MATH 571

Mathematical Models of Financial Derivatives

Homework One

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1. Consider a one-year forward contract whose underlying asset is a coupon paying bond with maturity date beyond the expiration date of the forward contract. Assume that the bond pays coupon semi-annually at the coupon rate of 8%, and the face value of the bond is \$100 (that is, each coupon payment is \$4). The current market price of the bond is \$94.6, and the previous coupon has just been paid. Taking the riskless interest rate to be at the constant value of 10%, find the forward price of this bond forward.

   Hint: The coupon payments may be considered as negative cost of carry.

2. (a) Why is the expected loss from the default on a swap less than the expected loss from the default on a loan with the same principal?

   (b) "A long forward contract subject to credit risk is a combination of a short position in a no-default put and a long position in a call subject to credit risk." Explain this statement.

3. Consider an interest rate swap of notional principal \$1 million and remaining life of 9 months, the terms of the swap specify that six-month LIBOR is exchanged for the fixed rate of 10% per annum (quoted with semi-annual compounding). The market prices of unit par zero coupon bonds with maturity dates 3 months and 9 months from now are \$0.972 and \$0.918, respectively, while the market price of unit par floating rate bond with maturity date 3 months from now is \$0.992. Find the value of the interest rate swap to the fixed-rate payer, assuming no default risk of the swap counterparty.

   Hint The discount factors over 3 months and 9 months from now are given by the zero coupon bond prices, and from which the implied 6-month LIBOR that is reset 3 months from now can be determined. The 6-month LIBOR payment to be paid 3 months from now has been fixed, whose value should be reflected in the price of the floating rate bond maturing 3 month from now.

4. Mr. Chan manages a pension fund and believes that his stock selection ability is excellent. However, he is worried because the market could go down. He considers entering an equity swap where each quarter i, up to quarter M, he pays counterparty B the previous quarter's total rate of return r_i on the S & P 500 index times some notional principal and receives payments at a fixed rate r on the same principal. The total rate of return includes dividends. Specifically, $1 + r_i = (S_i + d_i)/S_{i-1}$, where S_i and d_i are the values of the index at i and the dividends received from $i-1$ to i, respectively. Derive the value of such a swap by the following steps:

   (a) Let $V_{i-1}(S_i + d_i)$ denote the value function at time $i-1$ of receiving $S_i + d_i$ at time i. One may argue that

   $$V_{i-1}(S_i + d_i) = S_{i-1},$$

   use the result to find $V_{i-1}(r_i)$.

Hint: Note that $V_{i-1}(1) = d(i-1, i)$ and explain why

$$V_{i-1}(r_i) = 1 - d(i-1, i)$$

where $d(i-1, i)$ is the discount factor from $i-1$ to i.

(b) Find $V_0(r_i)$.

(c) Find $\sum_{i=1}^{M} V_0(r_i)$.

(d) Show that the value of the swap $= N\left[1 - d(0, M) - \frac{r}{4}\sum_{i=1}^{M} d(0, i)\right]$, where N is the swap notional.

5. Suppose the strike prices X_1 and X_2 satisfy $X_1 > X_2$, show that for European calls on a non-dividend paying asset, the difference in call values satisfies

$$B(\tau)(X_2 - X_1) \leq c(S, \tau; X_1) - c(S, \tau; X_2) \leq 0,$$

where $B(\tau)$ is the value of a pure discount bond with par value of unity and time to maturity τ. Furthermore, deduce that

$$-B(\tau) \leq \frac{\partial c}{\partial X}(S, \tau; X) \leq 0.$$

In other words, suppose the call price can be expressed as a differentiable function of the strike price, then the derivative must be non-positive and no greater in absolute value than the price of a pure discount bond of the same maturity. Do the above results also hold for European/American calls on a dividend paying asset?

6. Show that the put prices (European and American) are convex functions of the asset price, that is,

$$p(\lambda S_1 + (1-\lambda)S_2, X) \leq \lambda p(S_1, X) + (1-\lambda)p(S_2, X), \quad 0 \leq \lambda \leq 1,$$

where S_1 and S_2 denote the asset prices and X denotes the strike price.

Hint: Let $S_1 = h_1 X$ and $S_2 = h_2 X$, and note that the put price function is homogeneous of degree one in the asset price and the strike price, the above inequality can be expressed as

$$\begin{aligned}&[\lambda h_1 + (1-\lambda)h_2]p\left(X, \frac{X}{\lambda h_1 + (1-\lambda)h_2}\right)\\ &\leq \lambda h_1 p\left(X, \frac{X}{h_1}\right) + (1-\lambda)h_2 p\left(X, \frac{X}{h_2}\right).\end{aligned}$$

Use the result that the put prices are convex functions of the strike price.

7. Consider the following two portfolios:

Portfolio A: One European call option plus cash of the amount X.

Portfolio B: One American put option, one unit of the underlying asset minus cash of the amount D. The loan is in the form of a portfolio of bonds whose par values and dates of maturity match with the sizes and dates of the dividends.

Assume the underlying asset pays dividends and D denotes the present value of the dividends paid by the underlying asset during the life of the option. Show that if the American put is not exercised early, Portfolio B is worth $\max(S_T, X)$, which is less than the value of Portfolio A. Even when the American put is exercised prior to expiry, show that Portfolio A is always worth more than Portfolio B at the moment of exercise. Hence, deduce that

$$S - D - X < C - P.$$

Hint: $c < C$ for calls on a dividend paying asset and the cash amount in Portfolio A grows with time.

8. Show that the lower and upper bounds on the difference between the prices of American call and put options on a foreign currency are given by

$$SB_f(\tau) - X < C - P < S - XB(\tau),$$

where $B_f(\tau)$ and $B(\tau)$ are bond prices in the foreign and domestic currencies, respectively, both with par value of unity in the respective currency and time to maturity τ, S is the spot domestic currency price of one unit of foreign currency.

Hint: To show the left inequality, consider the values of the following two portfolios: the first one contains a European currency call option plus X dollars of domestic currency, the second portfolio contains an American currency put option plus $B_f(\tau)$ units of foreign currency. To show the right inequality, we choose the first portfolio to contain an American currency call option plus $XB(\tau)$ dollars of domestic currency, and the second portfolio to contain a European currency put option plus one unit of foreign currency.

9. Suppose the strike price is growing at the riskless interest rate, show that the price of an American put option is the same as that of its European counterpart. Find the value of an American put option when (i) strike price $X = 0$, (ii) asset price $S = 0$.

10. Let q and r denote the dividend yield of the asset and riskfree interest rate, respectively. Explain why, when $q \geq r$, an American call on a continuous dividend paying asset which is optimally held to expiration will have zero value at expiration.