6. Numerical methods for option pricing

Binomial model revisited

• Under the risk neutral measure, $\ln \frac{S_{t+\Delta t}}{S_t}$ becomes normally distributed with mean $\left(r - \frac{\sigma^2}{2}\right)\Delta t$ and variance $\sigma^2\Delta t$, where $r$ is the riskless interest rate and $\sigma^2$ is the variance rate.

• The mean and variance of $\frac{S_{t+\Delta t}}{S_t}$ are $R$ and $R^2(e^{\sigma^2\Delta t} - 1)$, respectively, where $R = e^{r\Delta t}$.

• For the one-period binomial option model under the risk neutral measure, the mean and variance of the asset price ratio $\frac{S_{t+\Delta t}}{S}$ are

$$pu + (1 - p)d \quad \text{and} \quad pu^2 + (1 - p)d^2 - [pu + (1 - p)d]^2,$$

respectively.
• By equating the mean and variance of the asset price ratio in both continuous and discrete models, we obtain

\[ pu + (1 - p)d = R \]

\[ pu^2 + (1 - p)d^2 - R^2 = R^2(e^{\sigma^2 \Delta t} - 1). \]

The first equation leads to \( p = \frac{R - d}{u - d} \), the usual risk neutral probability.

• A convenient choice of the third condition is the tree-symmetry condition

\[ u = \frac{1}{d}, \]

so that the lattice nodes associated with the binomial tree are symmetrical. Writing \( \tilde{\sigma}^2 = R^2 e^{\sigma^2 \Delta t} \), the solution is found to be

\[ u = \frac{1}{d} = \frac{\tilde{\sigma}^2 + 1 + \sqrt{(\tilde{\sigma}^2 + 1)^2 - 4R^2}}{2R}, \quad p = \frac{R - d}{u - d}. \]
• By expanding $u$ in Taylor series in powers of $\sqrt{\Delta t}$, we obtain

$$u = 1 + \sigma \sqrt{\Delta t} + \frac{\sigma^2}{2} \Delta t + \frac{4r^2 + 4\sigma^2 r + 3\sigma^4}{8\sigma} \Delta t^2 + O(\Delta t^2).$$

• Observe that the first three terms in the above Taylor series agree with those of $e^{\sigma \sqrt{\Delta t}}$ up to $O(\Delta t)$ term.

• This suggests the judicious choice of the following set of parameter values

$$u = e^{\sigma \sqrt{\Delta t}}, \quad d = e^{-\sigma \sqrt{\Delta t}}, \quad p = \frac{R - d}{u - d}.$$ 

• With this new set of parameters, the variance of the price ratio $\frac{S_{t+\Delta t}}{S_t}$ in the continuous and discrete models agree up to $O(\Delta t)$. 
Continuous limits of the binomial model

We consider the asymptotic limit $\Delta t \to 0$ of the binomial formula

\[ c = [pc_u^\Delta t + (1 - p)c_d^\Delta t]e^{-r\Delta t}. \]

In the continuous analog, the binomial formula can be written as

\[ c(S, t - \Delta t) = [pc(uS, t) + (1 - p)c(dS, t)]e^{-r\Delta t}. \]

Assuming sufficient continuity of $c(S, t)$, we perform the Taylor expansion of the binomial scheme at $(S, t)$ as follows:
\[-c(S, t - \Delta t) + [pc(uS, t) + (1 - p)c(dS, t)]e^{-r\Delta t}\]
\[= \frac{\partial c}{\partial t}(S, t)\Delta t - \frac{1}{2} \frac{\partial^2 c}{\partial t^2}(S, t)\Delta t^2 + \cdots - (1 - e^{-r\Delta t})c(S, t)\]
\[+ e^{-r\Delta t}\left\{ [p(u - 1) + (1 - p)(d - 1)]S\frac{\partial c}{\partial S}(S, t)\right.\]
\[+ \frac{1}{2} [p(u - 1)^2 + (1 - p)(d - 1)^2] S^2 \frac{\partial^2 c}{\partial S^2}(S, t)\]
\[+ \frac{1}{6} [p(u - 1)^3 + (1 - p)(d - 1)^3] S^3 \frac{\partial^3 c}{\partial S^3}(S, t) + \cdots \right\}.

By observing that
\[1 - e^{-r\Delta t} = r\Delta t + O(\Delta t^2),\]

it can be shown that
\[e^{-r\Delta t} \left[ p(u - 1) + (1 - p)(d - 1) \right] = r\Delta t + O(\Delta t^2),\]
\[e^{-r\Delta t} \left[ p(u - 1)^2 + (1 - p)(d - 1)^2 \right] = \sigma^2 \Delta t + O(\Delta t^2),\]
\[e^{-r\Delta t} \left[ p(u - 1)^3 + (1 - p)(d - 1)^3 \right] = O(\Delta t^2).\]
Combining the results, we obtain

\[-c(S, t - \Delta t) + [pc(uS, t) + (1 - p)c(dS, t)] e^{-r\Delta t} \]

\[= \left[ \frac{\partial c}{\partial t}(S, t) + rS \frac{\partial c}{\partial S}(S, t) + \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2}(S, t) - rc(S, t) \right] \Delta t + O(\Delta t^2).\]

Since \(c(S, t)\) satisfies the binomial formula, so we obtain

\[0 = \frac{\partial c}{\partial t}(S, t) + rS \frac{\partial c}{\partial S}(S, t) + \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2}(S, t) - rc(S, t) + O(\Delta t).\]

In the limit \(\Delta t \to 0\), the binomial call value \(c(S, t)\) satisfies the Black-Scholes equation.
**Discrete dividend models**

Consider the naive construction of the binomial tree. Let $S$ be the asset price at the current time which is $n\triangle t$ from expiry, and suppose a discrete dividend of amount $D$ is paid at time between one time step and two time steps from the current time.

The nodes in the binomial tree at two time steps from the current time would correspond to asset prices

$$u^2S - D, \quad S - D \quad \text{and} \quad d^2S - D,$$

since the asset price drops by the same amount as the dividend right after the dividend payment.
• Extending one time step further, there will be six nodes 

\[(u^2S - D)u, (u^2S - D)d, (S - D)u, (S - D)d, (d^2S - D)u, (d^2S - D)d\]

instead of four nodes as in the usual binomial tree without discrete dividend.

• This is because \((u^2S - D)d \neq (S - D)u\) and \((S - D)d \neq (d^2S - D)u\), so the interior nodes do not recombine.
Binomial tree with single discrete dividend.
• Splitting the asset price $S_t$ into two parts: the risky component $\tilde{S}_t$ that is stochastic and the remaining part that will be used to pay the discrete dividend (assumed to be deterministic) in the future.

• Suppose the dividend date is $t^*$, then at the current time $t$, the risky component $\tilde{S}_t$ is given by

$$\tilde{S}_t = \begin{cases} 
S_t - De^{-r(t^*-t)}, & t < t^* \\
S_t, & t > t^*. 
\end{cases}$$

• Let $\tilde{\sigma}$ denote the volatility of $\tilde{S}_t$ and assume $\tilde{\sigma}$ to be constant rather than the volatility of $S_t$ itself to be constant.
• Assume that a discrete dividend $D$ is paid at time $t^*$, which lies between the $k^{th}$ and $(k + 1)^{th}$ time step.

• At the tip of the binomial tree, the risky component $\tilde{S}$ is related to the asset price $S$ by

$$\tilde{S} = S - De^{-kr\Delta t}.$$  

• The total value of asset price at the $(n, j)^{th}$ node, which corresponds to $n$ time steps from the tip and $j$ upward jumps, is given by

$$\tilde{S}u^j a^{n-j} + De^{-(k-n)r\Delta t} \mathbf{1}_{\{n \leq k\}}, \quad n = 1, 2, \ldots, N \quad \text{and} \quad j = 0, 1, \ldots, n.$$
Construction of a reconnecting binomial tree with single discrete dividend $D$. Here, $N = 4$ and $k = 2$, and let $\tilde{S}$ denote the risky component of the asset value at the tip of the binomial tree. The asset value at nodes $P, Q$ and $R$ are $\tilde{S} + D e^{-2r \Delta t}$, $\tilde{S} u + D e^{-r \Delta t}$ and $\tilde{S} d$, respectively.
Early exercise feature and callable feature

- Without the early exercise privilege, risk neutral valuation leads to the usual binomial formula

\[ V_{cont} = \frac{pV^u_{\Delta t} + (1 - p)V^d_{\Delta t}}{R}. \]

- The following simple dynamic programming procedure is applied at each binomial node

\[ V = \max(V_{cont}, h(S)), \]

where \( h(S) \) is the exercise payoff when the asset price assumes the value \( S \).
• The intrinsic value of a vanilla put option is $X - S^m_j$ at the $(n, j)$ node, where $X$ is the strike price. The dynamic programming procedure applied at each node is given by

$$
P^n_j = \max \left( \frac{pP^{n+1}_{j+1} + (1 - p)P^{n+1}_j}{R}, X - S^m_j \right),$$

where $n = N - 1, \ldots, 0$, and $j = 0, 1, \cdots, n$. 
Callable American call

- The callable feature entitles the issuer to buy back the American option at any time at a predetermined call price.
- Upon call, the holder can choose either to exercise the call or receive the call price as cash.
- Let the call price be $K$. The dynamic programming procedure applied at each node is modified as follows

$$C^n_j = \min \left( \max \left( \frac{pc_{j+1}^{n+1} + (1 - p)c_j^{n+1}}{R}, S_j^n - X \right), \max(K, S_j^n - X) \right).$$
• The first term \( \max\left( \frac{pc_{n+1} + (1 - p)c_{j}^{n+1}}{R}, S_{j}^{n} - X \right) \) represents the optimal strategy of the holder, given no call of the option by the issuer.

• Upon call by the issuer, the payoff is given by the second term \( \max(K, S_{j}^{n} - X) \) since the holder can either receive cash amount \( K \) or exercise the option.

• From the perspective of the issuer, he chooses to call or restrain from calling so as to minimize the option value with reference to the possible actions of the holder. The value of the callable call is given by taking the minimum value of the above two terms.
Trinomial schemes

In a trinomial model, the asset price $S$ is assumed to jump to either $uS, mS$ or $dS$ after one time period $\Delta t$, where $u > m > d$. We consider a trinomial formula of option valuation of the form

$$V = \frac{p_1 V_u \Delta t + p_2 V_m \Delta t + p_3 V_d \Delta t}{R}, \quad R = e^{r\Delta t}.$$  

There are 6 unknowns: $p_1, p_2, p_3, u, m$ and $d$. We take $m = 1, u = 1/d$. We obtain 3 equations by (i), equating mean, (ii) equating variance, (iii) setting sum of probabilities $= 1$. We are left with one free parameter.
Write

$$\ln S_{t+\Delta t} = \ln S_t + \zeta,$$

where $\zeta$ is a normal random variable with mean $\left( r - \frac{\sigma^2}{2} \right) \Delta t$ and variance $\sigma^2 \Delta t$. We approximate $\zeta$ by an approximate discrete random variable $\zeta^a$ with the following distribution

$$\zeta^a = \begin{cases} 
u & \text{with probability } p_1 \\ 0 & \text{with probability } p_2 \\ -\nu & \text{with probability } p_3 \end{cases}$$

where $\nu = \lambda \sigma \sqrt{\Delta t}$ and $\lambda \geq 1$. The corresponding values for $u, m$ and $d$ in the trinomial scheme are: $u = e^\nu$, $m = 1$ and $d = e^{-\nu}$. 
To find the probability values $p_1, p_2$ and $p_3$, the mean and variance of $\zeta^a$ are chosen to be equal to those of $\zeta$. These lead to

$$E[\zeta^a] = v(p_1 - p_3) = \left(r - \frac{\sigma^2}{2}\right) \Delta t$$

$$\text{var}(\zeta^a) = v^2(p_1 + p_3) - v^2(p_1 - p_3)^2 = \sigma^2 \Delta t.$$ 

We see that $v^2(p_1 - p_3)^2 = O(\Delta t^2)$. We may drop this term so that

$$v^2(p_1 + p_3) = \sigma^2 \Delta t,$$

while still maintaining $O(\Delta t)$ accuracy.
Lastly, the probabilities must be summed to one so that

\[ p_1 + p_2 + p_3 = 1. \]

We then solve together to obtain

\[ p_1 = \frac{1}{2\lambda^2} + \frac{(r - \frac{\sigma^2}{2})\sqrt{\Delta t}}{2\lambda\sigma}, \]

\[ p_2 = 1 - \frac{1}{\lambda^2}, \]

\[ p_3 = \frac{1}{2\lambda^2} - \frac{(r - \frac{\sigma^2}{2})\sqrt{\Delta t}}{2\lambda\sigma}, \]

here \( \lambda \) is a free parameter.
Multi-state options

- We assume the joint density of the prices of the two underlying assets $S_1$ and $S_2$ to be bivariate lognormal.
- Let $\sigma_i$ be the volatility of asset price $S_i$, $i = 1, 2$ and $\rho$ be the correlation coefficient between the two lognormal diffusion processes.
- Let $S_i$ and $S_i^{\Delta t}$ denote, respectively, the price of asset $i$ at the current time and one period $\Delta t$ later.
- Under the risk neutral measure, we have

$$\ln \frac{S_i^{\Delta t}}{S_i} = \zeta_i, \quad i = 1, 2,$$

where $\zeta_i$ is a normal random variable with mean $\left( r - \frac{\sigma_i^2}{2} \right) \Delta t$ and variance $\sigma_i^2 \Delta t$. 
The instantaneous correlation coefficient between $\zeta_1$ and $\zeta_2$ is $\rho$. The joint bivariate normal process $\{\zeta_1, \zeta_2\}$ is approximated by a pair of joint discrete random variables $\{\zeta^a_1, \zeta^a_2\}$ with the following distribution

<table>
<thead>
<tr>
<th>$\zeta^a_1$</th>
<th>$\zeta^a_2$</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>$v_2$</td>
<td>$p_1$</td>
</tr>
<tr>
<td>$v_1$</td>
<td>$-v_2$</td>
<td>$p_2$</td>
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<td>$-v_1$</td>
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<td>$p_3$</td>
</tr>
<tr>
<td>$-v_1$</td>
<td>$v_2$</td>
<td>$p_4$</td>
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<tr>
<td>0</td>
<td>0</td>
<td>$p_5$</td>
</tr>
</tbody>
</table>

where $v_i = \lambda_i \sigma_i \sqrt{\Delta t}$, $i = 1, 2$. 
Equating the corresponding means give

\[ E[\zeta^a_1] = v_1(p_1 + p_2 - p_3 - p_4) = \left( r - \frac{\sigma^2_1}{2} \right) \Delta t \]  

(i)

\[ E[\zeta^a_2] = v_2(p_1 - p_2 - p_3 + p_4) = \left( r - \frac{\sigma^2_2}{2} \right) \Delta t. \]  

(ii)

By equating the variances and covariance to \( O(\Delta t) \) accuracy, we have

\[ \text{var}(\zeta^a_1) = v_1^2(p_1 + p_2 + p_3 + p_4) = \sigma^2_1 \Delta t \]  

(iii)

\[ \text{var}(\zeta^a_2) = v_2^2(p_1 + p_2 + p_3 + p_4) = \sigma^2_2 \Delta t \]  

(iv)

\[ E[\zeta^a_1 \zeta^a_2] = v_1 v_2(p_1 - p_2 + p_3 - p_4) = \sigma_1 \sigma_2 \rho \Delta t. \]  

(v)

In order that Eqs. (iii and iv) are consistent, we must set \( \lambda_1 = \lambda_2 \).
Writing $\lambda = \lambda_1 = \lambda_2$, we have the following four independent equations for the five probability values

\begin{align*}
p_1 + p_2 - p_3 - p_4 &= \frac{(r - \frac{\sigma_1^2}{2})\sqrt{\Delta t}}{\lambda \sigma_1} \\
p_1 - p_2 - p_3 + p_4 &= \frac{(r - \frac{\sigma_2^2}{2})\sqrt{\Delta t}}{\lambda \sigma_2} \\
p_1 + p_2 + p_3 + p_4 &= \frac{1}{\lambda^2} \\
p_1 - p_2 + p_3 - p_4 &= \frac{\rho}{\lambda^2}.
\end{align*}

Since the probabilities must be summed to one, this gives the remaining condition as

\[ p_1 + p_2 + p_3 + p_4 + p_5 = 1. \]
The solution of the above linear algebraic system of equations gives

\[ p_1 = \frac{1}{4} \left[ \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{r - \frac{\sigma_1^2}{2}}{\sigma_1} + \frac{r - \frac{\sigma_2^2}{2}}{\sigma_2} \right) + \frac{\rho}{\lambda^2} \right] \]

\[ p_2 = \frac{1}{4} \left[ \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{r - \frac{\sigma_1^2}{2}}{\sigma_1} - \frac{r - \frac{\sigma_2^2}{2}}{\sigma_2} \right) - \frac{\rho}{\lambda^2} \right] \]

\[ p_3 = \frac{1}{4} \left[ \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( -\frac{r - \frac{\sigma_1^2}{2}}{\sigma_1} - \frac{r - \frac{\sigma_2^2}{2}}{\sigma_2} \right) + \frac{\rho}{\lambda^2} \right] \]

\[ p_4 = \frac{1}{4} \left[ \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( -\frac{r - \frac{\sigma_1^2}{2}}{\sigma_1} + \frac{r - \frac{\sigma_2^2}{2}}{\sigma_2} \right) - \frac{\rho}{\lambda^2} \right] \]

\[ p_5 = 1 - \frac{1}{\lambda^2}, \quad \lambda \geq 1 \text{ is a free parameter.} \]
Two-state trinomial model

• For convenience, we write $u_i = e^{v_i}$, $d_i = e^{-v_i}$, $i = 1, 2$.

• Let $V_{u_1u_2}^{Δt}$ denote the option price at one time period later with asset prices $u_1S_1$ and $u_2S_2$, and similar meaning for $V_{u_1d_2}^{Δt}$, $V_{d_1u_2}^{Δt}$ and $V_{d_1d_2}^{Δt}$.

• We let $V_{0,0}^{Δt}$ denote the option price one period later with no jumps in asset prices.

• The corresponding 5-point formula for the two-state trinomial model can be expressed as

$$V = (p_1 V_{u_1u_2}^{Δt} + p_2 V_{u_1d_2}^{Δt} + p_3 V_{d_1d_2}^{Δt} + p_4 V_{d_1u_2}^{Δt} + p_5 V_{0,0}^{Δt})/R.$$ 

• When $λ = 1$, we have $p_5 = 0$ and the above 5-point formula reduces to the 4-point formula.
Forward shooting grid methods

• For path dependent options, the option value also depends on the path function $F_t = F(S, t)$ defined specifically for the given nature of path dependence, say, the minimum asset price realized along a specific asset price path.

• Since option value depends also on $F_t$, we find the value of the path dependent option at each node in the lattice tree for all alternative values of $F_t$ that can occur.

• The approach of appending an auxiliary state vector at each node in the lattice tree to model the correlated evolution of $F_t$ with $S_t$ is commonly called the forward shooting grid (FSG) method.
• Consider a trinomial tree whose probabilities of upward, zero and downward jump of the asset price are denoted by $p_u, p_0$ and $p_d$, respectively.

• Let $V_{j,k}^n$ denote the numerical option value of the exotic path dependent option at the $n^{th}$-time level ($n$ time steps from the tip of the tree). Also, $j$ denotes the $j$ upward jumps from the initial asset value and $k$ denotes the numbering index for the various possible values of the augmented state variable $F_t$ at the $(n,j)^{th}$ node.

• Let $G$ denote the function that describes the correlated evolution of $F_t$ with $S_t$ over the time interval $\Delta t$, that is,

$$F_{t+\Delta t} = G(F_t, S_{t+\Delta t}).$$
• Let $g(k, j)$ denote the grid function which is considered as the discrete analog of the evolution function $G$.

• The trinomial version of the FSG scheme can be represented as follows

$$V_{j,k}^n = \left[ p_u V_{j+1,k,g(k,j+1)}^{n+1} + p_0 V_{j,g(k,j)}^{n+1} + p_d V_{j-1,k,g(k,j-1)}^{n+1} \right] e^{-r\Delta t},$$

where $e^{-r\Delta t}$ is the discount factor over time interval $\Delta t$.

• To price a specific path dependent option, the design of the FSG algorithm requires the specification of the grid function $g(k, j)$. 
Cumulative Parisian feature

- Let $M$ denote the prespecified number of cumulative breaching occurrences that is required to activate knock-out, and let $k$ be the integer variable that counts the number of breaching so far.

- Let $B$ denote the down barrier associated with the knock-out feature.

- Let $x_j$ denote the value of $x = \ln S$ that corresponds to $j$ upward moves in the trinomial tree.

- When $n\Delta t$ happens to be a monitoring instant, the index $k$ increases its value by 1 if the asset price $S$ falls on or below the barrier $B$, that is, $x_j \leq \ln B$. 
To incorporate the cumulative Parisian feature, the appropriate choice of the grid function $g_{cum}(k, j)$ is defined by

$$g_{cum}(k, j) = k + 1_{\{x_j \leq \ln B\}}.$$ 

$$V_{n-1}^{j,k} = \begin{cases} 
    p_u V_{j+1,k}^n + p_0 V_{j,k}^n + p_d V_{j-1,k}^n, & \text{if } n\Delta t \text{ is not a monitoring instant} \\
    p_u V_{j+1,g_{cum}(k,j+1)}^n + p_0 V_{j,g_{cum}(k,j)}^n + p_d V_{j-1,g_{cum}(k,j-1)}^n, & \text{if } n\Delta t \text{ is a monitoring instant}
\end{cases}.$$
Schematic diagram that illustrates the construction of the grid function $g_{cum}(k, j)$ that models the cumulative Parisian feature. The down barrier $\ln B$ is placed mid-way between two horizontal rows of trinomial nodes. Here, the $n^{th}$-time level is a monitoring instant.
Finite difference algorithms

Black-Scholes equation: \[ \frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0. \]

Use the transformed variables: \( \tau = T - t, x = \ln S, \)

\[
\frac{\partial}{\partial t} = -\frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial S} = \frac{1}{S} \frac{\partial}{\partial x} \quad \text{or} \quad S \frac{\partial}{\partial S} = \frac{\partial}{\partial x} \\
\frac{\partial^2}{\partial x^2} = S \frac{\partial}{\partial S} \left( S \frac{\partial}{\partial S} \right) = S^2 \frac{\partial^2}{\partial S^2} + S \frac{\partial}{\partial S} \text{ so that } S^2 \frac{\partial^2}{\partial S^2} = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}.
\]

Black-Scholes equation

\[ \frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2} + \left( r - \frac{\sigma^2}{2} \right) \frac{\partial V}{\partial x} - rV, \quad \tau > 0, -\infty < x < \infty. \]
Preliminary procedure

Transform the domain of the continuous problem

\[ \{(x, \tau) : -\infty < x < \infty, \tau \geq 0\} \]

into a discretized domain. Infinite of \( x = \ln S \) is approximated by a finite truncated interval \([M_1, M_2]\), \( M_1 \) and \( M_2 \) are sufficiently large. The discretized domain is overlaid with a uniform system of meshes \((j\Delta x, n\Delta \tau), j = 0, 1, \cdots, N + 1, n = 1, 2, \cdots\), with \((N + 1)\Delta x = M_1 + M_2\).

Step width \( \Delta x \) and time step \( \Delta \tau \) are independent. Option values are computed only at the grid points.
Finite difference mesh with uniform stepwidth $\Delta x$ and time step $\Delta \tau$. Numerical option values are computed at the node points $(j\Delta x, n\Delta \tau), j = 1, 2, \cdots, N, n = 1, 2, \cdots$. Option values along the boundaries: $j = 0$ and $j = N + 1$ are prescribed by the boundary conditions of the option model. The “initial” values $V_j^0$ along the zeroth time level, $n = 0$, are given by the terminal payoff function.
Let $V^n_j$ denote the numerical approximation of $V(j\triangle x, n\triangle \tau)$. The continuous temporal and spatial derivatives are approximated by the following finite difference operators

$$\frac{\partial V}{\partial \tau}(j\triangle x, n\triangle \tau) \approx \frac{V^{n+1}_j - V^n_j}{\Delta \tau} \quad \text{(forward difference)}$$

$$\frac{\partial V}{\partial x}(j\triangle x, n\triangle \tau) \approx \frac{V^n_{j+1} - V^n_{j-1}}{2\Delta x} \quad \text{(centered difference)}$$

$$\frac{\partial^2 V}{\partial x^2}(j\triangle x, n\triangle \tau) \approx \frac{V^n_{j+1} - 2V^n_j + V^n_{j-1}}{\Delta x^2} \quad \text{(centered difference)}$$

By taking

$$W^{n+1}_j = e^{r(n+1)\Delta \tau}V^{n+1}_j \quad \text{and} \quad W^n_j = e^{rn\Delta \tau}V^n_j,$$

then canceling $e^{rn\Delta \tau}$, we obtain the following explicit Forward-Time-Centered-Space (FTCS) finite difference scheme

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\[ V_j^{n+1} = \left[ V_j^n + \frac{\sigma^2}{2} \frac{\Delta \tau}{\Delta x^2} (V_{j+1}^n - 2V_j^n + V_{j-1}^n) \right. \\
+ \left. \left( r - \frac{\sigma^2}{2} \right) \frac{\Delta \tau}{2\Delta x} (V_{j+1}^n - V_{j-1}^n) \right] e^{-r\Delta \tau}. \]

- Suppose we are given “initial” values \( V_j^0, j = 0, 1, \cdots, N + 1 \) along the zeroth time level, we can use the explicit scheme to find values \( V_j^1, j = 1, 2, \cdots, N \) along the first time level \( \tau = \Delta \tau \).

- The values at the two ends \( V_0^1 \) and \( V_{N+1}^1 \) are given by the numerical boundary conditions specified for the option model.
Two-level four-point explicit schemes

\[ V_j^{n+1} = b_1 V_{j+1}^n + b_0 V_j^n + b_{-1} V_{j-1}^n, \quad j = 1, 2, \cdots, N, \quad n = 0, 1, 2, \cdots. \]

The above FTCS scheme corresponds to

\[
\begin{align*}
    b_1 &= \left[ \frac{\sigma^2}{2} \frac{\Delta \tau}{\Delta x^2} + \left(r - \frac{\sigma^2}{2}\right) \frac{\Delta \tau}{2 \Delta x} \right] e^{-r \Delta \tau}, \\
    b_0 &= \left[ 1 - \frac{\sigma^2}{\Delta x^2} \frac{\Delta \tau}{\Delta x} \right] e^{-r \Delta \tau}, \\
    b_{-1} &= \left[ \frac{\sigma^2}{2} \frac{\Delta \tau}{\Delta x^2} - \left(r - \frac{\sigma^2}{2}\right) \frac{\Delta \tau}{2 \Delta x} \right] e^{r \Delta \tau}.
\end{align*}
\]

Both the binomial and trinomial schemes are members of the family when the reconnecting condition \( ud = 1 \) holds.
Suppose we write $\triangle x = \ln u$, then $\ln d = -\triangle x$; the binomial scheme can be expressed as

$$V^{n+1}(x) = \frac{pV^n(x + \triangle x) + (1 - p)V^n(x - \triangle x)}{R}, \quad x = \ln S, \text{ and } R = e^{r\Delta \tau},$$

where $V^{n+1}(x), V^n(x + \triangle x)$ and $V^n(x - \triangle x)$ are analogous to $c, c_{u}^{\Delta t}$ and $c_{d}^{\Delta t}$, respectively. The corresponds to

$$b_1 = p/R, \quad b_0 = 0 \quad \text{and} \quad b_{-1} = (1 - p)/R.$$ 

In the Cox-Ross-Rubinstein scheme, they are related by $\Delta x = \ln u = \sigma \sqrt{\Delta \tau}$ or $\sigma^2 \Delta \tau = \Delta x^2$. In the trinomial scheme, their relation is given by $\lambda^2 \sigma^2 \Delta \tau = \Delta x^2$, where the free parameter $\lambda$ can be chosen arbitrarily.
Numerical stability and oscillation phenomena

- A numerical scheme must be consistent in order that the numerical solution converges to the exact solution of the underlying differential equation. However, consistency is only a necessary but not sufficient condition for convergence.

- The roundoff errors incurred during numerical calculations may lead to the blow up of the solution and erode the whole computation.
**Order of accuracy**

Suppose the leading truncation terms are $O(\Delta \tau^k, \Delta x^m)$, then the numerical scheme is said to be $k^{\text{th}}$ order time accurate and $m^{\text{th}}$ order space accurate. The explicit FTCS scheme is first order time accurate and second order space accurate. Suppose we choose $\Delta \tau$ to be the same order as $\Delta x^2$, that is, $\Delta \tau = \lambda \Delta x^2$ for some finite constant $\lambda$, then the leading truncation error terms become $O(\Delta \tau)$. 
• A necessary condition for the convergence of the numerical solution to the continuous solution is that the local truncation error of the numerical scheme must tend to zero for vanishing stepwidth and time step. In this case, the numerical scheme is said to be *consistent*.

• The *order of accuracy* of a scheme is defined to be the order in powers of $\Delta x$ and $\Delta \tau$ in the leading truncation error terms.
Since $V(j \Delta x, n \Delta \tau)$ satisfies the Black-Scholes equation, this leads to

$$T(j \Delta x, n \Delta \tau) = \frac{\Delta \tau}{2} \frac{\partial^2 V}{\partial \tau^2} (j \Delta x, n \Delta \tau) - \frac{\sigma^2}{24} \Delta x^2 \frac{\partial^4 V}{\partial x^4} (j \Delta x, n \Delta \tau)$$

$$- \left( r - \frac{\sigma^2}{2} \right) \frac{\Delta x^2}{3} \frac{\partial^3 V}{\partial x^3} (j \Delta x, n \Delta \tau) + O(\Delta \tau^2)$$

$$+ O(\Delta x^4).$$

The local truncation error measures the discrepancy that the continuous solution does not satisfy the numerical scheme at the node point.
We then expand each term by performing the Taylor expansion at the node point \((j\Delta x, n\Delta \tau)\). After some cancellation of terms, we obtain

\[
T(j\Delta x, n\Delta \tau) = \frac{\partial V}{\partial \tau}(j\Delta x, n\Delta \tau) + \frac{\Delta \tau}{2} \frac{\partial^2 V}{\partial \tau^2}(j\Delta x, n\Delta \tau) + O(\Delta \tau^2) \\
- \frac{\sigma^2}{2} \left[ \frac{\partial^2 V}{\partial x^2}(j\Delta x, n\Delta \tau) + \frac{\Delta x^2}{12} \frac{\partial^4 V}{\partial x^4}(j\Delta x, n\Delta \tau) + O(\Delta x^4) \right] \\
- \left( r - \frac{\sigma^2}{2} \right) \left[ \frac{\partial V}{\partial x}(j\Delta x, n\Delta \tau) + \frac{\Delta x^2}{3} \frac{\partial^3 V}{\partial x^3}(j\Delta x, n\Delta \tau) + O(\Delta x^4) \right] \\
+ rV(j\Delta x, n\Delta \tau).
\]
Truncation errors and order of convergence

The local truncation error of a given numerical scheme is obtained by substituting the exact solution of the continuous problem into the numerical scheme. Let \( V(j\Delta x, n\Delta \tau) \) denote the exact solution of the continuous Black-Scholes equation. The local truncation error at the node point \((j\Delta x, n\Delta \tau)\) of the explicit FTCS scheme is given by

\[
T(j\Delta x, n\Delta \tau) = \frac{V(j\Delta x, (n+1)\Delta \tau) - V(j\Delta x, n\Delta \tau)}{\Delta \tau} - \frac{\sigma^2}{2} \left[ V((j+1)\Delta x, n\Delta \tau) - 2V(j\Delta x, n\Delta \tau) + V((j-1)\Delta x, n\Delta \tau) \right] + \frac{\Delta x^2}{2} \frac{\Delta \tau}{2} - \left( r - \frac{\sigma^2}{2} \right) \frac{V((j+1)\Delta x, n\Delta \tau) - V((j-1)\Delta x, n\Delta \tau)}{2\Delta x} + rV(j\Delta x, n\Delta \tau).
\]
Monte Carlo simulation

A wide class of derivative pricing problems come down to the evaluation of the following expectation functional

\[ Ef[Z(T; t, z)]. \]

- \( Z \) denotes the stochastic process that describes the price evolution of one or more underlying financial variables such as asset prices and interest rates, under the respective risk neutral probability distributions.

- The process \( Z \) has the initial value \( z \) at time \( t \), and the function \( f \) specifies the value of the derivative at the expiration time \( T \).
• The Monte Carlo method is basically a numerical procedure for estimating the expected value of a random variable, and so it leads itself naturally to derivative pricing problem represented as expectations.

• The simulation procedure involves generating random variables with a given probability density and using the law of large numbers to take the average of these values as an estimate of the expected value of the random variable.
In the context of derivative pricing, the Monte Carlo procedure involves the following steps.

(i) Simulate sample paths of the underlying state variables in the derivative model such as asset prices and interest rates over the life of the derivative, according to the risk neutral probability distributions.

(ii) For each simulated sample path, evaluate the discounted cash flows of the derivative.

(iii) Take the sample average of the discounted cash flows over all sample paths.
• The numerical procedure requires the computation of the expected payoff of the call option at expiry, $E_t[\max(S_T - X, 0)]$, and discounted to the present value at time $t$, namely, $e^{-r(T-t)} E_t[\max(S_T - X, 0)]$. 

• Assuming lognormal distribution for the asset price movement, the price dynamics under the risk neutral measure is given by 

$$
\frac{S_{t+\Delta t}}{S_t} = e^{(r-\frac{\sigma^2}{2})\Delta t + \sigma \epsilon \sqrt{\Delta t}},
$$

where $\Delta t$ is the time step, $\sigma$ is the volatility and $r$ is the riskless interest rate.
• Here, $\epsilon$ denotes a normally distributed random variable with zero mean and unit variance, and so $\sigma \epsilon \sqrt{\Delta t}$ represents a discrete approximation to an increment in the Wiener process of the asset price with volatility $\sigma$ in time increment $\Delta t$.

• Suppose these are $N$ time steps between the current time $t$ and expiration time $T$, where $\Delta t = (T - t)/N$.

• The numerical procedure is repeated $N$ times to simulate the price path from $S_t$ to $S_T = S_t + N \Delta t$.

• The call price corresponding to this particular simulated asset price path is then computed using the discounted formula

$$c = e^{-r(T-t)} \max(S_T - X, 0).$$
• After repeating the above simulation for a sufficiently large number of runs, the expected call value is obtained by computing the average of the estimates of the call value found in the sample simulation.

• Let $c_i$ denote the estimate of the call value obtained in the $i^{th}$ simulation and $M$ be the total number of simulation runs.

• The expected call value is given by

$$\hat{c} = \frac{1}{M} \sum_{i=1}^{M} c_i,$$

and the variance of the estimate is computed by

$$\hat{s}^2 = \frac{1}{M - 1} \sum_{i=1}^{M} (c_i - \hat{c})^2.$$
For a sufficiently large value of $M$, the distribution
\[
\frac{\hat{c} - c}{\sqrt{\frac{s^2}{M}}},
\]
c is the true call value,

tends to the standard normal distribution. Note that the standard deviation of $\hat{c}$ is equal to $\hat{s}/\sqrt{M}$ and so the confidence limits of estimation can be reduced by increasing the number of simulation runs $M$. 
Advantages

1. The error is independent of the dimension of the option problem.

2. Its ease to accommodate complicated payoff in an option model. For example, the terminal payoff of an Asian option depends on the average of the asset price over certain time interval while that of a lookback option depends on the extremum value of the asset price over some period of time. It is quite straightforward to obtain the average or extremum value in the simulated price path in each simulated path.

Drawback

The demand for a large number of simulation trials in order to achieve a high level of accuracy.
Computational efficiency

• Suppose $W_T$ is the total amount of computational work units available to generate an estimate of the value of an option $V$.

• Assume that there are two methods for generating the Monte Carlo estimates for the option value, requiring $W_1$ and $W_2$ units of computation work respectively for each simulation run. For simplicity, assume $W_T$ is divisible by both $W_1$ and $W_2$. 
• Let $V_i^{(1)}$ and $V_i^{(2)}$ denote the estimator of $V$ in the $i^{th}$ simulation using Methods 1 and 2, respectively, and their respective standard deviations are $\sigma_1$ and $\sigma_2$.

• The sample means for estimating $V$ from the two methods using $W_T$ amount of work are, respectively,

$$\frac{W_1}{W_T} \sum_{i=1}^{W_T/W_1} V_i^{(1)} \quad \text{and} \quad \frac{W_2}{W_T} \sum_{i=1}^{W_T/W_2} V_i^{(2)}.$$
• By the law of large numbers, the above two estimators are approximately normally distributed with mean $V$ and their respective standard deviations are

$$\sigma_1\sqrt{\frac{W_1}{W_T}} \quad \text{and} \quad \sigma_2\sqrt{\frac{W_2}{W_T}}.$$

• The first method would be preferred over the second one provided that

$$\sigma_1^2W_1 < \sigma_2^2W_2.$$

• Alternatively speaking, a lower variance estimator is preferred only if the variance ratio $\sigma_1^2/\sigma_2^2$ is less than the work ratio $W_2/W_1$, when the aspect of computational efficiency is taken into account.
Antithetic variates method

- Suppose \( \{\varepsilon^{(i)}\} \) denotes the independent samples from the standard normal distribution for the \( i \)th simulation run of the asset price path so that

\[
S_T^{(i)} = S_t e^{\left(\frac{r-\sigma^2}{2}\right)(T-t)+\sigma\sqrt{\Delta t} \sum_{j=1}^{N} \varepsilon_{j}^{(i)}}, \quad i = 1, 2, \ldots, M,
\]

where \( \Delta t = \frac{T-t}{N} \) and \( M \) is the total number of simulation runs. Note that \( \varepsilon_{j}^{(i)} \) is randomly sampled from the standard normal distribution.

- An unbiased estimator of the price of a European call option with strike price \( X \) is given by

\[
\hat{c} = \frac{1}{M} \sum_{i=1}^{M} c_i = \frac{1}{M} \sum_{i=1}^{M} e^{-r(T-t)} \max(S_T^{(i)} - X, 0).
\]
We observe that if \( \{ \epsilon^{(i)} \} \) has a standard normal distribution, so does \( \{-\epsilon^{(i)}\} \), and the simulated price \( \tilde{S}_T^{(i)} \) obtained using \( \{-\epsilon^{(i)}\} \) is also a valid sample from the terminal asset price distribution. A new unbiased estimator of the call price can be obtained from

\[
\tilde{c} = \frac{1}{M} \sum_{i=1}^{M} \tilde{c}_i = \frac{1}{M} \sum_{i=1}^{M} e^{-r(T-t)} \max(\tilde{S}_T^{(i)} - X, 0).
\]

Normally we would expect \( c_i \) and \( \tilde{c}_i \) to be negatively correlated, that is, if one estimate overshoots the true value, the other estimate downshoots the true value. It seems sensible to take the average of these two estimates. Indeed, we take the antithetic variates estimate to be

\[
\bar{c}_{AV} = \frac{\hat{c} + \tilde{c}}{2}.
\]
Control variate method

- The control variate method is applicable when there are two similar options, $A$ and $B$. Option $A$ is the one whose price value is desired, while option $B$ is similar to option $A$ in nature but its analytic price formula is available.

- Let $V_A$ and $V_B$ denote the true value of option $A$ and option $B$ respectively, and let $\hat{V}_A$ and $\hat{V}_B$ denote the respective estimated value of option $A$ and option $B$ using the Monte Carlo simulation.

- How does the knowledge of $V_B$ and $\hat{V}_B$ help improve the estimate of the value of option $A$ beyond the available estimate $\hat{V}_A$?
• The control variate method aims to provide a better estimate of the value of option $A$ using the formula

$$\hat{V}_{A}^{cv} = \hat{V}_{A} + (V_{B} - \hat{V}_{B}),$$

where the error $V_{B} - \hat{V}_{B}$ is used as a control in the estimation of $V_{A}$.

• Consider the following relation between the variances of the above quantities

$$\text{var}(\hat{V}_{A}^{cv}) = \text{var}(\hat{V}_{A}) + \text{var}(\hat{V}_{B}) - 2 \text{cov}(\hat{V}_{A}, \hat{V}_{B}),$$

so that

$$\text{var}(\hat{V}_{A}^{cv}) < \text{var}(\hat{V}_{A})$$

provided that $\text{var}(\hat{V}_{B}) < 2 \text{cov}(\hat{V}_{A}, \hat{V}_{B})$. 
• The control variate technique reduces the variance of the estimator of $V_A$ when the covariance between $\hat{V}_A$ and $\hat{V}_B$ is large. This is true when the two options are strongly correlated.

• In terms of computational efforts, we need to compute two estimates $\hat{V}_A$ and $\hat{V}_B$.

• However, if the underlying asset price paths of the two options are identical, then there is only slight additional work to evaluate $\hat{V}_B$ along with $\hat{V}_A$ on the same set of simulated price paths.
• To facilitate the more optimal use of the control $V_B - \hat{V}_B$, we define the control variate estimate to be

$$\hat{V}_A^\beta = \hat{V}_A + \beta (V_B - \hat{V}_B),$$

where $\beta$ is a parameter with value other than 1.

• The new relation between the variances is now given by

$$\text{var} \left( \hat{V}_A^\beta \right) = \text{var} \left( \hat{V}_A \right) + \beta^2 \text{var}(\hat{V}_B) - 2\beta \text{cov} \left( \hat{V}_A, \hat{V}_B \right).$$

• The particular choice of $\beta$ which minimizes $\text{var} \left( \hat{V}_A^\beta \right)$ is found to be

$$\beta^* = \frac{\text{cov}(\hat{V}_A, \hat{V}_B)}{\text{var}(\hat{V}_B)}.$$