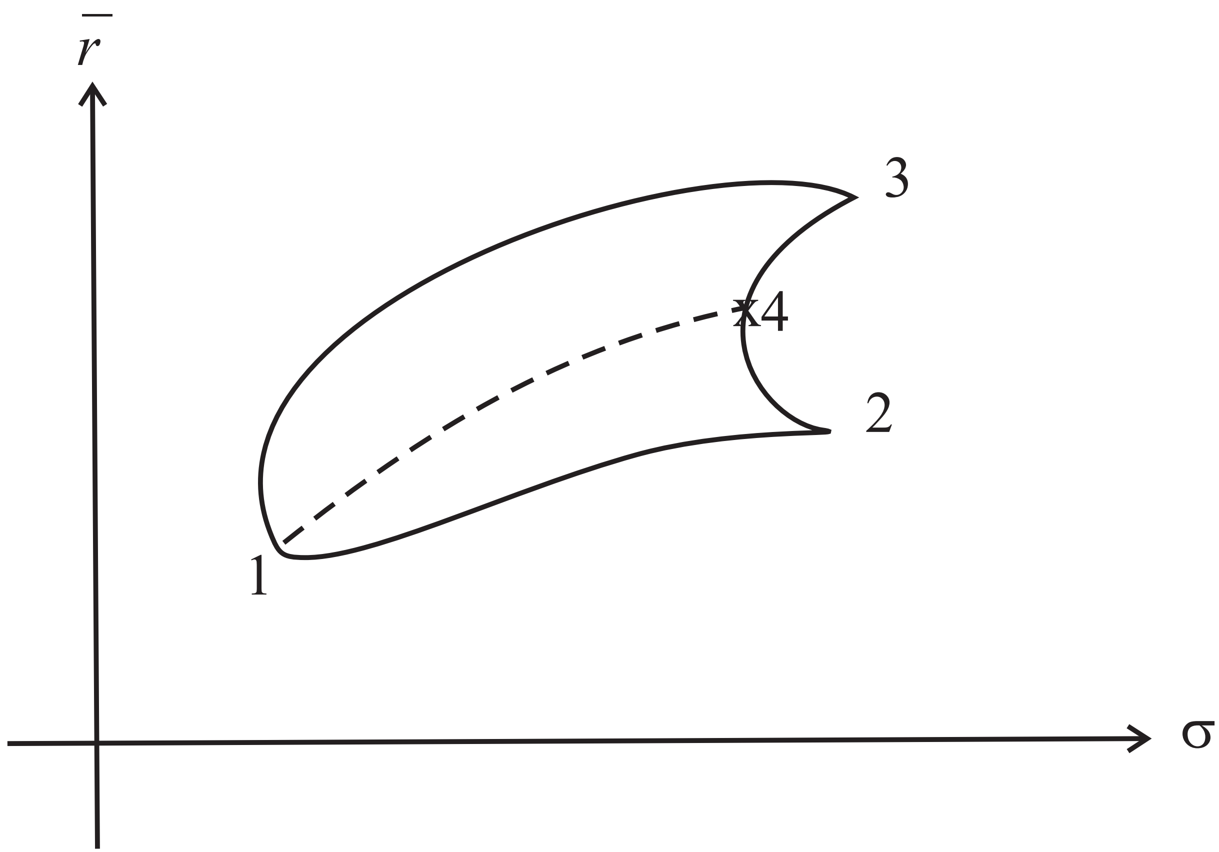
Since $\Omega^{-1}\mathbf{1} = a\boldsymbol{w}_g$ and $\Omega^{-1}\boldsymbol{\mu} = b\boldsymbol{w}_d$, the weight of any frontier fund (minimum variance fund) can be represented by

$$\boldsymbol{w}^* = (\lambda_1 a)\boldsymbol{w}_g + (\lambda_2 b)\boldsymbol{w}_d = \frac{c - b\mu_P}{\Delta}a\boldsymbol{w}_g + \frac{a\mu_P - b}{\Delta}b\boldsymbol{w}_d.$$

This is a consequence of the Two-Fund Theorem.

Feasible set

Given N risky assets, we can form various portfolios from these N assets. We plot the point $(\sigma_P, \overline{r}_P)$ that represents a particular portfolio in the $\sigma - \overline{r}$ diagram. The collection of these points constitutes the feasible set or feasible region.



Argument to show that the collection of the points representing $(\sigma_P, \overline{r}_P)$ *of a 3-asset portfolio generates a solid region in the* σ-$\overline{r}$ *plane*

Consider a 3-asset portfolio, the various combinations of assets 2 and 3 sweep out a curve between them (the particular curve taken depends on the correlation coefficient ρ_{23}).

A combination of assets 2 and 3 (labelled 4) can be combined with asset 1 to form a curve joining 1 and 4. As 4 moves between 2 and 3, the family of curves joining 1 and 4 sweep out a solid region.

Properties of the feasible regions

1. For a portfolio with at least 3 risky assets (not perfectly correlated and with different means), the feasible set is a solid two-dimensional region.

2. The feasible region is *convex to the left*. That is, given any two points in the region, the straight line connecting them does not cross the left boundary of the feasible region. This property must be observed since any combination of two portfolios also lies in the feasible region. Indeed, the left boundary of a feasible region is a hyperbola.

Locate the efficient and inefficient investment strategies

- Since investors prefer the lowest variance for the same expected return, they will focus on the set of portfolios with the smallest variance for a given mean, or the mean-variance frontier.

- The mean-variance frontier can be divided into two parts: an efficient frontier and an inefficient frontier.

- The efficient part includes the portfolios with the highest mean for a given variance.

- To find the efficient frontier, we must solve a quadratic programming problem.

Minimum variance set and efficient funds

The left boundary of a feasible region is called the *minimum variance set*. The most left point on the minimum variance set is called the *global minimum variance point*. The portfolios in the minimum variance set are called the *frontier funds*.

For a given level of risk, only those portfolios on the *upper half* of the efficient frontier with a higher return are desired by investors. They are called the *efficient funds*.

A portfolio $\boldsymbol{w}^*$ is said to be mean-variance efficient if there exists no portfolio $\boldsymbol{w}$ with $\mu_P \geq \mu_P^*$ and $\sigma_P^2 \leq {\sigma_P^*}^2$, except itself. That is, you cannot find a portfolio that has a higher return and lower risk than those of an efficient portfolio.

Example

Suppose there are three uncorrelated assets. Each has variance 1, and the mean values are $1, 2$ and 3, respectively.

We have $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$ and $\sigma_{12} = \sigma_{23} = \sigma_{13} = 0$. From the first order conditions, the governing equations are

$$\begin{aligned} w_1 - \lambda_2 - \lambda_1 &= 0 \\ w_2 - 2\lambda_2 - \lambda_1 &= 0 \\ w_3 - 3\lambda_2 - \lambda_1 &= 0 \\ w_1 + 2w_2 + 3w_3 &= \mu_P \\ w_1 + w_2 + w_3 &= 1. \end{aligned}$$

This leads to

$$\begin{aligned} 14\lambda_2 + 6\lambda_1 &= \mu_P \\ 6\lambda_2 + 3\lambda_1 &= 1. \end{aligned}$$

These two equations can be solved to yield $\lambda_2 = \frac{\mu_P}{2} - 1$ and $\lambda_1 = 2\frac{1}{3} - \mu_P$. Then

$$\begin{aligned} w_1 &= \frac{4}{3} - \frac{\mu_P}{2} \\ w_2 &= \frac{1}{3} \\ w_3 &= \frac{\mu_P}{2} - \frac{2}{3}. \end{aligned}$$

The standard deviation of r_P at the solution is $\sqrt{w_1^2 + w_2^2 + w_3^2}$, which by direct substitution gives

$$\sigma_P = \sqrt{\frac{7}{3} - 2\mu_P + \frac{\mu_P^2}{2}}.$$

The minimum-variance point is, by symmetry, at $\mu_P = 2$, with $\sigma_P = \sqrt{3}/3 = 0.58$.

Short sales not allowed

The problem cannot be reduced to a system of equations. The general solution is as follows:

$1 \leq \mu_P \leq \frac{4}{3}$	$\frac{4}{3} \leq \mu_P \leq \frac{8}{3}$	$\frac{8}{3} \leq v \leq 3$
$w_1 = 2 - \mu_P$	$\frac{4}{3} - \frac{\mu_P}{2}$	0
$w_2 = \mu_P - 1$	$\frac{1}{3}$	$3 - \mu_P$
$w_3 = 0$	$\frac{\mu_P}{2} - \frac{2}{3}$	$\mu_P - 2$
$\sigma_P = \sqrt{2\mu_P^2 - 6\mu_P + 5}$	$\sqrt{\frac{2}{3} - 2\mu_P + \frac{\mu_P^2}{2}}$	$\sqrt{2\mu_P^2 - 10\mu_P + 13}$.

- Under the constraint: $w_i \geq 0, i = 1, 2, 3, \mu_P$ can only lie between $1 \leq \mu_P \leq 3$ [recall $\mu_P = w_1 + 2w_2 + 3w_3$].

- Suppose $1 \leq \mu_P \leq \dfrac{4}{3}, w_3$ becomes negative in the minimum variance portfolio when short sales are allowed. When short sales are not allowed, we expect to have "$w_3 = 0$" in the minimum variance portfolio.

1.3 Two-fund Theorem

Take any two frontier funds (portfolios), then any frontier portfolio can be duplicated, in terms of mean and variance, as a combination of these two frontier funds. In other words, all investors seeking frontier portfolios need only invest in various combinations of these two funds.

Remark

Any convex combination (that is, weights are non-negative) of efficient portfolios is an efficient portfolio. Let $w_i \geq 0$ be the weight of Fund i whose rate of return is r_f^i. Recall that $\frac{b}{a}$ is the expected rate of return of the global minimum variance portfolio. Since $E\left[r_f^i\right] \geq \frac{b}{a}$ for all i as all funds are efficient, we have

$$\sum_{i=1}^{n} w_i E\left[r_f^i\right] \geq \sum_{i=1}^{n} w_i \frac{b}{a} = \frac{b}{a}.$$

Proof

Let $\boldsymbol{w}^1 = (w_1^1 \cdots w_n^1), \lambda_1^1, \lambda_2^1$ and $\boldsymbol{w}^2 = (w_1^2 \cdots w_n^2)^T, \lambda_1^2, \lambda_2^2$ be two known solutions to the minimum variance formulation with expected rates of return μ_P^1 and μ_P^2, respectively. Both solutions satisfy

$$\sum_{j=1}^{n} \sigma_{ij} w_j - \lambda_1 - \lambda_2 \overline{r}_i = 0, \quad i = 1, 2, \cdots, n \tag{1}$$

$$\sum_{i=1}^{n} w_i \overline{r}_i = \mu_P \tag{2}$$

$$\sum_{i=1}^{n} w_i = 1. \tag{3}$$

We would like to show that $\alpha \boldsymbol{w}_1 + (1-\alpha)\boldsymbol{w}_2$ is a solution corresponds to the expected rate of return $\alpha \mu_P^1 + (1-\alpha)\mu_P^2$.

1. The new weight vector $\alpha \boldsymbol{w}^1 + (1-\alpha)\boldsymbol{w}^2$ is a legitimate portfolio with weights that sum to one.

2. To check the condition on the expected rate of return, we note that

$$\begin{aligned} & \sum_{i=1}^{n} \left[\alpha w_i^1 + (1-\alpha) w_i^2\right] \overline{r}_i \\ = & \ \alpha \sum_{i=1}^{n} w_i^1 \overline{r}_i + (1-\alpha) \sum_{i=1}^{n} w_i^2 \overline{r}_i \\ = & \ \alpha \mu_P^1 + (1-\alpha)\mu_P^2. \end{aligned}$$

3. Eq. (1) is satisfied by $\alpha \boldsymbol{w}^1 + (1-\alpha)\boldsymbol{w}^2$ since the system of equations is linear with $\mu_P = \alpha \mu_P^1 + (1-\alpha)\mu_P^2$.

For convenience, we choose the two frontier funds to be $\boldsymbol{w}_g$ and $\boldsymbol{w}_d$. To obtain the optimal weight $\boldsymbol{w}^*$ for a given μ_P, we solve for α using $\alpha\mu_g + (1-\alpha)\mu_d = \mu_P$ and $\boldsymbol{w}^*$ is then given by $\alpha\boldsymbol{w}_g^* + (1-\alpha)\boldsymbol{w}_d^*$. Recall $\mu_g = b/a$ and $\mu_d = c/b$, so $\alpha = \dfrac{(c - b\mu_P)a}{\Delta}$.

Proposition

Any minimum variance portfolio with the target mean μ_P can be uniquely decomposed into the sum of two portfolios

$$\boldsymbol{w}_P^* = \alpha\boldsymbol{w}_g + (1-\alpha)\boldsymbol{w}_d$$

where $\alpha = \dfrac{c - b\mu_P}{\Delta}a$.

Indeed, any two minimum-variance portfolios $\boldsymbol{w}_u$ and $\boldsymbol{w}_v$ can be used to substitute for $\boldsymbol{w}_g$ and $\boldsymbol{w}_d$. Suppose

$$\begin{aligned}\boldsymbol{w}_u &= (1-u)\boldsymbol{w}_g + u\boldsymbol{w}_d \\ \boldsymbol{w}_v &= (1-v)\boldsymbol{w}_g + v\boldsymbol{w}_d\end{aligned}$$

we then solve for $\boldsymbol{w}_g$ and $\boldsymbol{w}_d$ in terms of $\boldsymbol{w}_u$ and $\boldsymbol{w}_v$. Recall

$$\boldsymbol{w}_P^* = \lambda_1\Omega^{-1}\mathbf{1} + \lambda_2\Omega^{-1}\boldsymbol{\mu}$$

so that

$$\begin{aligned}\boldsymbol{w}_P^* &= \lambda_1 a\boldsymbol{w}_g + (1-\lambda_1 a)\boldsymbol{w}_d \\ &= \frac{\lambda_1 a + v - 1}{v-u}\boldsymbol{w}_u + \frac{1-u-\lambda_1 a}{v-u}\boldsymbol{w}_v,\end{aligned}$$

whose sum of coefficients remains to be 1 and $\lambda_1 = \dfrac{c-b\mu_P}{\Delta}$.

Example

Mean, variances, and covariances of the rates of return of 5 risky assets are listed:

Security	covariance, σ_{ij}					mean, $\overline{r}_i$
1	2.30	0.93	0.62	0.74	−0.23	15.1
2	0.93	1.40	0.22	0.56	0.26	12.5
3	0.62	0.22	1.80	0.78	−0.27	14.7
4	0.74	0.56	0.78	3.40	−0.56	9.02
5	−0.23	0.26	−0.27	−0.56	2.60	17.68

Recall that $\boldsymbol{w}^*$ has the following closed form solution

$$\begin{aligned}\boldsymbol{w}^* &= \frac{c - b\mu_P}{\Delta}\Omega^{-1}\mathbf{1} + \frac{a\mu_P - b}{\Delta}\Omega^{-1}\boldsymbol{\mu} \\ &= \alpha\boldsymbol{w}_g + (1-\alpha)\boldsymbol{w}_d,\end{aligned}$$

where $\alpha = (c - b\mu_P)\dfrac{a}{\Delta}$.

We compute $\boldsymbol{w}_g^*$ and $\boldsymbol{w}_d^*$ through finding $\Omega^{-1}\mathbf{1}$ and $\Omega^{-1}\boldsymbol{\mu}$, then normalize by enforcing the condition that their weights are summed to one.

1. To find $\boldsymbol{v}^1 = \Omega^{-1}\mathbf{1}$, we solve the system of equations
$$\sum_{j=1}^{5} \sigma_{ij} v_j^1 = 1, \quad i = 1, 2, \cdots, 5.$$
Normalize the component v_i^1's so that they sum to one
$$w_i^1 = \frac{v_i^1}{\sum_{j=1}^{5} v_j^1}.$$
After normalization, this gives the solution to $\boldsymbol{w}_g$. Why?

We first solve for $\boldsymbol{v}^1 = \Omega^{-1}\mathbf{1}$ and later divide $\boldsymbol{v}^1$ by some constant k such that $\mathbf{1}^T\boldsymbol{v}^1/k = 1$. We see that k must be equal to a, where $a = \mathbf{1}^T\Omega^{-1}\mathbf{1}$. Actually, $a = \sum_{j=1}^{N} v_j^1$.

2. To find $\boldsymbol{v}^2 = \Omega^{-1}\boldsymbol{\mu}$, we solve the system of equations:

$$\sum_{j=1}^{5} \sigma_{ij} v_j^2 = \overline{r}_i, \quad i = 1, 2, \cdots, 5.$$

Normalize v_i^2's to obtain w_i^2. After normalization, this gives the solution to $\boldsymbol{w}_d$. Also, $b = \sum_{j=1}^{N} v_j^2$ and $c = \boldsymbol{\mu}^T\Omega^{-1}\boldsymbol{\mu} = \sum_{j=1}^{N} \overline{r}_j v_j^2$.

security	$\boldsymbol{v}^1$	$\boldsymbol{v}^2$	$\boldsymbol{w}_g$	$\boldsymbol{w}_d$
1	0.141	3.652	0.088	0.158
2	0.401	3.583	0.251	0.155
3	0.452	7.284	0.282	0.314
4	0.166	0.874	0.104	0.038
5	0.440	7.706	0.275	0.334
mean			14.413	15.202
variance			0.625	0.659
standard deviation			0.791	0.812

Recall $\boldsymbol{v}^1 = \Omega^{-1}\mathbf{1}$ and $\boldsymbol{v}^2 = \Omega^{-1}\boldsymbol{\mu}$ so that

- sum of components in $\boldsymbol{v}^1 = \mathbf{1}^T\Omega^{-1}\mathbf{1} = a$
- sum of components in $\boldsymbol{v}^2 = \mathbf{1}^T\Omega^{-1}\boldsymbol{\mu} = b$.

Note that $\boldsymbol{w}_g = \boldsymbol{v}^1/a$ and $\boldsymbol{w}_d = \boldsymbol{v}^2/b$.

Relation between $\boldsymbol{w}_g$ and $\boldsymbol{w}_d$

Both $\boldsymbol{w}_g$ and $\boldsymbol{w}_d$ are frontier funds with

$$\mu_g = \frac{\boldsymbol{\mu}^T \Omega^{-1} \mathbf{1}}{a} = \frac{b}{a} \quad \text{and} \quad \mu_d = \frac{\boldsymbol{\mu}^T \Omega^{-1} \boldsymbol{\mu}}{b} = \frac{c}{b}.$$

Difference in expected returns $= \mu_d - \mu_g = \dfrac{c}{b} - \dfrac{b}{a} = \dfrac{\Delta}{ab} > 0.$

Also, difference in variances $= \sigma_d^2 - \sigma_g^2 = \dfrac{c}{b^2} - \dfrac{1}{a} = \dfrac{\Delta}{ab^2} > 0.$

Since $\mu_d > \mu_g$ and $\sigma_d^2 > \sigma_g^2$, $\boldsymbol{w}_d$ is an efficient portfolio that lies on the upper half of the efficient frontier.

How about the covariance of the portfolio returns for any two minimum variance portfolios? Write

$$r_P^u = \boldsymbol{w}_u^T \boldsymbol{r} \text{ and } r_P^v = \boldsymbol{w}_v^T \boldsymbol{r}$$

where $\boldsymbol{r} = (r_1 \cdots r_N)^T$. Recall that for the two special efficient funds, $\boldsymbol{w}_g$ and $\boldsymbol{w}_d$, their covariance is given by

$$\begin{aligned}
\sigma_{gd} &= \text{cov}(r_P^g, r_P^d) = \text{cov}\left(\sum_{i=1}^N w_i^g r_i, \sum_{j=1}^N w_j^d r_j\right) \\
&= \sum_{i=1}^N \sum_{j=1}^N w_i^g w_j^d \text{cov}(r_i, r_j) \\
&= \boldsymbol{w}_g^T \Omega \boldsymbol{w}_d = \left(\frac{\Omega^{-1}\mathbf{1}}{a}\right)^T \Omega \left(\frac{\Omega^{-1}\boldsymbol{\mu}}{b}\right) \\
&= \frac{\mathbf{1}^T \Omega^{-1}\boldsymbol{\mu}}{ab} = \frac{1}{a} \quad \text{since} \quad b = \mathbf{1}^T \Omega^{-1}\boldsymbol{\mu}
\end{aligned}$$

and

$$(\Omega^{-1}\mathbf{1})^T = \mathbf{1}^T(\Omega^{-1})^T = \mathbf{1}^T(\Omega^T)^{-1} = \mathbf{1}^T\Omega^{-1}.$$

In general, consider two portfolios parametrized by u and v:

$$\boldsymbol{w}_u = (1-u)\boldsymbol{w}_g + u\boldsymbol{w}_d \text{ and } \boldsymbol{w}_v = (1-v)\boldsymbol{w}_g + v\boldsymbol{w}_d$$

so that

$$\begin{aligned}
\text{cov}(r_P^u, r_P^v) &= (1-u)(1-v)\sigma_g^2 + uv\sigma_d^2 + [u(1-v) + v(1-u)]\sigma_{gd} \\
&= \frac{(1-u)(1-v)}{a} + \frac{uvc}{b^2} + \frac{u+v-2uv}{a} \\
&= \frac{1}{a} + \frac{uv\Delta}{ab^2}.
\end{aligned}$$

For any portfolio $\boldsymbol{w}_P$,

$$\text{cov}(r_g, r_P) = \boldsymbol{w}_g^T \Omega \boldsymbol{w}_P = \frac{\mathbf{1}^T \Omega^{-1} \Omega \boldsymbol{w}_P}{a} = \frac{1}{a} = \text{var}(r_g).$$

For any Portfolio u, we can find another Portfolio v such that these two portfolios are uncorrelated. This can be done by setting

$$\frac{1}{a} + \frac{uv\Delta}{ab^2} = 0,$$

and solve for v. Portfolio v is the uncorrelated counterpart of Portfolio u.

1.4 Inclusion of the risk free asset: One-fund Theorem

Consider a portfolio with weight α for the risk free asset and $1 - \alpha$ for a risky asset. The risk free asset has the deterministic rate of return r_f. The mean of the expected rate of portfolio return is

$$\overline{r}_P = \alpha\overline{r}_f + (1-\alpha)\overline{r}_j \quad \text{(note that } r_f = \overline{r}_f\text{)}.$$

The covariance σ_{fj} between the risk free asset and any risky asset is zero since

$$E[(r_j - \overline{r}_j)\underbrace{(r_f - \overline{r}_f)}_{\text{zero}}] = 0.$$

Therefore, the variance of portfolio return σ_P^2 is

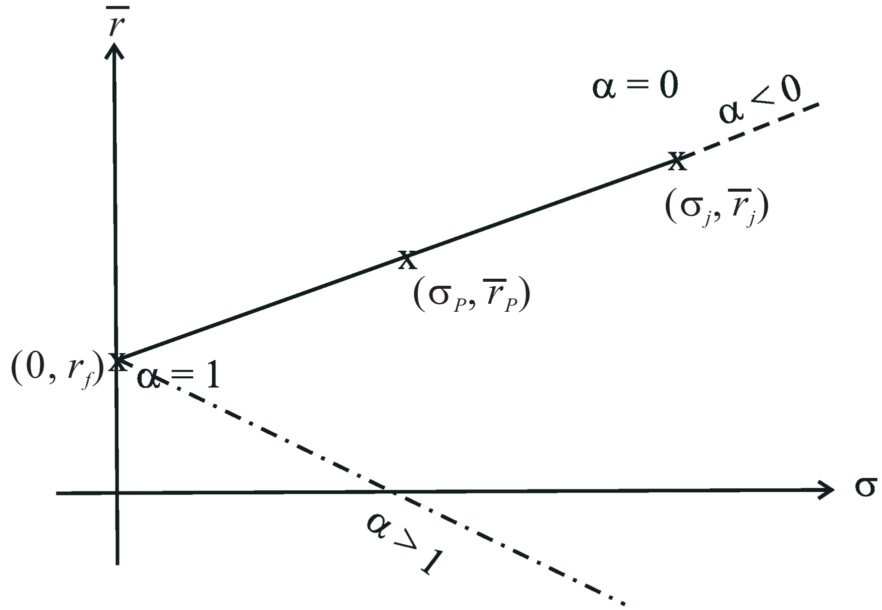
$$\sigma_P^2 = \alpha^2 \underbrace{\sigma_f^2}_{\text{zero}} + (1-\alpha)^2\sigma_j^2 + 2\alpha(1-\alpha)\underbrace{\sigma_{fj}}_{\text{zero}}$$

so that

$$\sigma_P = |1-\alpha|\sigma_j.$$

Since both r_P and σ_P are linear functions of α, so $(\sigma_P, \overline{r}_P)$ lies on a pair line segments in the σ-$\overline{r}$ diagram.

1. For $0 < \alpha < 1$, the points representing $(\sigma_P, \overline{r}_P)$ for varying values of α lie on the straight line segment joining $(0, r_f)$ and $(\sigma_j, \overline{r}_j)$.



2. If borrowing of the risk free asset is allowed, then α can be negative. In this case, the line extends beyond the right side of $(\sigma_j, \overline{r}_j)$ (possibly up to infinity).

3. When $\alpha > 1$, this corresponds to short selling of the risky asset. In this case, the portfolios are represented by a line with slope negative to that of the line segment joining $(0, r_f)$ and $(\sigma_j, \overline{r}_j)$ (see the lower dotted-dashed line).

- This can be seen as the mirror image with respect to the vertical $\overline{r}$-axis of the line segment that extends beyond the left side of $(0, r_f)$. This is due to the swapping in sign in $|1 - \alpha|\sigma_j$ when $\alpha > 1$.

- The holder bears the same risk, like long holding of the risky asset, while μ_P falls below r_f.

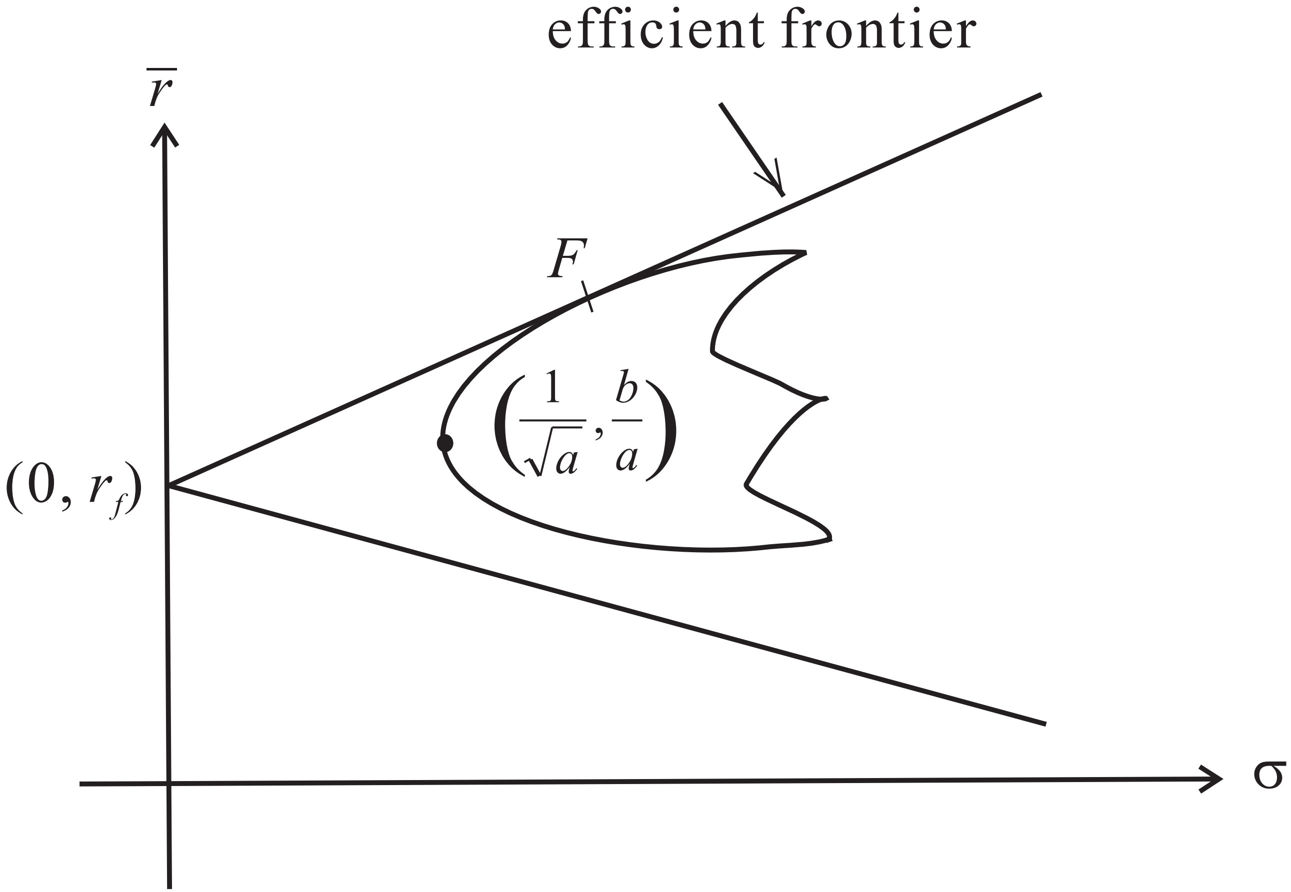
Consider a portfolio that starts with N risky assets originally, what is the impact of the inclusion of a risk free asset on the feasible region?

Lending and borrowing of the risk free asset is allowed

For each portfolio formed using the N risky assets, the new combinations with the inclusion of the risk free asset trace out the infinite straight line originating from the risk free point and passing through the point representing the original portfolio.

The totality of these lines forms an infinite triangular feasible region bounded by the two tangent lines through the risk free point to the original feasible region.

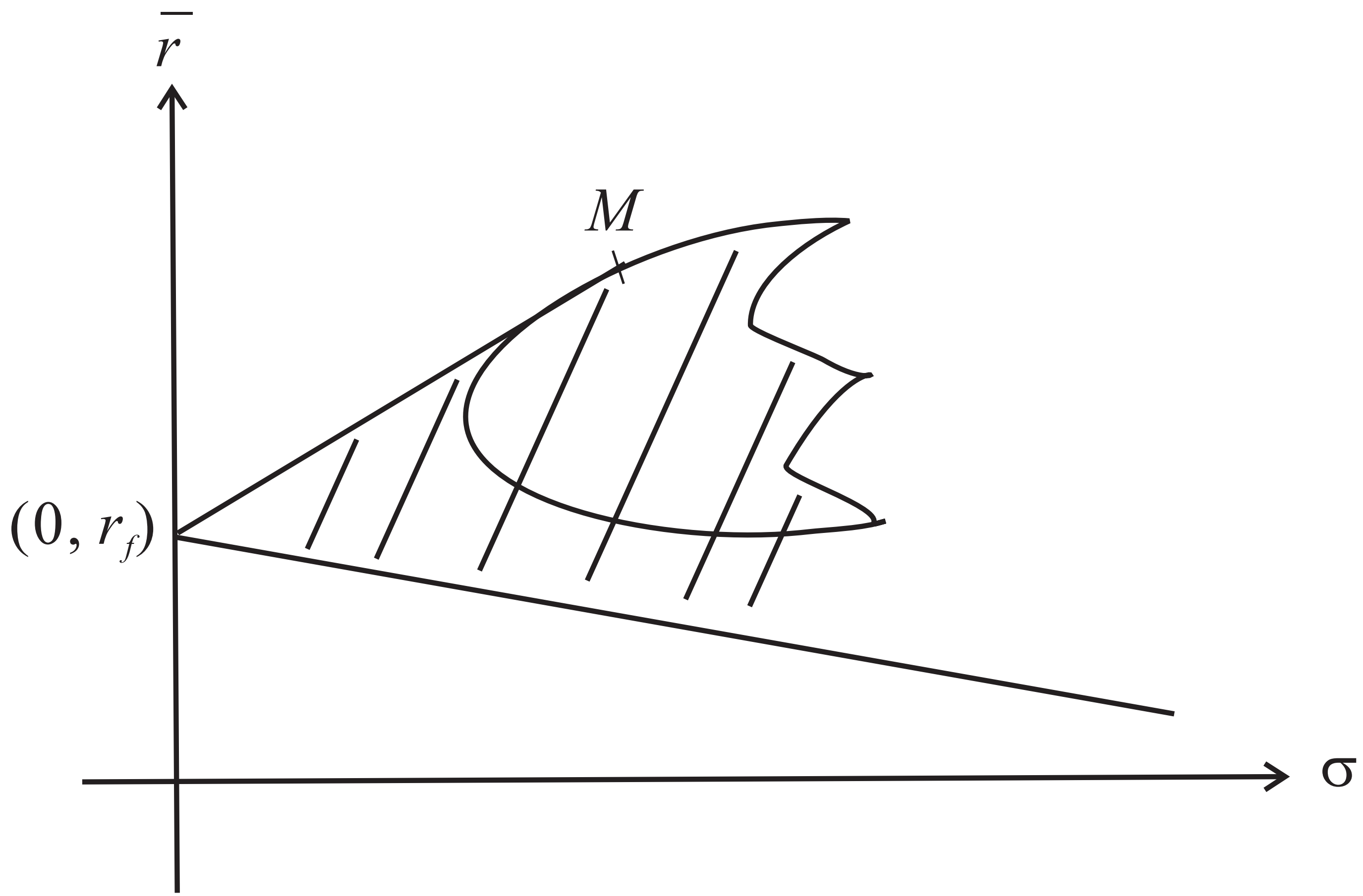
The new *efficient set* is the single straight line on the top of the new triangular feasible region. This tangent line touches the original feasible region at a point F, where F lies on the efficient frontier of the original feasible set.



This case corresponds to $r_f < \dfrac{b}{a}$, where the upper line of the symmetric double line pair touches the original feasible region.

No shorting of the risk free asset ($r_f < \mu_g$)

The line originating from the risk free point cannot be extended beyond the points in the original feasible region (otherwise entails borrowing of the risk free asset). The upper half line is extended up to the tangency point only while the lower half line can be extended to infinity.



One-fund Theorem

Any efficient portfolio (represented by a point on the upper tangent line) can be expressed as a combination of the risk free asset and the portfolio (or fund) represented by M.

"There is a single fund M of risky assets such that any efficient portfolio can be constructed as a combination of the fund M and the risk free asset."

The One-fund Theorem is based on the assumptions that

- every investor is a mean-variance optimizer
- they all agree on the probabilistic structure of asset returns
- a unique risk free asset exists.

Then everyone purchases a single fund, which is then called the *market portfolio*.

The proportion of wealth invested in the risk free asset is $1 - \sum_{i=1}^{N} w_i$.
Write r as the constant rate of return of the risk free asset.

Modified Lagrangian formulation

$$\text{minimize} \qquad \frac{\sigma_P^2}{2} = \frac{1}{2}\boldsymbol{w}^T \Omega \boldsymbol{w}$$

$$\text{subject to} \qquad \boldsymbol{\mu}^T \boldsymbol{w} + (1 - \mathbf{1}^T \boldsymbol{w}) r = \mu_P.$$

Define the Lagrangian: $L = \frac{1}{2}\boldsymbol{w}^T \Omega \boldsymbol{w} + \lambda[\mu_P - r - (\boldsymbol{\mu} - r\mathbf{1})^T \boldsymbol{w}]$

$$\frac{\partial L}{\partial w_i} = \sum_{j=1}^{N} \sigma_{ij} w_j - \lambda(\mu_i - r) = 0, \quad i = 1, 2, \cdots, N \tag{1}$$

$$\frac{\partial L}{\partial \lambda} = 0 \quad \text{giving} \quad (\boldsymbol{\mu} - r\mathbf{1})^T \boldsymbol{w} = \mu_P - r. \tag{2}$$

$(\boldsymbol{\mu} - r\mathbf{1})^T \boldsymbol{w}$ is interpreted as the weighted sum of the expected excess rate of return above the risk free rate r.

Solving (1): $\boldsymbol{w}^* = \lambda\Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1})$. Substituting into (2)

$$\mu_P - r = \lambda(\boldsymbol{\mu} - r\mathbf{1})^T\Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1}) = \lambda(c - 2br + ar^2).$$

By eliminating λ, the relation between μ_P and σ_P is given by the following pair of half lines ending at the risk free asset point $(0, r)$

$$\begin{aligned}\sigma_P^2 &= {\boldsymbol{w}^*}^T\Omega\boldsymbol{w}^* = \lambda({\boldsymbol{w}^*}^T\boldsymbol{\mu} - r{\boldsymbol{w}^*}^T\mathbf{1}) \\ &= \lambda(\mu_P - r) = (\mu_P - r)^2/(c - 2br + ar^2).\end{aligned}$$

Here, $1/\lambda = \dfrac{\mu_P - r}{\sigma_P^2}$ may be interpreted as the ratio of excess expected portfolio return above the riskless interest rate to the variance of portfolio return.

What is the relationship between this pair of half lines and the frontier boundary of the feasible region of the risky assets plus the risk free asset?

With the inclusion of the risk free asset, the set of minimum variance portfolios are represented by portfolios on the two half lines

$$L_{up} : \mu_P - r = \sigma_P \sqrt{ar^2 - 2br + c} \tag{3a}$$

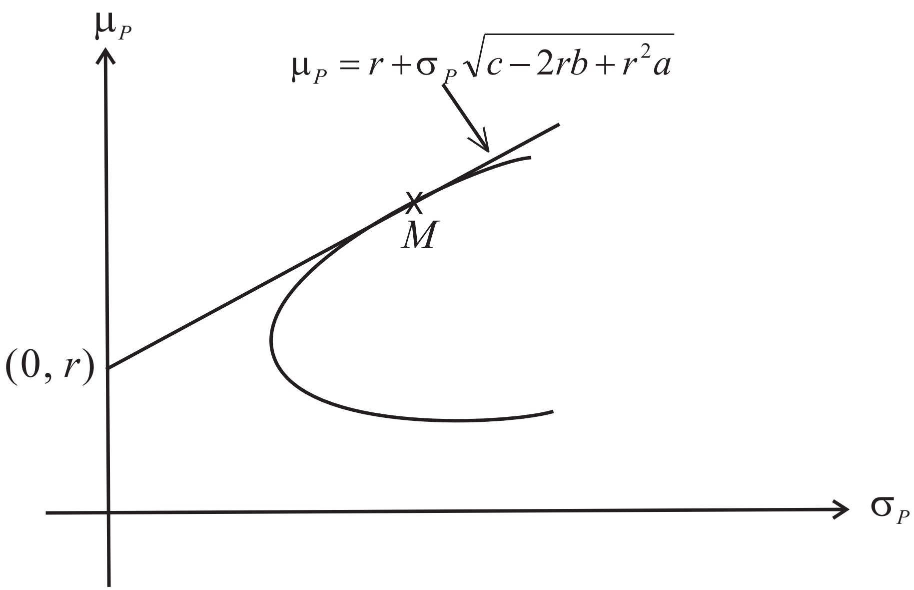
$$L_{low} : \mu_P - r = -\sigma_P \sqrt{ar^2 - 2br + c}. \tag{3b}$$

Recall that $ar^2 - 2br + c > 0$ for all values of r since $\Delta = ac - b^2 > 0$.

The minimum variance portfolios without the risk free asset lie on the hyperbola

$$\sigma_P^2 = \frac{a\mu_P^2 - 2b\mu_P + c}{\Delta}.$$

When $r < \mu_g = \dfrac{b}{a}$, the upper half line is a tangent to the hyperbola. The tangency portfolio is the tangent point to the efficient frontier (upper part of the hyperbolic curve) through the point $(0, r)$.



Solution of the tangency portfolio (provided $r < \frac{b}{a}$)

The tangency portfolio M is represented by the point $(\sigma_{P,M}, \mu_P^M)$, and the solution to $\sigma_{P,M}$ and μ_P^M are obtained by solving simultaneously

$$\begin{aligned} \sigma_P^2 &= \frac{a\mu_P^2 - 2b\mu_P + c}{\Delta} \\ \mu_P &= r + \sigma_P\sqrt{ar^2 - 2br + c}. \end{aligned}$$

We obtain

$$\mu_P^M = \frac{c - br}{b - ar} \quad \text{and} \quad \sigma_{P,M}^2 = \frac{ar^2 - 2br + c}{(b - ar)^2}.$$

Once μ_P^M is obtained, the corresponding values for λ_M and $\boldsymbol{w}_M^*$ are

$$\begin{aligned} \lambda_M &= \frac{\mu_P^M - r}{c - 2rb + r^2 a} = \frac{1}{b - ar} \quad \text{and} \\ \boldsymbol{w}_M^* &= \lambda_M \Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1}) = \frac{\Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1})}{b - ar}. \end{aligned}$$

Properties on the tangency portfolio

Recall $\mu_g = \dfrac{b}{a}$. When $r < \dfrac{b}{a}$, it can be shown that $\mu_P^M > \mu_g$. To prove the claim, we observe

$$\begin{aligned}\left(\mu_P^M - \frac{b}{a}\right)\left(\frac{b}{a} - r\right) &= \left(\frac{c-br}{b-ar} - \frac{b}{a}\right)\frac{b-ar}{a} \\ &= \frac{c-br}{a} - \frac{b^2}{a^2} + \frac{br}{a} \\ &= \frac{ac-b^2}{a^2} = \frac{\Delta}{a^2} > 0,\end{aligned}$$

so we deduce that $\mu_P^M > \dfrac{b}{a} > r$.

Also, we can deduce that $\sigma_{P,M} > \sigma_g$ as expected. Why?

Both Portfolio M and Portfolio g are portfolios generated by the universe of risky assets (with no inclusion of the risk free asset), and g is the global minimum variance portfolio.

Properties on the minimum variance portfolios for $r < b/a$

1. *Efficient portfolios*

   Any portfolio on the upper half line

   $$\mu_P = r + \sigma_P \sqrt{ar^2 - 2br + c}$$

   within the segment FM joining the two points $F(0, r)$ and M involves long holding of the market portfolio M and the risk free asset F, while those outside FM involves short selling of the risk free asset and long holding of the market portfolio.

2. Any portfolio on the lower half line

   $$\mu_P = r - \sigma_P \sqrt{ar^2 - 2br + c}$$

   involves short selling of the market portfolio and investing the proceeds in the risk free asset. This represents a non-optimal investment strategy since the investor faces risk but gains no extra expected return above r.

Degenerate case where $\mu_g = \dfrac{b}{a} = r$

- What happens when $r = b/a$? The half lines become

$$\mu_P = r \pm \sigma_P \sqrt{c - 2\left(\frac{b}{a}\right) b + \frac{b^2}{a}} = r \pm \sigma_P \sqrt{\frac{\Delta}{a}},$$

which correspond to the asymptotes of the hyperbolic left boundary of the feasible region with risky assets only.

- Under the scenario: $r = \dfrac{b}{a}$, efficient funds still lie on the upper half line, though the tangency portfolio does not exist. Recall that

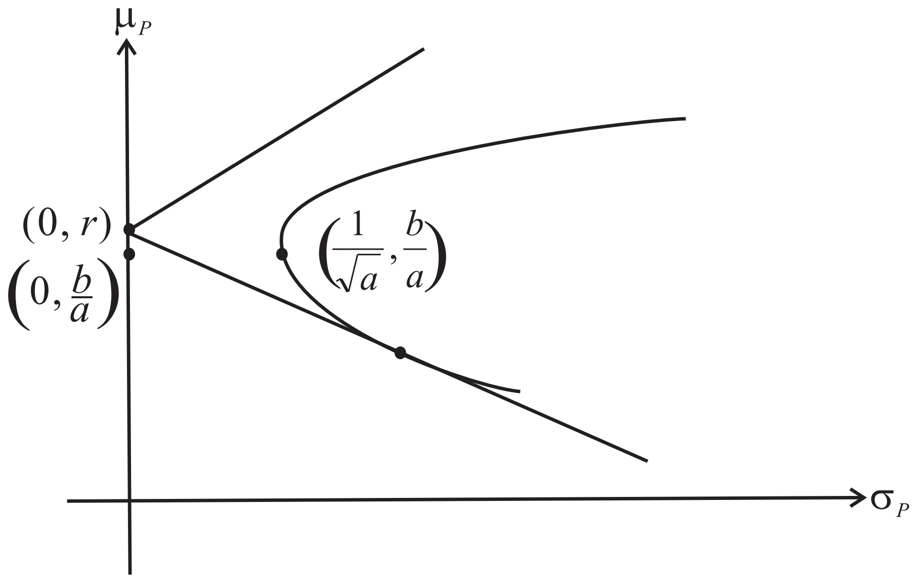
$$\boldsymbol{w}^* = \lambda \Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1}) \text{ so that}$$
$$\mathbf{1}^T \boldsymbol{w} = \lambda(\mathbf{1}^T \Omega^{-1} \boldsymbol{\mu} - r \mathbf{1}^T \Omega^{-1} \mathbf{1}) = \lambda(b - ra).$$

- When $r = b/a, \mathbf{1}^T \boldsymbol{w} = 0$ as λ is finite. Any minimum variance portfolio involves investing everything in the risk free asset and holding a portfolio of risky assets whose weights are summed to zero.

- The optimal weight vector $\boldsymbol{w}^*$ equals $\lambda\Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1})$ and the multiplier λ is determined by

$$\lambda = \left.\frac{\mu_P - r}{c - 2br + ar^2}\right|_{r=b/a} = \frac{\mu_P - r}{c - 2\left(\frac{b}{a}\right)b + \frac{b^2}{a}} = \frac{a(\mu_P - r)}{\Delta}.$$

In reality, we expect $r < \mu_g = \dfrac{b}{a}$. What happen when $r > \dfrac{b}{a}$? The lower half line touches the feasible region with risky assets only.



- Any portfolio on the upper half line involves short selling of the tangency portfolio and investing the proceeds in the risk free asset.

- It makes sense to short sell the tangency portfolio since it has an expected rate of return lower than the risk free asset.

Example (5 risky assets and one risk free asset)

Data of the 5 risky assets are given in the earlier example, and $r = 10\%$.

The system of linear equations to be solved is

$$\sum_{j=1}^{5} \sigma_{ij} v_j = \overline{r}_i - r = 1 \times \overline{r}_i - r \times 1, \quad i = 1, 2, \cdots, 5.$$

Recall that $\boldsymbol{v}^1$ and $\boldsymbol{v}^2$ in the earlier example are solutions to

$$\sum_{j=1}^{5} \sigma_{ij} v_j^1 = 1 \quad \text{and} \quad \sum_{j=1}^{5} \sigma_{ij} v_j^2 = \overline{r}_i, \text{ respectively.}$$

Hence, $v_j = v_j^2 - r v_j^1, j = 1, 2, \cdots, 5$ (numerically, we take $r = 10\%$).

Now, we have obtained $\boldsymbol{v}$ where

$$\boldsymbol{v} = \Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1}).$$

Note that the optimal weight vector for the 5 risky assets satisfies

$$\boldsymbol{w} = \lambda \boldsymbol{v} \quad \text{for some scalar } \lambda.$$

We determine λ by enforcing

$$\lambda(\boldsymbol{\mu} - r\mathbf{1})^T \boldsymbol{v} = \mu_P - r,$$

where μ_P is the target rate of return of the portfolio.

The weight of the risk free asset is $1 - \sum_{j=1}^{5} w_j$.

Interpretation of the tangency portfolio (market portfolio)

- The One-fund Theorem states that everyone purchases a single fund of risky assets and borrow or lend at the risk free rate.

- If everyone purchases the same fund of risky assets, what must that fund be? This fund must equal the *market portfolio*.

- The market portfolio is the summation of all assets. If everyone buys just one fund, and their purchases add up to the market, then that fund must be the market as well.

- In the situation where everyone follows the mean-variance methodology with the same estimates of parameters, the efficient fund of risky assets will be the market portfolio.

How does this happen? The answer is based on the equilibrium argument.

- If everyone else (or at least a large number of people) solves the problem, we do not need to. The return on an asset depends on both its initial price and its final price. The other investors solve the mean-variance portfolio problem using their common estimates, and they place orders in the market to acquire their portfolios.

- If orders placed do not match with what is available, the prices must change. The prices of the assets under heavy demand will increase while the prices of the assets under light demand will decrease. These price changes affect the estimates of asset returns directly, and hence investors will recalculate their optimal portfolio.

- This process continues until demand exactly matches supply, that is, it continues until an equilibrium prevails.

Summary

- In the idealized world, where every investor is a mean-variance investor and all have the same estimates, everyone buys the same portfolio and that must be equal to the market portfolio.

- Prices adjust to drive the market to efficiency. Then after other people have made the adjustments, we can be sure that the efficient portfolio is the market portfolio.

1.5 Addition of a risk tolerance factor

Maximize $\tau\mu_P - \dfrac{\sigma_P^2}{2}$, with $\tau \geq 0$, where τ is the risk tolerance.

Optimization problem: $\displaystyle\max_{\boldsymbol{w}\in\mathbb{R}^N} \tau\mu_P - \frac{\sigma_P^2}{2}$ subject to $\mathbf{1}^T\boldsymbol{w} = 1$.

- Instead of only minimizing risk as in the mean variance models, the new objective function represents the tradeoff between return and risk with weighted factor 2τ. When τ is high, the investor is more interested in expected return and has a high tolerance on risk.

- The tolerance factor τ is chosen by the investor and will be fixed in the formulation. The choice variables are the portfolio weights $w_i, i = 1, 2, \cdots, N$.

- The parameter τ is closely related to the relative risk aversion coefficient. Given an initial wealth W_0 and under a portfolio choice $\boldsymbol{w}$, the end-of-period wealth is $W_0(1 + r_P)$.

Write $\mu_P = E[r_P]$ and $\sigma_P^2 = \text{var}(r_P)$, and let u denote the utility function.

Consider the Taylor expansion of the terminal utility value

$$u[W_0(1+r_P)] \approx u(W_0) + W_0 u'(W_0) r_P + \frac{W_0^2}{2} u''(W_0) r_P^2 + \cdots .$$

Neglecting the third and higher order moments and noting $E[r_P^2] = \sigma_P^2 + \mu_P^2$.

Next, considering the expected utility value of the terminal wealth

$$\begin{aligned} E[u(W_0(1+\mu_P))] &\approx u(W_0) + W_0 u'(W_0)\mu_P + \frac{W_0^2}{2} u''(W_0)(\sigma_P^2 + \mu_P^2) + \cdots \\ &= u(W_0) - W_0^2 u''(W_0) \left[-\frac{u'(W_0)}{W_0 u''(W_0)} \mu_P - \frac{\sigma_P^2 + \mu_P^2}{2} \right] \\ &\quad + \cdots \end{aligned}$$

Neglecting μ_P^2 compared to σ_P^2 and recalling $R_R = -\dfrac{W_0 u''(W_0)}{u'(W_0)}$ as the relative risk aversion coefficient, we obtain the objective function: $\dfrac{1}{R_R}\mu_P - \dfrac{\sigma_P^2}{2}$. Note that the deterministic multiplier $-\dfrac{W_0 u''(W_0)}{u'(W_0)}$ [assume positive value since $u'(W_0) > 0$ and $u''(W_0) < 0$] and the constant $u(W_0)$ are immaterial. Lower relative risk aversion means higher risk tolerance.

Note that the expected utility can be expressed solely in terms of mean μ_P and variance σ_P^2 when

(i) u is a quadratic function [$u'''(W_0)$ and higher order derivatives do not appear], or

(ii) r_P is normal (third and higher order moments become irrelevant statistics).

Quadratic optimization problem

$$\max_{\boldsymbol{w}\in\mathbb{R}^N}\left[\tau\boldsymbol{\mu}^T\boldsymbol{w}-\frac{\boldsymbol{w}^T\Omega\boldsymbol{w}}{2}\right] \text{ subject to } \boldsymbol{w}^T\mathbf{1}=1.$$

The Lagrangian formulation becomes:

$$L(\boldsymbol{w};\lambda)=\tau\boldsymbol{\mu}^T\boldsymbol{w}-\frac{\boldsymbol{w}^T\Omega\boldsymbol{w}}{2}+\lambda(\boldsymbol{w}^T\mathbf{1}-1).$$

The first order conditions are

$$\begin{cases}\tau\boldsymbol{\mu}-\Omega\boldsymbol{w}^*+\lambda\mathbf{1}=0\\ \boldsymbol{w}^{*T}\mathbf{1}=1\end{cases}.$$

When τ is taken to be zero, the problem reduces to the minimization of portfolio return variance without regard to expected portfolio return. This gives the global minimum variance portfolio $\boldsymbol{w}_g^*$.

Express the optimal solution $\boldsymbol{w}^*$ as $\boldsymbol{w}_g + \tau \boldsymbol{z}^*, \tau \geq 0$.

1. When $\tau = 0$, the two first order conditions become

$$\Omega \boldsymbol{w} = \lambda_0 \mathbf{1} \quad \text{and} \quad \mathbf{1}^T \boldsymbol{w}_g = 1.$$

   Solving

$$\boldsymbol{w}_g = \lambda_0 \Omega^{-1} \mathbf{1} \quad \text{and} \quad 1 = \mathbf{1}^T \boldsymbol{w}_g = \lambda_0 \mathbf{1}^T \Omega^{-1} \mathbf{1}$$

   hence

$$\boldsymbol{w}_g = \frac{\Omega^{-1}\mathbf{1}}{\mathbf{1}^T \Omega^{-1} \mathbf{1}} \text{(independent of } \boldsymbol{\mu}\text{)}.$$

   The formulation does not depend on $\boldsymbol{\mu}$ when τ is taken to be zero.

2. When $\tau \geq 0$, we obtain $\boldsymbol{w} = \tau\Omega^{-1}\boldsymbol{\mu} + \lambda\Omega^{-1}\mathbf{1}$. To determine λ, we apply

$$1 = \mathbf{1}^T w = \tau \mathbf{1}^T \Omega^{-1}\boldsymbol{\mu} + \lambda \mathbf{1}^T \Omega^{-1}\mathbf{1} \text{ so that } \lambda = \frac{1 - \tau\mathbf{1}\Omega^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\Omega^{-1}\mathbf{1}}.$$

$$\begin{aligned} \boldsymbol{w}^* &= \tau\Omega^{-1}\boldsymbol{\mu} + \frac{1 - \tau\mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\Omega^{-1}\mathbf{1}}\Omega^{-1}\mathbf{1} \\ &= \tau\left(\Omega^{-1}\boldsymbol{\mu} - \frac{\mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\Omega^{-1}\mathbf{1}}\Omega^{-1}\mathbf{1}\right) + \boldsymbol{w}_g. \end{aligned}$$

We obtain $\boldsymbol{w}^* = \boldsymbol{w}_g + \tau\boldsymbol{z}^*$, where

$$\boldsymbol{z}^* = \Omega^{-1}\boldsymbol{\mu} - \frac{\mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\Omega^{-1}\mathbf{1}}\Omega^{-1}\mathbf{1} \text{ and } \mathbf{1}^T\boldsymbol{z}^* = 0.$$

Observe that $\text{cov}(r_{\boldsymbol{w}_g}, r_{\boldsymbol{z}^*}) = \boldsymbol{z}^{*T}\Omega\boldsymbol{w}_g = 0$,

$$\mu_{\boldsymbol{z}^*} = \boldsymbol{\mu}^T\boldsymbol{z}^* = c - \frac{b^2}{a} = \frac{\Delta}{a} > 0 \quad \text{and} \quad \sigma^2_{\boldsymbol{z}^*} = \frac{\Delta}{a} > 0.$$

Financial interpretation

The zero tolerance solution $\boldsymbol{w}_g$ leads to the global minimum risk position. This position is modified by investing in the self-financing portfolio $\boldsymbol{z}^*$ [note that $\mathbf{1}^T\boldsymbol{z}^* = 0$] so as to maximize $\tau\boldsymbol{\mu}^T\boldsymbol{w} - \dfrac{\boldsymbol{w}^T\Omega\boldsymbol{w}}{2}$.

Set of optimal portfolios

For a given value of τ, we have solved for $\boldsymbol{w}^*$ (with dependence on τ). We then compute μ_P and σ_P^2 corresponding to the optimal weight $\boldsymbol{w}^*$.

$$\begin{aligned}
\mu_P &= \boldsymbol{\mu}^T(\boldsymbol{w}_g + \tau\boldsymbol{z}^*) = \mu_g + \tau\mu_{\boldsymbol{z}^*} \\
\sigma_P^2 &= \sigma_g^2 + 2\tau\underbrace{\mathrm{cov}(r_{\boldsymbol{w}_g}, r_{\boldsymbol{z}^*})}_{\boldsymbol{z}^{*T}\Omega\boldsymbol{w}_g=0} + \tau^2\sigma_{\boldsymbol{z}^*}^2.
\end{aligned}$$

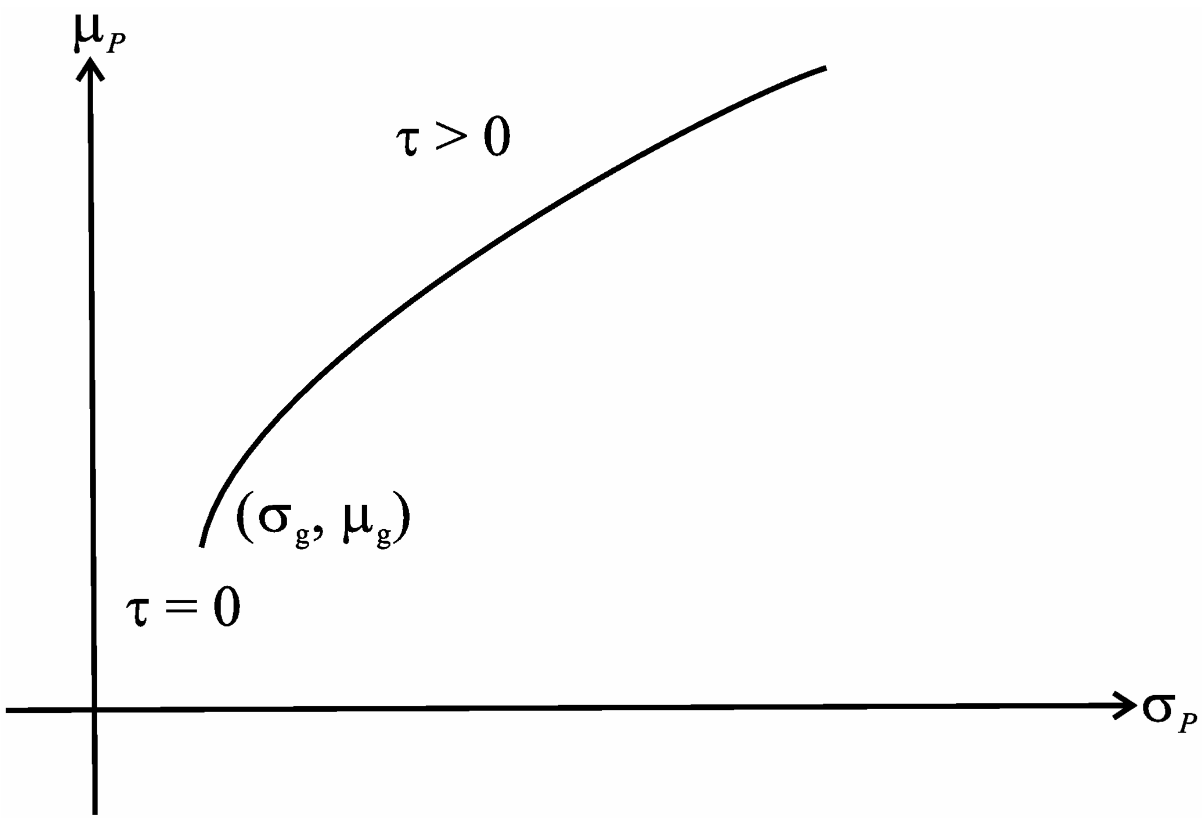
By eliminating τ, we obtain

$$\sigma_P^2 = \sigma_g^2 + \left(\frac{\mu_P - \mu_g}{\mu_{\boldsymbol{z}^*}}\right)^2 \sigma_{\boldsymbol{z}^*}^2 = \sigma_g^2 + \left(\frac{\mu_P - \mu_g}{\sigma_{\boldsymbol{z}^*}}\right)^2, \mu_{\boldsymbol{z}^*} = \sigma_{\boldsymbol{z}^*}^2 = \frac{\Delta}{a}.$$

This is an equation of a hyperbola in the σ_P-μ_P diagram.

$$\mu_P = \frac{b}{a} + \frac{\Delta}{a}\tau$$

$$\sigma_P^2 = \frac{1}{a} + \frac{\Delta}{a}\tau^2$$



The points representing these optimal portfolios in the σ_P-μ_P diagram lie on the upper half of the hyperbola. We expect that for a higher value of τ chosen by the investor, the optimal portfolio has higher μ_P and σ_P.

How to reconcile (和解) the mean-variance model and risk-tolerance model?

Recall that the left boundary of the feasible region of the risky assets is given by

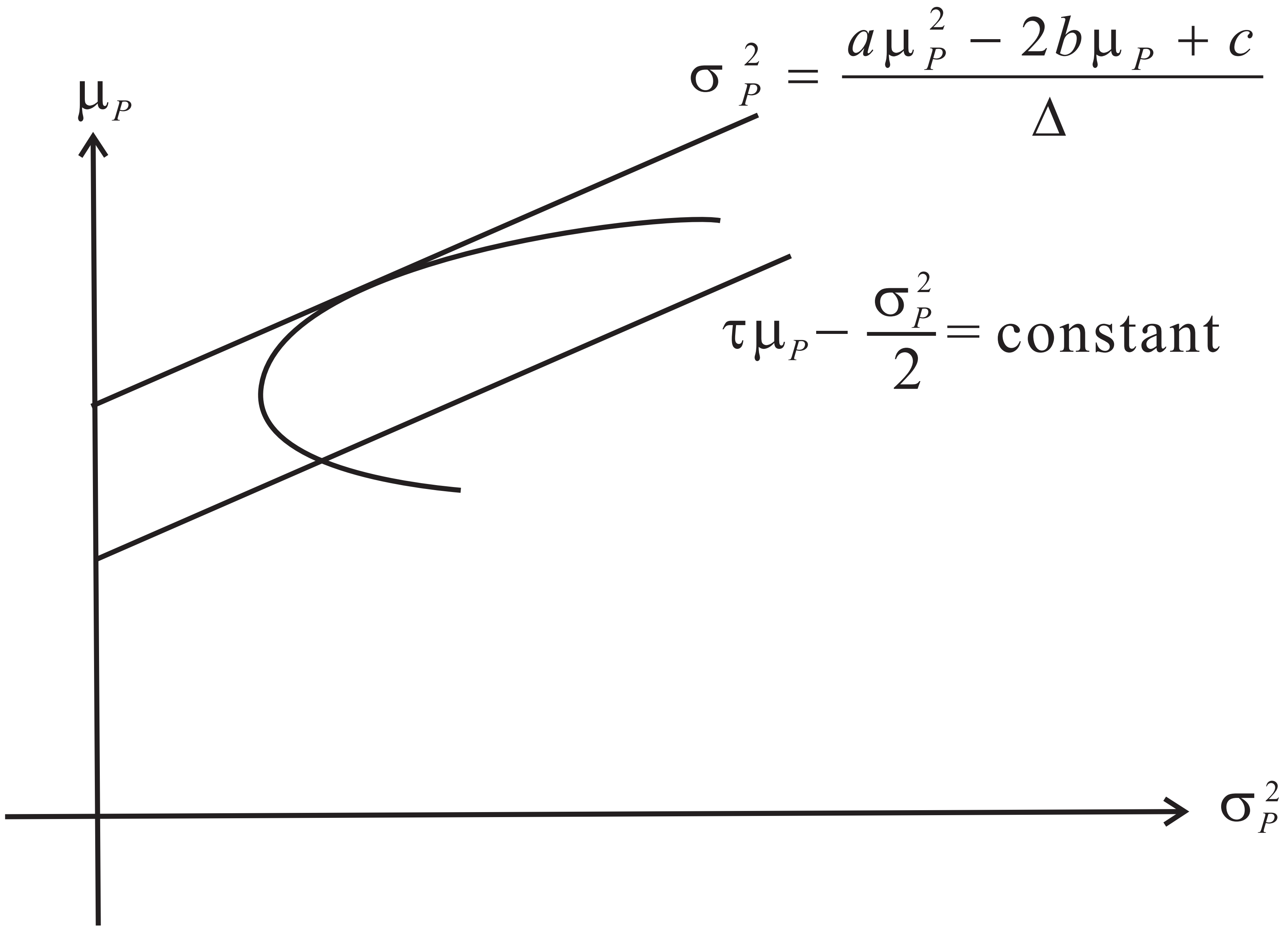
$$\sigma_P^2 = \frac{a\mu_P^2 - 2b\mu_P + c}{\Delta}, \qquad \Delta = ac - b^2. \tag{1}$$

The parabolic curve that traces all optimal portfolios of the risk-tolerance model in the σ_P^2-μ_P diagram is

$$\mu_P = \frac{b}{a} + \frac{\Delta}{a}\tau \quad \text{and} \quad \sigma_P^2 = \frac{1}{a} + \frac{\Delta}{a}\tau^2. \tag{2}$$

It can be shown that the solutions to μ_P and σ_P^2 in Eq. (2) satisfy the parabolic equation (1) since

$$\frac{a\left(\frac{b}{a} + \frac{\Delta}{a}\tau\right)^2 - 2b\left(\frac{b}{a} + \frac{\Delta}{a}\tau\right) + c}{\Delta} = \frac{1}{a} + \frac{\Delta}{a}\tau^2.$$


$\sigma_P^2 = \dfrac{a\mu_P^2 - 2b\mu_P + c}{\Delta}$
μ_P
$\tau\mu_P - \dfrac{\sigma_P^2}{2} = \text{constant}$
σ_P^2

The objective function line: $\tau\mu_P - \dfrac{\sigma_P^2}{2}$ = constant in the σ_P^2-μ_P diagram is pushed up as much as possible in the maximization procedure. However, the optimal portfolio must lie in the feasible region of risky assets. Recall that the feasible region is bounded on the left by the parabolic curve: $\sigma_P^2 = \dfrac{a\mu_P^2 - 2b\mu_P + c}{\Delta}$.

The objective function $\tau\mu_P - \dfrac{\sigma_P^2}{2}$ is maximized when the objective function line touches the left boundary of the feasible region.

Another version of the Two-fund Theorem

Given μ_P, the efficient fund under the mean-variance model is given by

$$\boldsymbol{w}^* = \boldsymbol{w}_g^* + \frac{a}{\Delta}\left(\mu_P - \frac{b}{a}\right)\boldsymbol{z}^*, \quad \mu_P > \mu_g = \frac{b}{a},$$

where $\boldsymbol{w}_g^* = \dfrac{\Omega^{-1}\mathbf{1}}{a}, \boldsymbol{z}^* = \Omega^{-1}\boldsymbol{\mu} - \dfrac{b}{a}\Omega^{-1}\mathbf{1}$.

This implies that any efficient fund can be generated by the two funds: global minimum variance fund $\boldsymbol{w}_g^*$ and the self-financing fund $\boldsymbol{z}^*$.

Proof

1. Note that $\boldsymbol{w}^*$ is of the form $\lambda_1 \Omega^{-1}\mathbf{1} + \lambda_2 \Omega^{-1}\boldsymbol{\mu}$.

2. Consider the expected portfolio return:

$$\boldsymbol{\mu}^T \boldsymbol{w}^* = \mu_g + \frac{a}{\Delta}\left(\mu_P - \frac{b}{a}\right)\mu_{\boldsymbol{z}^*} = \frac{b}{a} + \frac{a}{\Delta}\left(\mu_P - \frac{b}{a}\right)\frac{\Delta}{a} = \mu_P.$$

3. Consider the sum of weights:

$$\mathbf{1}^T \boldsymbol{w}^* = \mathbf{1}^T \boldsymbol{w}_g^* + \frac{a}{\Delta}\left(\mu_P - \frac{b}{a}\right)\mathbf{1}^T \boldsymbol{z}^* = 1.$$

Comparing the first order conditions of the mean-variance model and risk-tolerance model

1. $\Omega \boldsymbol{w}^* = \lambda_1 \mathbf{1} + \lambda_2 \boldsymbol{\mu}$
   $\mathbf{1}^T \boldsymbol{w}^* = 1$
   $\mathbf{1}^T \boldsymbol{\mu} = \mu_P$

2. $\Omega \boldsymbol{w}^* = \lambda \mathbf{1} + \tau \boldsymbol{\mu}$
   $\mathbf{1}^T \boldsymbol{w}^* = 1$

We observe that

$$\begin{aligned}\lambda_1 &= \lambda = \frac{c - b\mu_P}{\Delta} \\ \lambda_2 &= \frac{a\mu_P - b}{\Delta} \text{ is simply } \tau.\end{aligned}$$

The specification of risk tolerance τ is somewhat equivalent to the specification of μ_P.

Summary

1. The objective function $\tau\boldsymbol{\mu}^T\boldsymbol{w} - \dfrac{\boldsymbol{w}^T\Omega\boldsymbol{w}}{2}$ represents a balance of maximizing return $\tau\boldsymbol{\mu}^T\boldsymbol{w}$ against risk $\dfrac{\boldsymbol{w}^T\Omega\boldsymbol{w}}{2}$, where the weighing factor τ is related to the reciprocal of the relative risk aversion coefficient R_R.

2. The optimal solution takes the form

$$\boldsymbol{w}^* = \boldsymbol{w}_g + \tau\boldsymbol{z}^*$$

where $\boldsymbol{w}_g$ is the portfolio weight of the global minimum variance portfolio and the weights in $\boldsymbol{z}^*$ are summed to zero.

Note that

$$\boldsymbol{z}^* = \Omega^{-1}\boldsymbol{\mu} - \frac{b}{a}\Omega^{-1}\mathbf{1} = b(\boldsymbol{w}_d - \boldsymbol{w}_g).$$

Alternatively,

$$\boldsymbol{w}^* = \boldsymbol{w}_g + \frac{ab}{\Delta}\left(\mu_P - \frac{b}{a}\right)(\boldsymbol{w}_d - \boldsymbol{w}_g)$$

and τ and μ_P are related by

$$\tau = \frac{a}{\Delta}\left(\mu_P - \frac{b}{a}\right).$$

3. The additional variance above σ_g^2 is given by

$$\tau^2\sigma_{\boldsymbol{z}^*}^2 = \tau^2\frac{\Delta}{a}, \quad \Delta = ac - b^2.$$

Also, $\text{cov}(r_{\boldsymbol{w}_g}, r_{\boldsymbol{z}^*}) = 0$, that is, $r_{\boldsymbol{w}_g}$ and $r_{\boldsymbol{z}^*}$ are uncorrelated

4. The efficient frontier of the mean-variance model coincides with the set of optimal portfolios of the risk-tolerance model. The risk tolerance τ and expected portfolio return μ_P are related by

$$\frac{\mu_P - \mu_g}{\mu_{\boldsymbol{z}^*}} = \frac{\mu_P - \mu_g}{\sigma_{\boldsymbol{z}^*}^2} = \tau.$$

5. A new version of the Two-fund Theorem can be established where any efficient fund can be generated by the two funds: $\boldsymbol{w}_g^*$ and $\boldsymbol{z}^*$.

1.6 Asset-liability model

Liabilities of a pension fund = future benefits − future contributions

Market value can hardly be determined since liabilities are not readily marketable, unlike tradeable assets. Assume that some specific accounting rules are used to calculate an initial value L_0. If the same rule is applied one period later, a value for L_1 results. Note that L_1 is random.

Rate of growth of the liabilities $= r_L = \dfrac{L_1 - L_0}{L_0}$, where r_L is expected to depend on the changes of interest rate structure, mortality and other stochastic factors.

Let A_0 be the initial value of assets. The investment strategy of the pension fund is given by the portfolio choice $\boldsymbol{w}$. Let $r_{\boldsymbol{w}}$ denote the rate of growth of the asset portfolio.

Surplus optimization

Depending on the portfolio choice $\boldsymbol{w}$, the surplus gain after one period

$$S_1 - S_0 = [A_0(1 + r\boldsymbol{w}) - L_0(1 + r_L)] - (A_0 - L_0) = A_0 r\boldsymbol{w} - L_0 r_L.$$

The rate of return on the surplus is defined by

$$r_S = \frac{S_1 - S_0}{A_0} = r\boldsymbol{w} - \frac{1}{f_0} r_L$$

where $f_0 = A_0/L_0$ is the initial funding ratio.

Maximization formulation:-

$$\max_{\boldsymbol{w} \in \mathbb{R}^N} \left\{ \tau E\left[r\boldsymbol{w} - \frac{1}{f_0} r_L \right] - \frac{1}{2} \text{var}\left(r\boldsymbol{w} - \frac{1}{f_0} r_L \right) \right\}$$

subject to $\displaystyle\sum_{i=1}^{N} w_i = 1$. Since $E\left[\frac{1}{f_0} r_L\right]$ and var(r_L) are independent of $\boldsymbol{w}$ so that they can be omitted from the objective function.

We rewrite the quadratic maximization formulation as

$$\max_{\boldsymbol{w}\in\mathbb{R}^N}\left\{\tau E[r\boldsymbol{w}] - \frac{\mathrm{var}(r\boldsymbol{w})}{2} + \frac{1}{f_0}\mathrm{cov}(r\boldsymbol{w}, r_L)\right\}$$

subject to $\sum_{i=1}^{N} w_i = 1$. Recall that

$$\mathrm{cov}(r\boldsymbol{w}, r_L) = \mathrm{cov}\left(\sum_{i=1}^{N} w_i r_i, r_L\right) = \sum_{i=1}^{N} w_i \mathrm{cov}(r_i, r_L).$$

Final maximization formulation:-

$$\max_{\boldsymbol{w}\in\mathbb{R}^N}\left\{\tau\boldsymbol{\mu}^T\boldsymbol{w} + \boldsymbol{\gamma}^T\boldsymbol{w} - \frac{\boldsymbol{w}^T\Omega\boldsymbol{w}}{2}\right\} \text{ subject to } \mathbf{1}^T\boldsymbol{w} = 1,$$

where $\boldsymbol{\gamma}^T = (\gamma_1 \cdots \gamma_N)$ with $\gamma_i = \frac{1}{f_0}\mathrm{cov}(r_i, r_L)$,

$$\boldsymbol{\mu}^T = (\mu_1 \cdots \mu_N) \text{ with } \mu_i = E[r_i],\ \sigma_{ij} = \mathrm{cov}(r_i, r_j).$$

Remarks

1. The additional term $\boldsymbol{\gamma}^T \boldsymbol{w}$ in the objective function arises from the correlation cov(r_i, r_L) multiplied by the factor L_0/A_0.

2. Compared to the earlier risk tolerance model, we just need to replace $\boldsymbol{\mu}$ by $\boldsymbol{\mu} + \frac{1}{\tau}\gamma$. The efficient portfolios are of the form

$$\boldsymbol{w}^* = \boldsymbol{w}_g + \boldsymbol{z}^L + \tau \boldsymbol{z}^*, \quad \tau \geq 0,$$

where $\boldsymbol{z}^L = \Omega^{-1}\gamma - \dfrac{\mathbf{1}^T \Omega^{-1}\gamma}{\mathbf{1}^T \Omega^{-1}\mathbf{1}} \Omega^{-1}\mathbf{1}$ with $\displaystyle\sum_{i=1}^{N} z_i^L = 0$.