MATH 571
Mathematical Models of Financial Derivatives

Homework Two

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1. Show that a dominant trading strategy exists if and only if there exists a trading strategy satisfying $V_0 < 0$ and $V_1(\omega) \geq 0$ for all $\omega \in \Omega$.

Hint: Consider the dominant trading strategy $\mathcal{H} = (h_0 \; h_1 \cdots h_M)^T$ satisfying $V_0 = 0$ and $V_1(\omega) > 0$ for all $\omega \in \Omega$. Take $G^*_{\min} = \min_{\omega} G^*(\omega) > 0$ and define a new trading strategy with $\widehat{h}_m = h_m, m = 1, \cdots, M$ and $\widehat{h}_0 = -G^*_{\min} - \sum_{m=1}^{M} h_m S^*_m(0)$.

2. Consider a portfolio with one risky security and the riskfree security. Suppose the price of the risky asset at time 0 is 4 and the possible values of the $t = 1$ price are $1.1, 2.2$ and 3.3 (3 possible states of the world at the end of a single trading period). Let the riskfree interest rate r be 0.1 and take the price of the riskfree security at $t = 0$ to be unity.
   (a) Show that the trading strategy: $h_0 = 4$ and $h_1 = -1$ is a dominant trading strategy that starts with zero wealth and ends with positive wealth with certainty.
   (b) Find the discounted gain G^* over the single trading period.
   (c) Find a trading strategy that starts with negative wealth and ends with non-negative wealth with certainty.

3. Show that if the law of one price does not hold, then every payoff in the asset span can be bought at any price.

4. Construct a securities model such that it satisfies the law of one price but admits a dominant trading strategy.

Hint: Construct a securities model where the null space of the payoff matrix $S^*(1)$ has zero dimension while the positivity of the linear measure does not hold.

5. Given the discounted terminal payoff matrix

$$\widehat{S}^*(1) = \begin{pmatrix} 1 & 6 & 3 \\ 1 & 2 & 2 \\ 1 & 12 & 6 \end{pmatrix},$$

and the current discounted price vector $\widehat{\mathbf{S}}^*(0) = (1 \quad 3 \quad 2)$, find the state price of the Arrow security with discounted payoff $\mathbf{e}_k, k = 1, 2, 3$. Do we observe positivity of the state prices? Does the securities model admit any arbitrage opportunities? If so, find one such example.

6. Define the pricing functional $F(\mathbf{x})$ on the asset span $\mathcal{S}$ by $F(\mathbf{x}) = \{y : y = \mathbf{S}^*(0)\mathbf{h}$ for some $\mathbf{h}$ such that $\mathbf{x} = S^*(1)\mathbf{h}$, where $\mathbf{x} \in \mathcal{S}\}$. Show that if the law of one price holds, then F is a *linear* functional.

7. Construct a securities model with 2 risky securities and the riskfree security and 3 possible states of world such that the law of one price holds but there are dominant trading strategies.

8. Suppose a betting game has 3 possible outcomes. If a gambler bets on outcome i, then he receives a net gain of d_i dollars for one dollar betted, $i = 1, 2, 3$. The payoff matrix thus takes the form (discounting is immaterial in a betting game)

$$S(1;\Omega) = \begin{pmatrix} d_1+1 & 0 & 0 \\ 0 & d_2+1 & 0 \\ 0 & 0 & d_3+1 \end{pmatrix}.$$

Find the condition on d_i such that a risk neutral probability measure exists for the above betting game (visualized as an investment model).

9. Consider the following securities model with discounted terminal payoff of the securities given by the payoff matrix

$$\widehat{S}^*(1;\Omega) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 1 & 5 & 6 & 7 \end{pmatrix},$$

where the first column gives the discounted payoff of the riskfree security. Let the initial price vector $\widehat{\mathbf{S}}^*(0)$ be $(1 \quad 3 \quad 5 \quad 9)$. Does the law of one price hold for this securities model? Show that the contingent claim with discounted payoff $\begin{pmatrix} 6 \\ 8 \\ 12 \end{pmatrix}$ is attainable and find the set of all possible trading securities that generate the payoff. Can we find the price at $t = 0$ of this contingent claim?

10. in this problem, we would like to derive the binomial formula using the *riskless hedging principle*. Suppose we have a call which is one period from expiry and would like to create a perfectly hedged portfolio with a long position of one unit of the underlying asset and a short position of m units of call. Let c_u and c_d denote the payoff of the call at expiry corresponding to the upward and downward movement of the asset price, respectively. Show that the number of calls to be sold short in the portfolio should be

$$m = \frac{S(u-d)}{c_u - c_d}$$

in order that the portfolio is perfectly hedged. The hedged portfolio should earn the risk-free interest rate. Let R denote the growth factor of the value of a perfectly hedged portfolio in one period. Show that the binomial option pricing formula for the call as deduced from the riskless hedging principle is given by

$$c = \frac{pc_u + (1-p)c_d}{R} \quad \text{where} \quad p = \frac{R-d}{u-d}.$$

11. Let Π_u and Π_d denote the state price corresponding to the state of the asset value going up and going down, respectively. The state prices can also be interpreted as state contingent discount rates. Assuming absence of arbitrage opportunities, all securities (including the money market account, asset and call option) would have returns with the same state contingent discount rates Π_u and Π_d. Hence, the respective relations for the money market account, asset price and call option value with Π_u and Π_d are given by

$$\begin{aligned} 1 &= \Pi_u R + \Pi_d R \\ S &= \Pi_u uS + \Pi_d dS \\ c &= \Pi_u c_u + \Pi_d c_d. \end{aligned}$$

By solving for Π_u and Π_d from the first two equations and substituting the solutions into the third equation, show that the binomial call price formula over one period is given by

$$c = \frac{pc_u + (1-p)c_d}{R} \quad \text{where} \quad p = \frac{R-d}{u-d}.$$