MATH 571
Mathematical Models of Financial Derivatives

Homework Three

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1. Consider the sample space $\Omega = \{-3, -2, -1, 1, 2, 3\}$ and the algebra $\mathcal{F} = \{\phi, \{-3, -2\}, \{-1, 1\}, \{2, 3\}, \{-3, -2, -1, 1\}, \{-3, -2, 2, 3\}, \{-1, 1, 2, 3\}, \Omega\}$. For each of the following random variables, determine whether it is $\mathcal{F}$-measurable:

   (i) $X(\omega) = \omega^2$, (ii) $X(\omega) = \max(\omega, 2)$.

   Find a random variable that is $\mathcal{F}$-measurable.

2. Let $X, X_1, \cdots, X_n$ be random variables defined on $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_1 \subset \mathcal{F}_2$ are sub-algebras of $\mathcal{F}$. Prove the following properties on conditional expectations:

   (a) $E[XI_B] = E[I_B E[X|\mathcal{F}]]$ for all $B \in \mathcal{F}$,

   (b) $E[\max(X_1, \cdots, X_n)|\mathcal{F}] \geq \max(E[X_1|\mathcal{F}], \cdots, E[X_n|\mathcal{F}])$.

3. Let $X = \{X_t; t = 0, 1, \cdots, T\}$ be a stochastic process adapted to the filtration $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \cdots, T\}$. Does the property: $E[X_{t+1} - X_t|\mathcal{F}_t] = 0, t = 0, 1, \cdots, T-1$ imply that X is a martingale?

4. Consider the binomial experiment with probability of success $p, 0 < p < 1$. We let N_k denote the number of successes after k independent trials. Define the discrete process Y_k by $N_k - kp$, the excess number of successes above the mean kp. Show that Y_k is a martingale.

5. Consider the two-period securities model in the lecture note of Topic 3, p.15. Suppose the riskless interest rate r violates the restriction $r < 0.2$, say, $r = 0.3$. Construct an arbitrage opportunity associated with the securities model.

6. Deduce the price formula for a European put option with terminal payoff $\max(X - S, 0)$ for the n-period binomial model.