MATH 571 — Mathematical Models of Financial Derivatives

Topic 1 – Introduction to Derivative Instruments

1.1 Basic derivative instruments: bonds, forward contracts, swaps and options

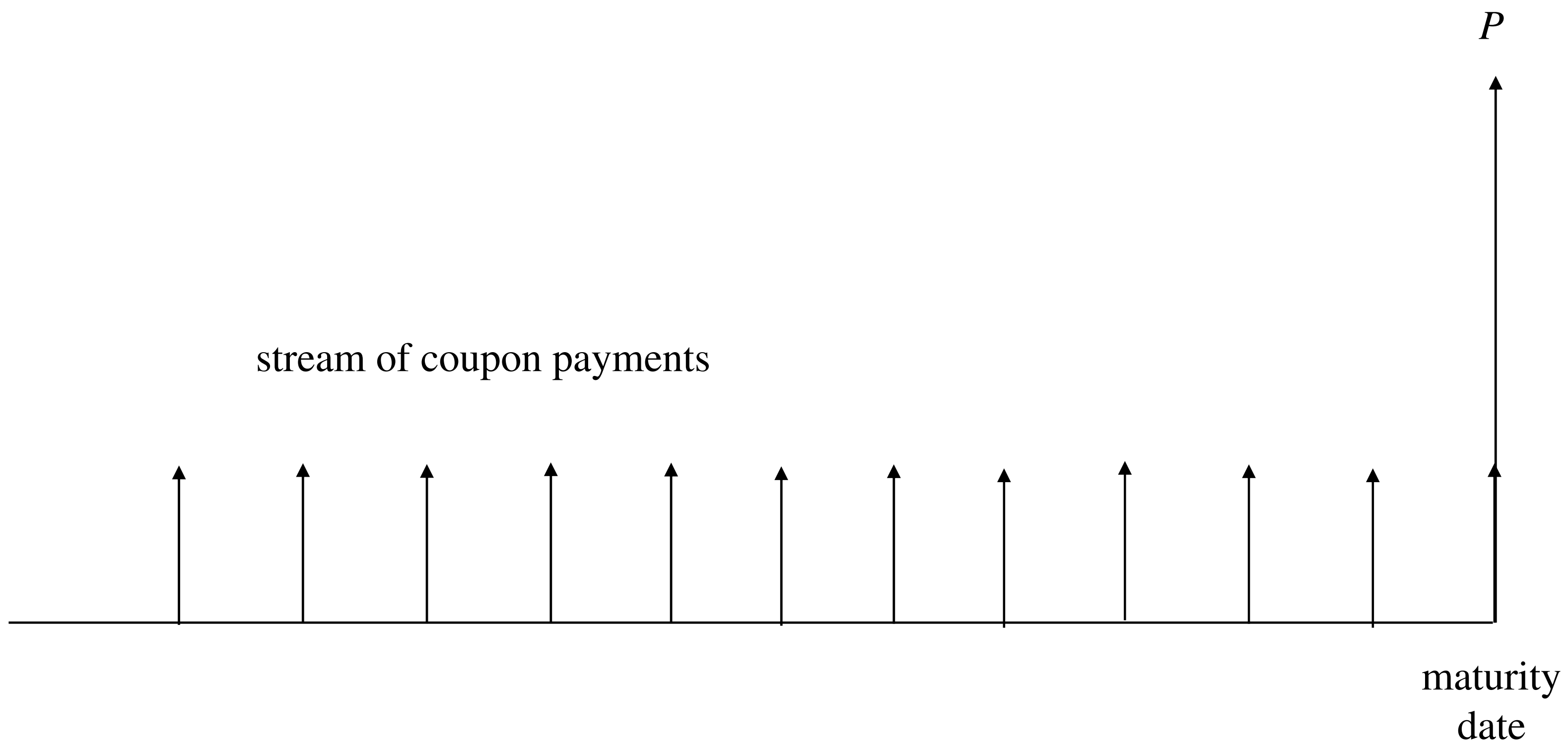
1.2 Rational boundaries of option values

1.3 Early exercise policies of American options

1.1 Basic derivative instruments: bonds, forward contracts, swaps and options

A bond is a *debt instrument* requiring the issuer to repay to the lender/investor the amount borrowed (par or face value) plus interests over a specified period of time.

Specify (i) the maturity date when the principal is repaid;
(ii) the coupon payments over the life of the bond



- The *coupon rate* offered by the bond issuer represents the *cost of raising capital*. It depends on the prevailing risk free interest rate and the creditworthiness of the bond issuer. It is also affected by the values of the embedded options in the bond, like conversion right in convertible bonds.

- Assume that the bond issuer does not *default* or *redeem* the bond prior to maturity date, an investor holding this bond until maturity is assured of a *known* cash flow pattern. This is why bond products are also called *Fixed Income Products/Derivatives*.

  Pricing of a bond

  Based on the current information of the interest rates (yield curve) and the embedded option provisions, find the cash amount that the bond investor should pay at the current time so that the deal is fair to both counterparties.

Features in bond indenture

1. *Floating rate bond*

   The coupon rates are reset periodically according to some pre-determined financial benchmark, like LIBOR + spread, where LIBOR is the LONDON INTER-BANK OFFERED RATE.

2. Amortization feature – principal repaid over the life of the bond.

3. Callable feature (callable bonds)

   The issuer has the *right to buy back* the bond at a specified price. Usually this call price falls with time, and often there is an initial call protection period wherein the bond cannot be called.

4. Put provision – grants the bondholder the right to sell back to the issuer at par value on designated dates.

5. Convertible bond – gives the bondholder the right to *exchange the bond* for a *specified number of shares* of the issuer's firm.

   ⋆ Bond holders can take advantage of the future growth of the issuer's company.

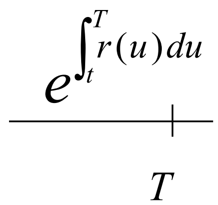
   ⋆ Issuer can raise capital at a lower cost.

6. Exchangeable bond – allows the bondholder to exchange the bond par for a specified number of common stocks of another corporation.

Short rate

Let $r(t)$ denote the short rate, which is in general stochastic. This is the interest rate that is applied over the next infinitesimal Δt time interval. The short rate is a mathematical construction, not a market reality.

Money market account: $M(t)$



You put \$1 at time t and let it earn interest at the rate $r(t)$ *continuously* over the period (t, T). Governing differential equation:

$$dM(t) = r(t)M(t)\, dt.$$

$$\int_t^T \frac{dM(u)}{M(u)} = \int_t^T r(u)\,du$$

so that

$$M(T) = M(t)e^{\int_t^T r(u)\,du}.$$

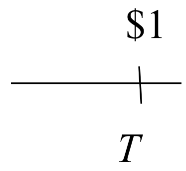
Here, $e^{\int_t^T r(u)\,du}$ is seen to be the growth factor of the money market account. If r is constant, then

$$\text{growth factor } = e^{r\tau}, \quad \tau = T - t.$$

If $r(t)$ is stochastic, then $M(T)$ is also stochastic.

The reciprocal of the growth factor is called the discount factor $e^{-\int_t^T r(u)\,du}$.

Discount bond price



$$\tau = T - t = \text{ time to bond's maturity}$$

The price that an investor on the zero-coupon (discount) bond with unit par is willing to pay at time t if the bond promises to pay him back $1 at a later time (maturity date) T.

This fair value is called the discount bond price $B(t,T)$, which is given by the expectation of the discount factor based on current information: $E_t\left[e^{-\int_t^T r(u)\,du}\right]$. If r is constant, then $B(t,T) = e^{-r\tau}, \tau = T - t$.

Forward contract 遠期合約

The buyer of the forward contract agrees to pay the delivery price K dollars at future time T to purchase a commodity whose value at time T is S_T. The pricing question is how to set K?

How about

$$E[\exp(-rT)(S_T - K)] = 0$$

so that $K = E[S_T]$?

This is the *expectation pricing* approach, which cannot enforce a price. When the expectation calculation $E[S_T]$ is performed, the distribution of the asset price process comes into play.

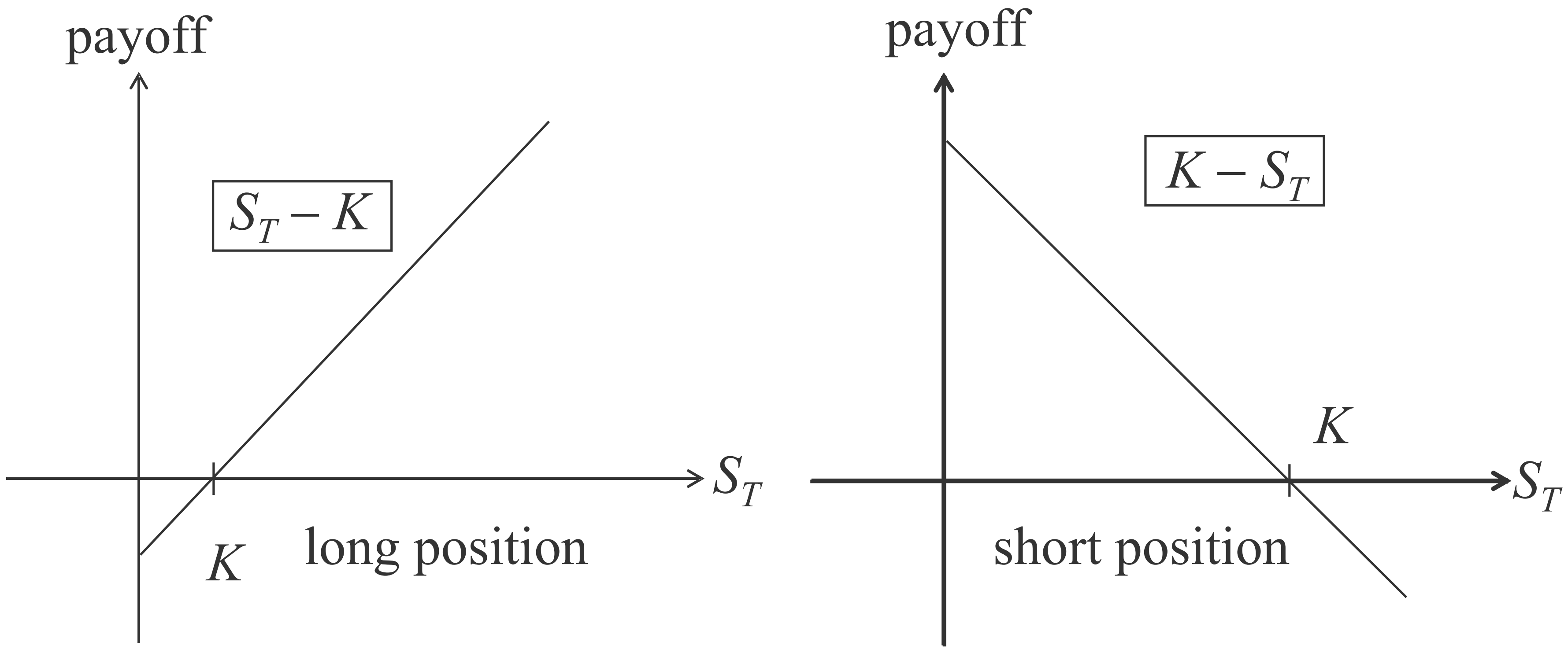
Objective of the buyer:

To hedge against the price fluctuation of the underlying commodity.

- Intension of a purchase to be decided earlier, actual transaction to be done later.
- The forward contract needs to specify the delivery price, amount, quality, delivery date, means of delivery, etc.

Potential default of either party (counterparty risk): writer or buyer.

Terminal payoff from a forward contract



K = delivery price, S_T = asset price at maturity

Zero-sum game between the writer (short position) and buyer (long position).

Forward contracts have been extended to include underlying assets other than physical commodities, e.g. on exchange rate of foreign currency or interest rate instrument (like bonds) or stock index.

Hang Seng index futures 恒生指數期貨

- The underlying is the Hang Seng index.
- One index point corresponds to $50.

Settlement

- On the last but one trading day at the end of the month.
- Take the average of the index value at 5-minute intervals as the settlement value.

Is the forward price related to the expected price of the commodity on the delivery date? Provided that the underlying asset can be held for hedging by the writer, then

$$\begin{aligned} & \text{forward price} \\ = \; & \text{spot price} + \underbrace{\text{cost of fund} + \text{storage cost}}_{\text{cost of carry}} \end{aligned}$$

- Cost of fund is the interest accrued over the period of the forward contract.

- Cost of carry is the total cost incurred to acquire and hold the underlying asset, say, including the cost of fund and storage cost.

- Dividends paid to the holder of the asset are treated as negative contribution to the cost of carry.

Numerical example on arbitrage

– spot price of oil is US$19
– quoted 1-year forward price of oil is US$25
– 1-year US dollar interest rate is 5% pa
– storage cost of oil is 2% per annum, paid at maturity

Any arbitrage opportunity? *Yes*

Sell the forward and expect to receive US$25 one year later. Borrow \$19 now to acquire oil, pay back $\$19(1+0.05) = \$19.95$ a year later. Also, one needs to spend $\$0.38 = \$19 \times 2\%$ as the storage cost.

$$\begin{aligned} & \text{total cost of replication (dollar value at maturity)} \\ = \; & \text{spot price} + \text{cost of fund} + \text{storage cost} \\ = \; & \$20.33 < \$25 \text{ to be received.} \end{aligned}$$

Close out all positions by delivering the oil to honor the forward. At maturity of the forward contract, guaranteed riskless profit = \$4.67.

Value and price of a forward contract

Let $f(S, \tau)$ = value of forward, $F(S, \tau)$ = forward price,

$$\begin{aligned} \tau &= \text{time to expiration,} \\ S &= \text{spot price of the underlying asset.} \end{aligned}$$

Further, we let

$$B(\tau) = \text{value of an unit par discount bond with time to maturity } \tau$$

- If the interest rate r is constant and interests are compounded continuously, then $B(\tau) = e^{-r\tau}$.

- Assuming no dividend to be paid by the underlying asset and no storage cost.

We construct a "static" replication of the forward contract by a portfolio of the underlying asset and bond.

Portfolio A: long one forward and a discount bond with par value K

Portfolio B: one unit of the underlying asset

Both portfolios become one unit of asset at maturity. Let $\Pi_A(t)$ denote the value of Portfolio A at time t. Note that $\Pi_A(T) = \Pi_B(T)$. By the "law of one price",*, we must have $\Pi_A(t) = \Pi_B(t)$. The forward value is given by

$$f = S - KB(\tau).$$

The forward price is defined to be the delivery price which makes $f = 0$, so $K = S/B(\tau)$. Hence, the forward price is given by

$$F(S, \tau) = S/B(\tau) = \text{ spot price + cost of fund.}$$

*Suppose $\Pi_A(t) > \Pi_B(t)$, then an arbitrage can be taken by selling Portfolio A and buying Portfolio B. An upfront positive cash flow is resulted at time t but the portfolio values are offset at maturity T.

Discrete dividend paying asset

$D =$ present value of all dividends received from holding the asset during the life of the forward contract.

We modify Portfolio B to contain one unit of the asset plus borrowing of D dollars. The loan of D dollars will be repaid by the dividends received by holding the asset. We then have

$$f + KB(\tau) = S - D$$

so that

$$f = S - [D + KB(\tau)].$$

Setting $f = 0$ to solve for K, we obtain $F = (S - D)/B(\tau)$.

The "net" asset value is reduced by the amount D due to the anticipation of the dividends. Unlike holding the asset, the holder of the forward will not receive the dividends. As a fair deal, he should pay a lower delivery price at forward's maturity.

Cost of carry

Additional costs to hold the commodities, like storage, insurance, deterioration, etc. These can be considered as negative dividends. Treating U as $-D$, we obtain

$$F = (S + U)e^{r\tau},$$

$U =$ present value of total cost incurred during the remaining life of the forward to hold the asset.

Suppose the costs are paid continuously, we have

$$F = Se^{(r+u)\tau},$$

where $u =$ cost per annum as a proportion of the spot price.

In general, $F = Se^{b\tau}$, where b is the cost of carry per annum. Let q denote the continuous dividend *yield* per annum paid by the asset. With both continuous holding cost and dividend yield, the cost of carry $b = r + u - q$.

Forward price formula with discrete carrying costs

Suppose an asset has a holding cost of $c(k)$ per unit in period k, and the asset can be sold short. Suppose the initial spot price is S. The theoretical forward price F is

$$F = \frac{S}{d(0, M)} + \sum_{k=0}^{M-1} \frac{c(k)}{d(k, M)},$$

where $d(0, k)d(k, M) = d(0, M)$.

The terms on the right hand side represent the future value at maturity of the total costs required for holding the underlying asset for hedging. Note that holding costs can be visualized as negative dividends.

Proportional carrying charge

Forward contract written at time 0 and there are M periods until delivery. The carrying charge in period k is $qS(k-1)$, where q is a proportional constant. Show that the *forward price* is

$$F = \frac{S(0)/(1-q)^M}{d(0,M)}.$$

- We expect the forward price increases when the carrying charges become higher.

Borrow $\alpha S(0)$ dollars at current time to buy α units of assets and long one forward. Here, α is to be determined.

Sell out q portion of asset at each period in order to pay for the carrying charge. After M period, α units becomes $\alpha(1-q)^M$.

The goal is to make available one unit of the asset for delivery at maturity. We set $\alpha(1-q)^M$ to be one and obtain $\alpha = \dfrac{1}{(1-q)^M}$.

The portfolio of longing the forward with delivery price K and a bond with par K is equivalent to long α units of the asset. This gives

$$f + Kd(0, M) = \alpha S(0).$$

By setting $f = 0$ to obtain the forward price K, we obtain

$$K = \left. \frac{S(0)}{(1-q)^M} \right/ d(0, M).$$

Futures contracts 期貨合約

A futures contract is a legal agreement between a buyer (seller) and an established exchange or its clearing house in which the buyer (seller) agrees to take (make) delivery of a financial entity at a specified price at the end of a designated period of time. Usually the exchange specifies certain standardized features.

Mark to market the account

Pay or receive from the writer the change in the futures price through the margin account so that payment required on the maturity date is simply the spot price on that date.

Credit risk is limited to one-day performance period

Roles of the clearinghouse

- Eliminate the **counterparty risk** through the margin account.

- Provide the **platform** for parties of a futures contract to unwind their position prior to the settlement date.

  Margin requirements

  Initial margin – paid at inception as deposit for the contract.

  Maintenance margin – minimum level before the investor is required to deposit additional margin.

Example (Margin)

Suppose that Mr. Chan takes a long position of one contract in corn ($5,000$ kilograms) for March delivery at a price of \$2.10 (per kilogram). And suppose the broker requires margin of \$800 with a maintenance margin of \$600.

- The next day the price of this contract drops to \$2.07. This represents a loss of $0.03 \times 5,000 = \$150$. The broker will take this amount from the margin account, leaving a balance of \$650. The following day the price drops again to \$2.05. This represents an additional loss of \$100, which is again deducted from the margin account. As this point the margin account is \$550, which is below the maintenance level.

- The broker calls Mr. Chan and tells him that he must deposit at least \$250 in his margin account, or his position will be closed out.

Difference in payment schedules may lead to difference in futures and forward prices since different interest rates are applied on intermediate payments.

Equality of futures and forward prices under constant interest rate

Let F_i and G_i denote the forward price and futures price at the end of the i^{th} day, respectively, $\delta =$ constant interest rate per day

Gain/loss of futures on the i^{th} day $= G_i - G_{i-1}$ and this amount will grow to $(G_i - G_{i-1})e^{\delta(n-i)}$ at maturity.

Suppose the investor keeps changing the amount of futures held, say, α_i units at the end of the i^{th} day. Recall that it costs nothing for him to enter into a futures. The accumulated value on the maturity day is given by

$$\sum_{i=1}^{n} \alpha_i (G_i - G_{i-1}) e^{\delta(n-i)}.$$

Portfolio A: long a bond with par F_0 maturing on the n^{th} day
long one unit of forward contract with delivery price F_0

Portfolio B: long a bond with par G_0 maturity on the n^{th} day
long $\alpha_i = e^{-\delta(n-i)}$ units of futures on the i^{th} day

$$
\begin{aligned}
\text{value of portfolio } A &= F_0 + S_n - F_0 = S_n = \text{ asset price on the } n^{\text{th}} \text{ day} \\
\text{value of portfolio } B &= G_0 + \sum_{i=1}^{n} e^{-\delta(n-i)}(G_i - G_{i-1})e^{\delta(n-i)} \\
&= G_0 + \sum_{i=1}^{n} [G_i - G_{i-1}] = G_0 + G_n - G_0 = G_n.
\end{aligned}
$$

Note that $G_n = S_n$ since futures price = asset price at maturity.

Both portfolios have the same value at maturity so they have the same value at initiation. Recall that the initial value of a forward and a futures are both zero, thus we obtain $F_0 = G_0$.

Proposition 1

Consider an asset with a price $\widetilde{S}_T$ at time T. The futures price of the asset, $G_{t,T}$, is the time-t spot price of an asset which has a payoff

$$\frac{\widetilde{S}_T}{B_{t,t+1}\widetilde{B}_{t+1,t+2}\cdots\widetilde{B}_{T-1,T}}$$

at time T. Note that quantities with "tilde" at top indicate stochastic variables.

Proof We start with $\frac{1}{B_{t,t+1}}$ long futures contracts at time t. The gain/loss from the futures position day τ earns/pays the overnight rate $\frac{1}{\widetilde{B}_{\tau,\tau+1}}$. Also, invest $G_{t,T}$ in a one-day risk free bond and roll the cash position over on each day at the one-day rate. The investment of $G_{t,T}$ is equivalent to the price paid to acquire the asset.

As an illustrative example, take $t = 0$ and $T = 3$.

1. Take $1/B_{0,1}$ long futures at $t = 0$;

   $1/B_{0,1}B_{1,2}$ long futures at $\tau = 1$;

   $1/B_{0,1}B_{1,2}B_{2,3}$ long futures at $\tau = 2$.

2. Invest $G_{0,3}$ in one-day risk free bond and roll over the net cash position

Time	Profits from futures	Bond position	Net Position
0	—	$G_{0,3}$	$G_{0,3}$
1	$\frac{1}{B_{0,1}}(G_{1,3} - G_{0,3})$	$\frac{G_{0,3}}{B_{0,1}}$	$\frac{G_{1,3}}{B_{0,1}}$
2	$\frac{1}{B_{0,1}B_{1,2}}(G_{2,3} - G_{1,3})$	$\frac{G_{1,3}}{B_{0,1}B_{1,2}}$	$\frac{G_{2,3}}{B_{0,1}B_{1,2}}$
3	$\frac{1}{B_{0,1}B_{1,2}B_{2,3}}(G_{3,3} - G_{2,3})$	$\frac{G_{2,3}}{B_{0,1}B_{1,2}B_{2,3}}$	$\frac{G_{3,3}}{B_{0,1}B_{1,2}B_{2,3}} = \frac{S_3}{B_{0,1}B_{1,2}B_{2,3}}$

Note that $G_{3,3} = S_3$.

Proposition 2

Consider an asset with a price $\widetilde{S}_T$ at time T. The forward price of the asset, $F_{t,T}$, is the time-t spot price of an asset which has a payoff $\widetilde{S}_T/B_{t,T}$ at time T.

Remark

1. The importance of these propositions stems from the observation that they turn futures price and forward price into the price of a physical asset that could exist.

2. When the interest rates are deterministic, we have $G_{t,T} = F_{t,T}$. This is a sufficient condition for equality of the two prices. The necessary and sufficient condition is that the discount process and the underlying price process are uncorrelated.

Pricing issues

We consider the discrete-time model and assume the existence of a risk neutral pricing measure Q. Based on the risk neutral valuation principle, the time-t price of a security is given by discounted expectation, where

$$\begin{aligned} G_{t,T} &= E_Q\left[B_{t,t+1}\widetilde{B}_{t+1,t+2}\cdots\widetilde{B}_{T-1,T}\frac{\widetilde{S}_T}{B_{t,t+1}\widetilde{B}_{t,t+1}\widetilde{B}_{t+1,t+2}\cdots\widetilde{B}_{T-1,T}}\right] \\ &= E_Q[\widetilde{S}_T]. \end{aligned}$$

The result remains valid for the continuous-time counterpart. For the forward price, it has been shown that

$$F_{t,T} = \frac{S_t}{B_{t,T}}.$$

Recall the use of the forward pricing measure Q^T where the discount bond price is used as the numeraire, it is well known that

$$F_{t,T} = \frac{S_t}{B_{t,T}} = E_{Q^T}\left[\frac{\widetilde{S}_T}{B_{T,T}}\right] = E_{Q^T}[\widetilde{S}_T].$$

Difference in futures price G_t and forward price F_t

- When physical holding of the underlying index (say, snow fall amount) for hedging is infeasible, then the buyer sets

$$\text{forward price } = E_P[S_T],$$

where P is the subjective probability measure of the buyer.
- When physical holding is possible and there is no margin requirement (static replication), then

$$\text{forward price } = E_{Q^T}[S_T],$$

where Q^T is the forward measure that uses the discount bond price as the numeraire.
- When physical holding of the asset is subject to daily settlement through the margin requirement (dynamic rebalancing)

$$\text{futures price } = E_Q[S_T],$$

where Q is the risk neutral measure that uses the money market account as the numeraire.

From the Numeraire Invariance Theorem, we have

$$\begin{aligned}\frac{S_t}{M_t} &= E_Q\left[\frac{S_T}{M_T}\middle|\mathcal{F}_t\right] \Leftrightarrow S_t = E_Q\left[e^{-\int_t^T r(u)\,du} S_T|\mathcal{F}_t\right]\\ \frac{S_t}{B(t,T)} &= E_{Q^T}\left[\frac{S_T}{B(T,T)}\middle|\mathcal{F}_t\right] = E_{Q^T}[S_T|\mathcal{F}_t]\end{aligned}$$

so that

$$\begin{aligned}G_t - F_t &= E_Q\left[S_T|\mathcal{F}_t\right] - \frac{S_t}{B(t,T)}\\ &= \frac{E_Q[S_T|\mathcal{F}_t]E_Q\left[e^{-\int_t^T r(u)\,du}\middle|\mathcal{F}_t\right] - E_Q\left[e^{-\int_t^T r(u)\,du}S_T\middle|\mathcal{F}_t\right]}{B(t,T)}\\ &= -\frac{\text{cov}_Q\left[e^{-\int_t^T r(u)\,du}, S_T\middle|\mathcal{F}_t\right]}{B(t,T)}.\end{aligned}$$

Hence, the spread becomes zero when the discount process and the price process of the underlying asset are uncorrelated under the risk neutral measure Q.

Example

A particular stock index futures expires on March 15. On March 13, the contract is selling at 353.625. Assume that there is a forward contract that also expires on March 15. The current forward price is 353. The risk-free interest rate is 5 percent per year, and you can assume that this interest rate will remain in effect for the next two days.

a. Identify the existence of an arbitrage opportunity. What transactions should be executed on March 13?

   Buy the low-priced forward and short the high-priced futures.

b. On March 14 the futures price is 353.35, and on March 15 the futures expires at 350.125. Show that the arbitrage works.

Date	March 13	March 14	March 15
futures price	353.625	353.35	350.125
Gain (shorting futures)	—	0.275	3.225

Gain on shorting futures $= 0.275 \times e^{0.05/365} + 3.225 \approx 3.5$.

Gain on longing forward $= 350.125 - 353 = -2.875$.

Arbitrage profit $= 3.5 - 2.875 = 0.625$.

c. Given your answer in part b, what effect will this have on market prices?

The low-priced forward increases in price while the high-priced futures drops in price. The two prices converge.

Index futures arbitrage

If the delivery price is higher than the no-arbitrage price, arbitrage profits can be made by buying the basket of stocks that is underlying the stock index and shorting the index futures contract.

Difficulties in actual implementation and associated costs

1. Require significant amount of capital e.g. Shorting 2,000 Hang Seng index futures requires the purchase of 2.5 billion worth of stock (based on Hang Seng Index of $25,000$ and \$50 for each index point).

2. Timing risk

   Stock prices move quickly, there exist time lags in the buy-in and buy-out processes.

3. Settlement

   The unwinding of the stock positions must be done in 47 steps on the settlement date.

4. Stocks must be bought or sold in board lot. One can only approximate the proportion of the stocks in the index calculation formula.

5. Dividend amounts and dividend payment dates of the stocks are uncertain. Note that dividends cause the stock price to drop and affect the index value.

6. Transaction costs in the buy-in and buy-out of the stocks; and interest losses in the margin account.

Currency forward

The underlying is the exchange rate X, which is the domestic currency price of one unit of foreign currency.

$$\begin{aligned} r_d &= \text{constant domestic interest rate} \\ r_f &= \text{constant foreign interest rate} \end{aligned}$$

Portfolio A: Hold one currency forward with delivery price K and a domestic bond of par K maturing on the delivery date of forward.

Portfolio B: Hold a foreign bond of unit par maturing on the delivery date of forward.

Let $\Pi_A(t)$ and $\Pi_A(T)$ denote the value of Portfolio A at time t and T, respectively.

Remark

Exchange rate is not a tradable asset. However, it comes into existence when we convert the price of the foreign bond (tradeable asset) from the foreign currency world into the domestic currency world.

On the delivery date, the holder of the currency forward has to pay K domestic dollars to buy one unit of foreign currency. Hence, $\Pi_A(T) = \Pi_B(T)$, where T is the delivery date.

Using the law of one price, $\Pi_A(t) = \Pi_B(t)$ must be observed at the current time t.

Note that

$$B_d(\tau) = e^{-r_d\tau}, B_f(\tau) = e^{-r_f\tau},$$

where $\tau = T - t$ is the time to expiry. Let f be the value of the currency forward in domestic currency,

$$f + KB_d(\tau) = XB_f(\tau),$$

where $XB_f(\tau)$ is the value of the foreign bond in domestic currency.

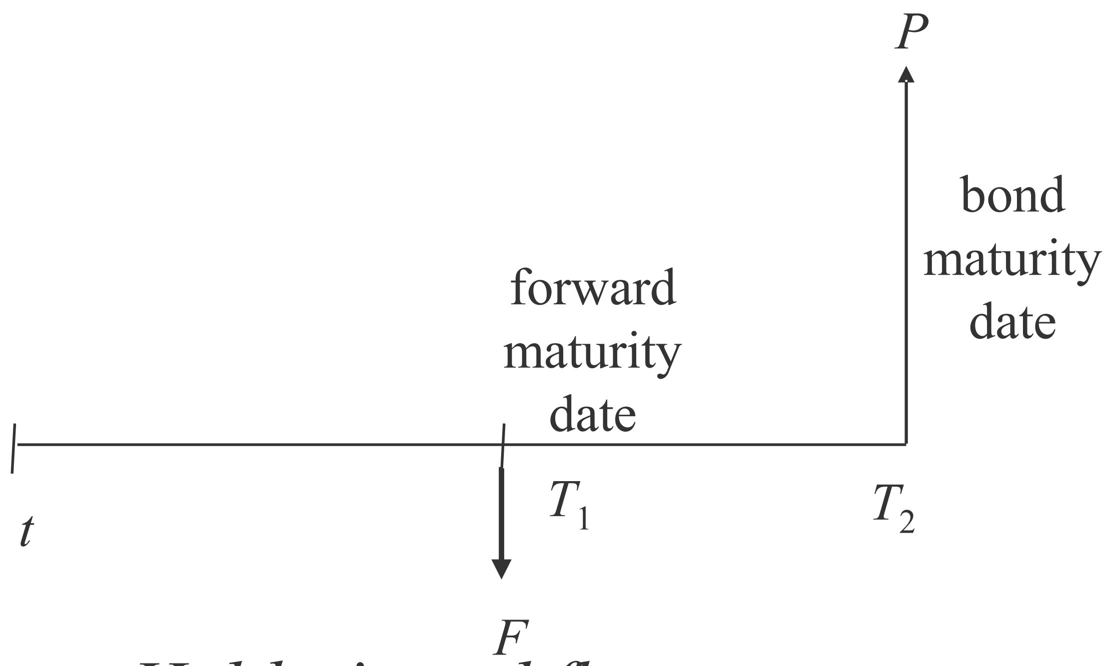
By setting $f = 0$,

$$K = \frac{XB_f(\tau)}{B_d(\tau)} = Xe^{(r_d - r_f)\tau}.$$

We may recognize r_d as the cost of fund and r_f as the dividend yield. This result is the well known *Interest Rate Parity Relation*.

Bond forward

The underlying asset is a zero-coupon bond of maturity T_2 with a settlement date T_1, where $t < T_1 < T_2$.



Holder's cashflows

The holder pays the delivery price F of the bond forward on the forward maturity date T_1 to receive a bond with par value P on the maturity date T_2.

Bond forward price in terms of traded bond prices

Let $B_t(T)$ denote the traded price of unit par discount bond at current time t with maturity date T.

$$
\begin{aligned}
& \text{Present value of the net cashflows} \\
= \; & -FB_t(T_1) + PB_t(T_2).
\end{aligned}
$$

To determine the forward price F, we set the above value zero and obtain

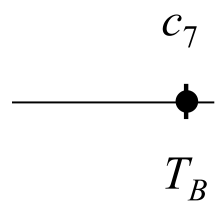
$$F = PB_t(T_2)/B_t(T_1).$$

Here, $PB_t(T_2)$ can be visualized as the spot price of the discount bond. The forward price is given in terms of the known market bond prices observed at time t with maturity dates T_1 and T_2.

Forward on a coupon-paying bond

The underlying is a coupon-paying bond with maturity date T_B.

Note that the bond is a traded security whose value changes with respect to time.



Let T_F be the delivery date of the bond forward, where $T_F < T_B$. Let t_i be the coupon payment date of the bond on which deterministic coupon c_i is paid. Let t be the current time, where $t < T_F < T_B$. Some of the coupons have been paid at earlier times. Let F be the forward price, the amount paid by the forward contract holder at time T_F to buy the bond.

Based on the forward price formula: $F = \frac{S-D}{B(\tau)}$, we deduce that

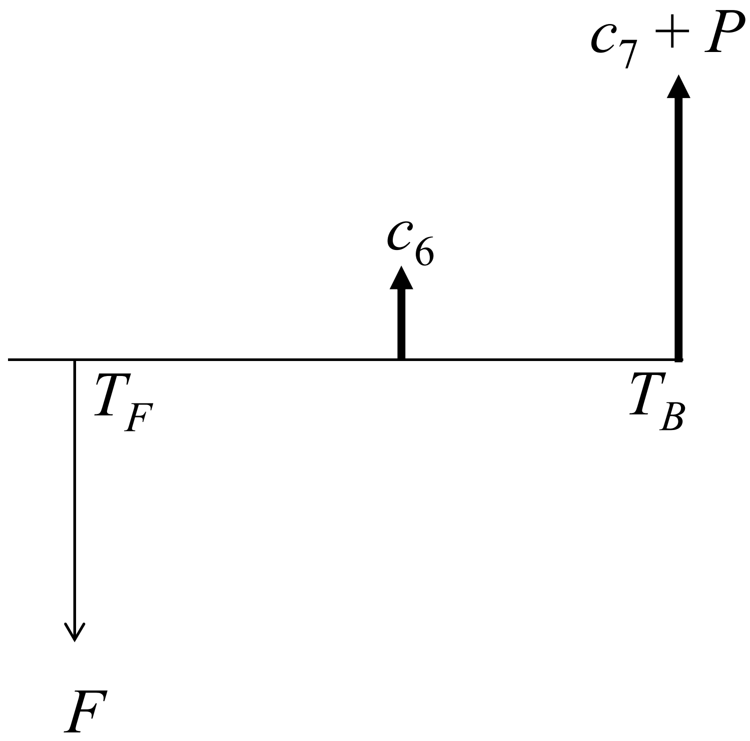
$$F = \frac{\text{spot price of bond}}{B_t(T_F)} - \frac{c_4 B_t(t_4)}{B_t(T_F)} - \frac{c_5 B_t(t_5)}{B_t(T_F)}.$$

Let P be the par value of the bond. After receiving the bond at T_F, the bond forward holder is entitled to receive c_6, c_7 and P once he has received the underlying bond. By considering the cash flows after T_F, he pays F at T_F and receives c_6 at t_6, $c_7 + P$ at T_B.

$$\begin{aligned} & \text{Present value at time } t \\ = \; & -FB_t(T_F) + c_6 B_t(t_6) + c_7 B_t(T_B) + PB_t(T_B). \end{aligned}$$

Hence, the bond forward price

$$F = \frac{c_6 B_t(t_6) + c_7 B_t(T_B) + PB_t(T_B)}{B_t(T_F)}.$$



At T_B, the bondholder receives par plus the last coupon.

Example — Bond forward

- A 10-year bond is currently selling for \$920.

- Currently, hold a forward contract on this bond that has a delivery date in 1 year and a delivery price of \$940.

- The bond pays coupons of \$80 every 6 months, with one due 6 months from now and another just before maturity of the forward.

- The current interest rates for 6 months and 1 year (compounded semi-annually) are 7% and 8%, respectively (annual rates compounded every 6 months).

- What is the current value of the forward?

Let $d(0,k)$ denote the discount factor over the $(0,k)$ semi-annual period. Consider the future value of the cash flows associated with holding the bond one year later and payment of F_0 under the forward contract. The current forward price of the bond

$$
\begin{aligned}
F_0 &= \frac{\text{spot price}}{d(0,2)} - \frac{c(1)d(0,1)}{d(0,2)} - \frac{c(2)d(0,2)}{d(0,2)} \\
&= 920(1.04)^2 - \frac{80(1.04)^2}{1.035} - \frac{80(1.04)^2}{(1.04)^2} = 831.47.
\end{aligned}
$$

The difference in the forward prices is discounted to the present value. The current value of the forward contract $= \frac{831.47-940}{(1.04)^2} =$ -100.34.

Implied forward interest rate

The forward price of a forward on a discount bond should be related to the implied forward interest rate $R(t;T_1,T_2)$. The implied forward rate is the interest rate over $[T_1,T_2]$ as implied by time-t discount bond prices. The bond forward buyer pays F at T_1 and receives P at T_2 and she is expected to earn $R(t;T_1,T_2)$ over $[T_1,T_2]$, so

$$F[1+R(t;T_1,T_2)(T_2-T_1)] = P.$$

Together with

$$F = PB_t(T_2)/B_t(T_1),$$

we obtain

$$R(t;T_1,T_2) = \frac{1}{T_2-T_1}\left[\frac{B_t(T_1)}{B_t(T_2)} - 1\right].$$

Forward rate agreement (FRA)

The FRA is an agreement between two counterparties to exchange floating and fixed interest payments on the future settlement date T_2.

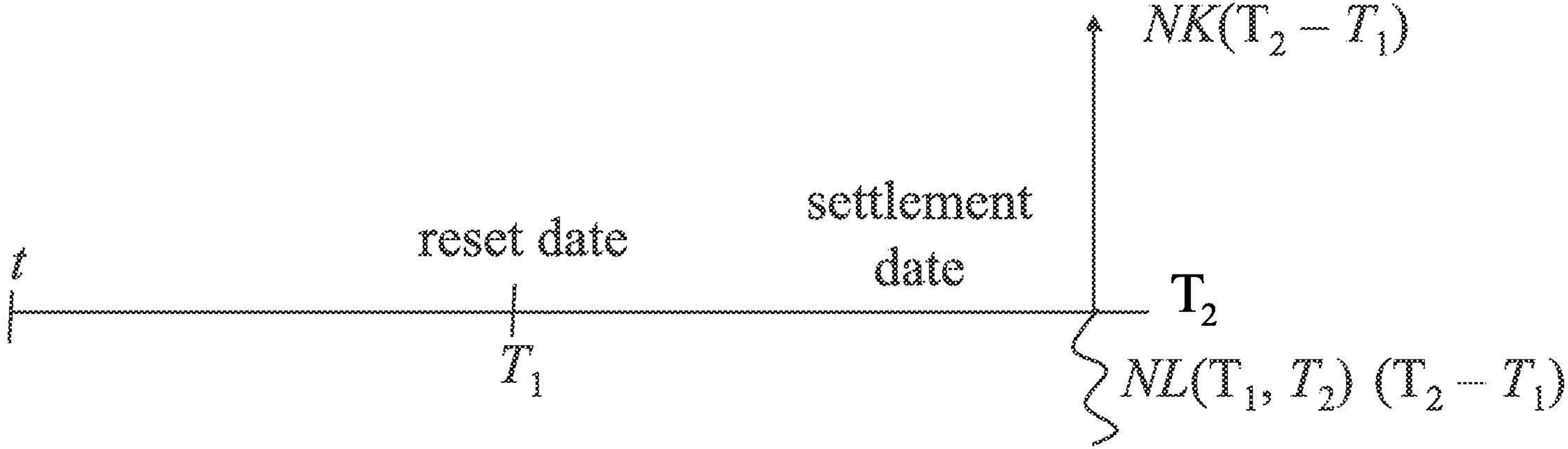
- The floating rate will be the LIBOR rate $L[T_1, T_2]$ as observed on the future reset date T_1.

Question

Should the fixed rate be set equal to the implied forward rate over the same period as observed today?

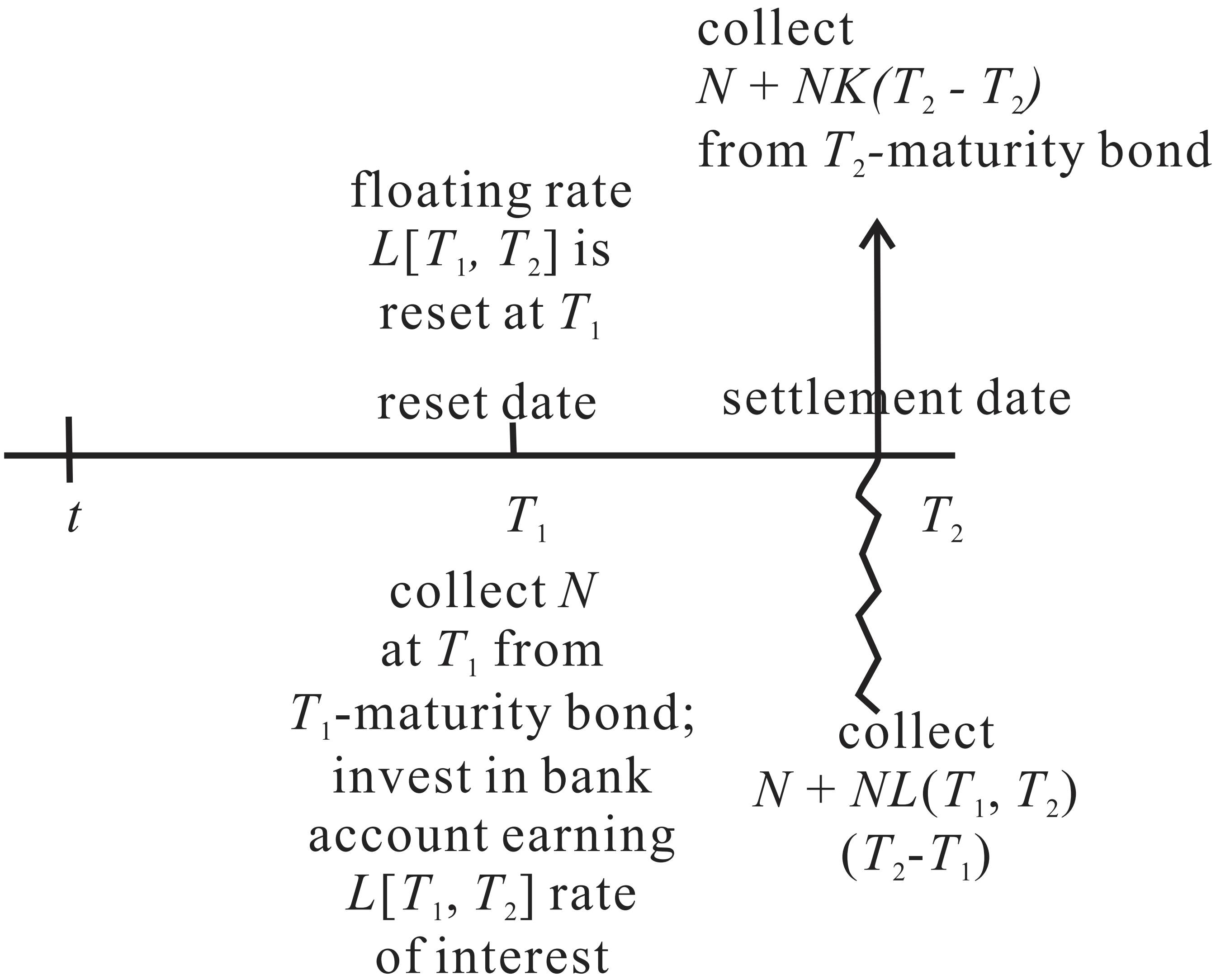
Determination of the forward price of LIBOR

$$
\begin{aligned}
L[T_1, T_2] &= \text{LIBOR rate observed at future time } T_1 \\
&\quad \text{for the accrual period } [T_1, T_2] \\
K &= \text{fixed rate} \\
N &= \text{notional of the FRA}
\end{aligned}
$$



Cash flow of the fixed rate receiver

Cash flow of the fixed rate receiver



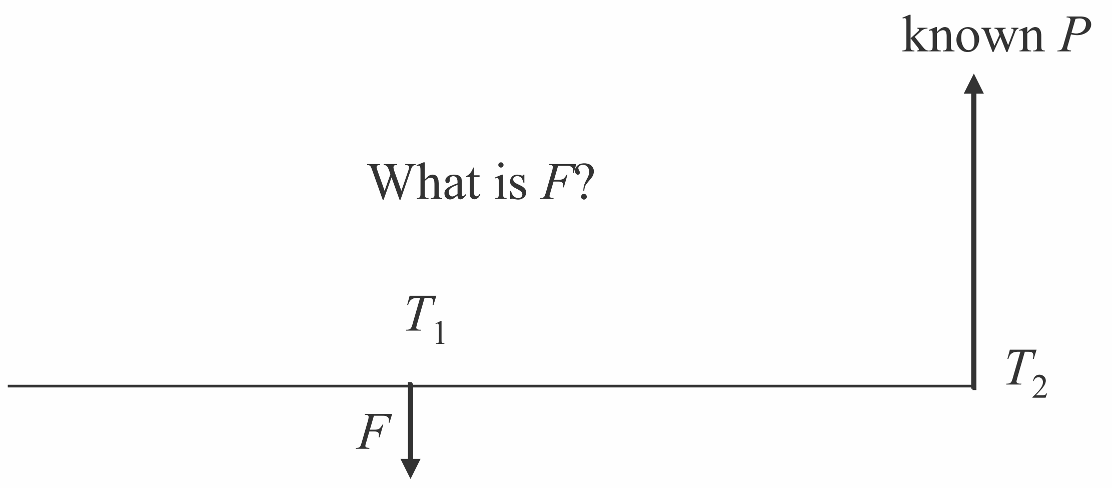
Adding N to both parties at time T_2, the cash flows of the fixed rate payer can be replicated by

(i) long holding of the T_2-maturity zero coupon bond with par $N[1+K(T_2 - T_1)]$.

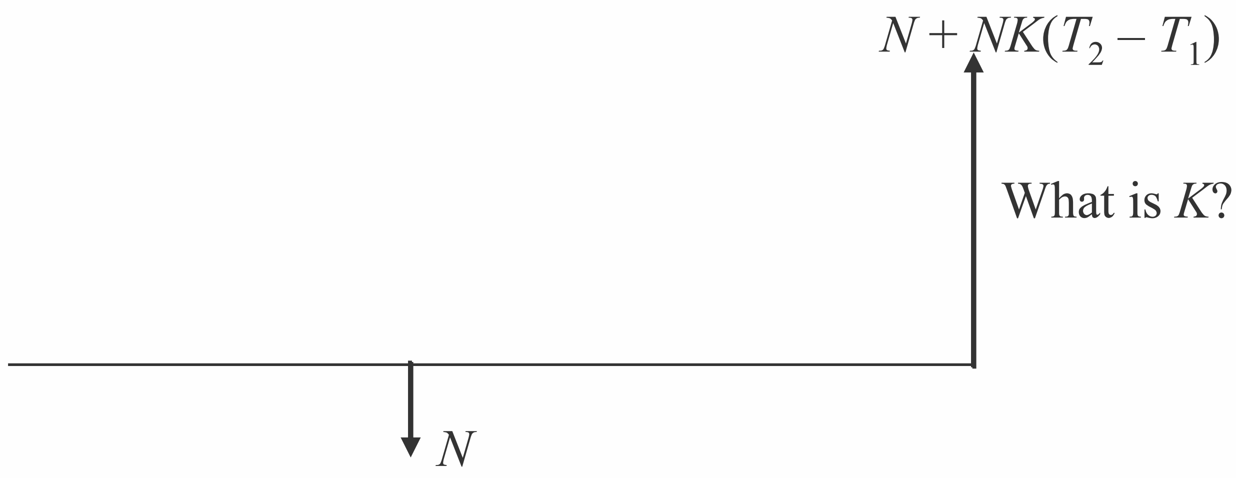
(ii) short holding of the T_1-maturity zero coupon bond with par N.

It is assumed that the par amount N collected at T_1 will be put in a deposit account that earns $L[T_1, T_2]$.

Comparison between bond forward and FRA



bond forward contract – determination of F



forward rate agreement – determination of K

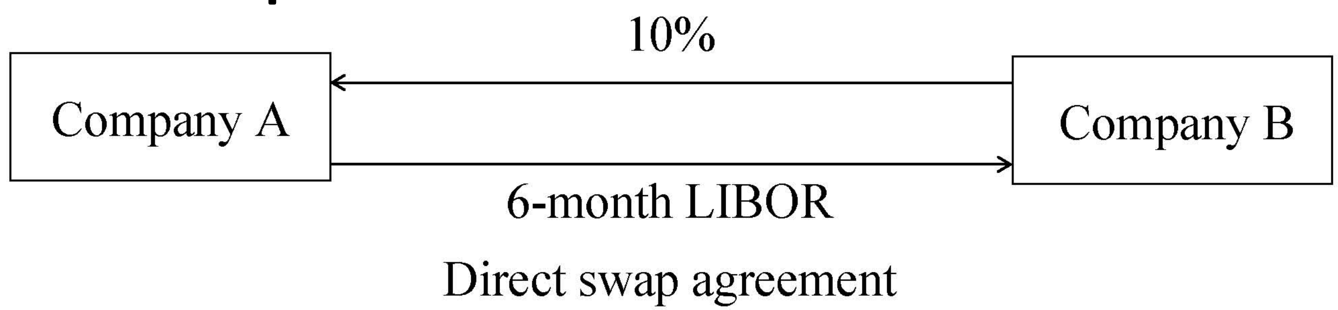
$$\begin{aligned} & \text{Value of the replicating portfolio at the current time} \\ = \; & N\{[1 + K(T_2 - T_1)]B_t(T_2) - B_t(T_1)\}. \end{aligned}$$

We find K such that the above value is zero.

$$K = \underbrace{\frac{1}{T_2 - T_1}\left[\frac{B_t(T_1)}{B_t(T_2)} - 1\right]}_{\text{forward rate over } [T_1, T_2]}.$$

The fair fixed rate K is seen to be the forward price of the LIBOR rate $L[T_1, T_2]$ over the time period $[T_1, T_2]$.

Interest rate swaps



In an interest swap, the two parties agree to exchange periodic interest payments.

- The interest payments exchanged are calculated based on some predetermined dollar principal, called the notional amount.

- One party is the fixed-rate payer and the other party is the floating-rate payer. The floating interest rate is based on some reference rate (the most common index is the LONDON INTERBANK OFFERED RATE, LIBOR).

Example

Notional amount = \$50 million
fixed rate = 10%
floating rate = 6-month LIBOR

Tenor = 3 years, semi-annual payments

6-month period	Cash flows		
	Net (float-fix)	floating rate bond	fixed rate bond
0	0	-50	50
1	$\text{LIBOR}_1/2 \times 50 - 2.5$	$\text{LIBOR}_1/2 \times 50$	-2.5
2	$\text{LIBOR}_2/2 \times 50 - 2.5$	$\text{LIBOR}_2/2 \times 50$	-2.5
3	$\text{LIBOR}_3/2 \times 50 - 2.5$	$\text{LIBOR}_3/2 \times 50$	-2.5
4	$\text{LIBOR}_4/2 \times 50 - 2.5$	$\text{LIBOR}_4/2 \times 50$	-2.5
5	$\text{LIBOR}_5/2 \times 50 - 2.5$	$\text{LIBOR}_5/2 \times 50$	-2.5
6	$\text{LIBOR}_6/2 \times 50 - 2.5$	$\text{LIBOR}_6/2 \times 50$	-2.5

- One interest rate swap contract can effectively establish a payoff equivalent to a package of forward contracts.

A swap can be interpreted as a package of cash market instruments – a portfolio of forward rate agreements.

- Buy \$50 million par of a 3-year floating rate bond that pays 6-month LIBOR semi-annually.
- Finance the purchase by borrowing \$50 million for 3 years at 10% interest rate paid semi-annually.

The fixed-rate payer has a cash market position equivalent to a long position in a floating-rate bond and a short position in a fixed rate bond (borrowing through issuance of a fixed rate bond).

Application to asset/liability management

- Holding a 5-year term commercial loans of $50 million with a fixed interest rate of 10%, that is, interest of $2.5 million received semi-annually and par received at the end of 5 years.

- To fund its loan portfolio, the bank issues 6-month certificates of deposit with floating interest rate of LIBOR + 40 bps (100 bps = 1%).

Risk	6-month LIBOR may be 9.6% or greater.
Possible strategy	Swap the fixed rate income into a floating rate cash stream.

Life insurance company's position

- Has committed to pay a 9% rate for the next 5 years on a guaranteed investment contract (GIC) of amount \$50 million.
- Can invest \$50 million in an attractive 5-year floating-rate instrument with floating interest rate of 6-month LIBOR +160 bps.

Risk	6-month LIBOR may fall to 7.4%.
Possible strategy	Swap the floating rate income into a fixed rate cash stream.

Choice of swap for the bank

- Every six months, the bank will pay 8.45% (annualized rate).
- Every six month, the bank will receive LIBOR.

Outcome

To be received	10.00%+ 6-month LIBOR
To be paid	8.45% + 0.4%+ 6-month LIBOR
spread income	1.15% or 115 basis points

Choice of swap for the insurance company

- Every six months, the insurance company will pay LIBOR.
- Every six months, the insurance company will receive 8.45%.

Outcome

To be received	8.45% + 1.6%+ 6-month LIBOR
To be paid	9.00%+ 6-month LIBOR
spread income	1.05% or 105 basis points

Creation of structured notes using swaps

Company A wants to issue $100 million of 5-year fixed-rate medium-term notes (MTN).

- Issue $100 million of 5-year inverse-floating-rate MTN with coupon rate 13% − LIBOR.
- Enter into 5-year interest rate swap with notional $100 million.

A pays LIBOR and receives 7%

Outcome

To MTN holders	13.00% − LIBOR
To swap counterparty	LIBOR
From swap counterparty	7%
Net payment	6%

Comparative advantages in borrowing rates motivate the construction of an interest rate swap

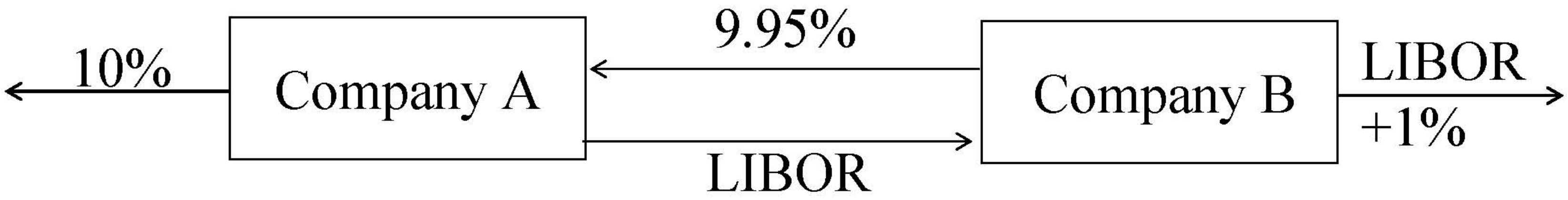
	fixed	floating
Company A	10.00%	6-month LIBOR +0.30%
Company B	11.20%	6-month LIBOR +1.00%

- Company A wants to borrow at floating rate while Company B wants to borrow at fixed rate.

- Company A is seen to have higher credit rating than Company B.

- Company A has a comparative advantage in the fixed rate markets while Company B has an advantage in the floating rate markets.

Direct swap agreement between A and B



Company A

- pay 10% per annum to an outside lender
- receives 9.95% per annum from B
- pays LIBOR to B

Net payment: A pays LIBOR $+0.05\%; 0.25\%$ less.

Company B

- pays LIBOR +1.00 per annum to an outside lender
- receives LIBOR from A
- pays 9.95% per annum to A

Net payment: B pays 10.95% per annum; 0.25% less.

The total gain is 0.5% per annum. Why?

Difference between interest rates in floating rate markets = 0.7%, while difference between interest rates in fixed rate markets = 1.2%.

The net difference is 0.5%, and it is shared by the two companies.

Interest rate swap using financial intermediary

Net gain to $A = 0.2\%$

Net gain to $B = 0.2\%$

Net gain to the financial intermediary $= 0.1\%$. The financial intermediary has to bear the counterparty risks of the two companies. The settlement clauses of interest rate swaps upon default can be quite complicated.

- The financial institution has two separate contracts. If one of the companies defaults, the financial institution still has to honor its agreement with the other company.

Exploiting comparative advantages

- Initial motivation for the interest rate swap market was borrower exploitation of "credit arbitrage" opportunities because of differences between the quality spread between the lower- and higher-rated credits in the floating and fixed rate loans.

Query As with any arbitrage opportunity, the more it is exploited, the smaller it becomes.

Explanation The difference in quality spread persists due to differences in regulations and tax treatment in different countries.

Valuation of interest rate swaps

- When a swap is entered into, it typically has zero value.

- Valuation involves finding the fixed swap rate K such that the fixed and floating legs have equal value at inception.

- Consider a swap with payment dates $t_1, t_2, \cdots, t_n$ (tenor structure) set in the term of the swap. L_{i-1} is the LIBOR observed at t_{i-1} but payment is made at t_i. Write $\delta_i \approx t_i - t_{i-1}$ as the accrual period over $[t_{i-1}, t_i]$.

- The fixed payment at t_i is $KN\delta_i$ while the floating payment at t_i is $L_{i-1}N\delta_i, i = 1, 2, \cdots n$. Here, N is the notional.

Day count convention

For the 30/360 day count convention of the time period $(D_1, D_2]$ with D_1 excluded but D_2 included, the year fraction is

$$\frac{\max(30 - d_1, 0) + \min(d_2, 30) + 360 \times (y_2 - y_1) + 30 \times (m_2 - m_1 - 1)}{360}$$

where d_i, m_i and y_i represent the day, month and year of date $D_i, i = 1, 2$.

For example, the year fraction between *Feb 27, 2006* and *July 31, 2008*

$$= \frac{30 - 27 + 30 + 360 \times (2008 - 2006) + 30 \times (7 - 2 - 1)}{360}$$

$$= \frac{33}{360} + 2 + \frac{4}{12}.$$

Replication of cash flows

- The fixed payment at t_i is $KN\delta_i$. The fixed payments are packages of discount bonds with par $KN\delta_i$ at maturity date $T_i, i = 1, 2, \cdots, n$.

- To replicate the floating leg payments at current time $t, t < T_0$, we long T_0-maturity discount bond with par N and short T_n-maturity discount bond with par N. The N dollars collected at T_0 can generate the floating leg payments $L_{i-1}N\delta_i$ at all $T_i, i = 1, 2, \cdots, n$. The remaining N dollars at T_n is used to pay the par of the T_n-maturity bond.

- Let $B(t,T)$ be the time-t value of the discount bond with maturity t.

- Sum of percent value of the floating leg payments

$$= N[B(t, T_0) - B(t, T_n)];$$

sum of present value of fixed leg payments

$$= NK \sum_{i=1}^{n} \delta_i B(t, T_i).$$

The swap rate K is given by equating the present values of the two sets of payments:

$$K = \frac{B(t, T_0) - B(t, T_n)}{\sum_{i=1}^{n} \delta_i B(t, T_i)}.$$

The interest rate swap reduces to a FRA when $n = 1$. As a check, we obtain

$$K = \frac{B(t, T_0) - B(t, T_1)}{(T_1 - T_0) B(t, T_1)}.$$

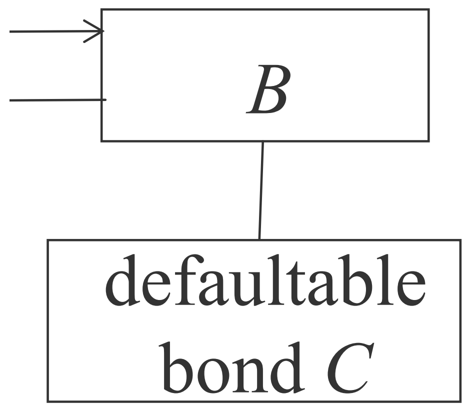
Asset swap

- Combination of a defaultable bond with an interest rate swap.

  B pays the notional amount upfront to acquire the asset swap package.

1. A fixed defaultable coupon bond issued by C with coupon $\overline{c}$ payable on coupon dates.

2. A fixed-for-floating swap.



The asset swap spread s^A is adjusted to ensure that the asset swap package has an initial value equal to the notional.

Remarks

1. Asset swaps are more liquid than the underlying defaultable bond.

2. An asset swaption gives B the right to enter an asset swap package at some future date T at a predetermined asset swap spread s^A.

Hedge based pricing – *approximate hedge and replication strategies*

Provide hedge strategies that cover much of the risks involved in credit derivatives – independent of any specific pricing model.

Basic instruments

1. Default free bond

$$C(t) = \text{time-}t \text{ price of default-free bond with fixed-coupon } \overline{c}$$
$$B(t,T) = \text{time-}t \text{ price of default-free zero-coupon bond}$$

2. Defaultable bond

$$\overline{C}(t) = \text{time-}t \text{ price of defaultable bond with fixed-coupon } \overline{c}$$

3. Interest rate swap

$$
\begin{aligned}
s(t) &= \text{forward swap rate at time } t \text{ of a standard fixed-for-floating} \\
&= \frac{B(t,t_n) - B(t,t_N)}{A(t;t_n,t_N)}, \quad t \le t_n
\end{aligned}
$$

where $A(t;t_n,t_N) = \sum_{i=n+1}^{N} \delta_i B(t,t_i)$ = value of the annuity payment stream paying δ_i on each date t_i. The first swap payment starts on t_{n+1} and the last payment date is t_N.

The forward swap rate is market observable. It may occur that the swap rate markets may not agree exactly with the bond markets.

Asset swap packages

An asset swap package consists of a defaultable coupon bond $\overline{C}$ with coupon $\overline{c}$ and an interest rate swap. The bond's coupon is swapped into LIBOR plus the asset swap rate s^A. Asset swap package is sold at par. Asset swap transactions are driven by the desire to strip out unwanted structured features from the underlying asset.

Payoff streams to the buyer of the asset swap package

time	defaultable bond	interest rate swap	net
$t=0^\dagger$	$-\overline{C}(0)$	$-1+\overline{C}(0)$	-1
$t=t_i$	$\overline{c}^*$	$-\overline{c}+L_{i-1}+s^A$	$L_{i-1}+s^A+(\overline{c}^*-\overline{c})$
$t=t_N$	$(1+\overline{c})^*$	$-\overline{c}+L_{N-1}+s^A$	$1^*+L_{N-1}+s^A+(\overline{c}^*-\overline{c})$
default	recovery	unaffected	recovery

$\star$ denotes payment contingent on survival.

† The value of the interest rate swap at $t=0$ is not zero. The sum of the values of the interest rate swap and defaultable bond is equal to par at $t=0$.

The additional asset spread s^A serves as the compensation for bearing the potential loss upon default.

$s(0)$ = fixed-for-floating swap rate (market quote)

$A(0)$ = value of an annuity paying at $1 per annum (calculated based on observable default free bond prices)

The value of asset swap package is set at par at $t = 0$, so that

$$\overline{C}(0) + \underbrace{A(0)s(0) + A(0)s^A(0) - A(0)\overline{c}}_{\text{swap arrangement}} = 1.$$

The present value of the floating coupons is given by $A(0)s(0)$. The swap continues even after default so that $A(0)$ appears in all terms associated with the swap arrangement.

Solving for $s^A(0)$

$$s^A(0) = \frac{1}{A(0)}[1 - \overline{C}(0)] + \overline{c} - s(0).$$

Rearranging the terms,

$$\overline{C}(0) + A(0)s^A(0) = \underbrace{[1 - A(0)s(0)] + A(0)\overline{c}}_{\text{default-free bond}} \equiv C(0)$$

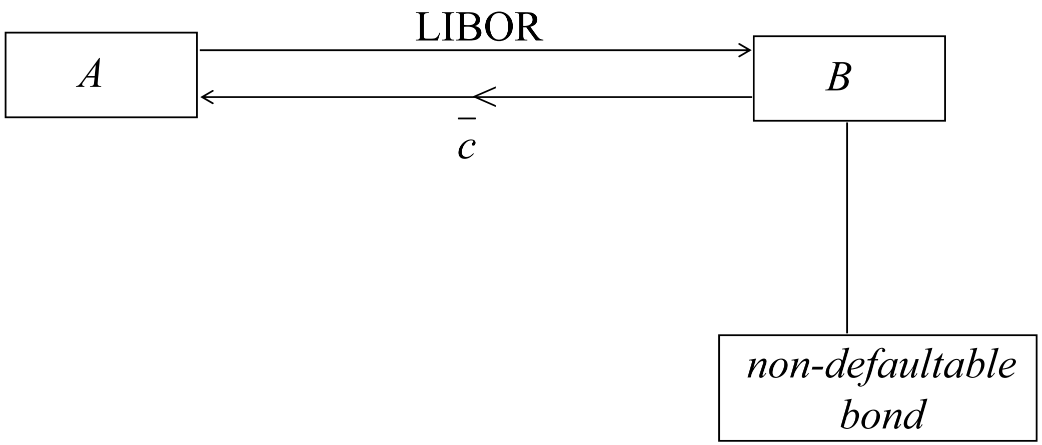
where the right-hand side gives the value of a default-free bond with coupon $\overline{c}$. Note that $1 - A(0)s(0)$ is the present value of receiving \$1 at maturity t_N. We obtain

$$s^A(0) = \frac{1}{A(0)}[C(0) - \overline{C}(0)].$$

The difference in the bond prices is equal to the present value of annuity stream at the rate $s^A(0)$.

Alternative proof

A combination of the non-defaultable counterpart (bond with coupon rate $\overline{c}$) plus an interest rate swap (whose floating leg is LIBOR while the fixed leg is $\overline{c}$) becomes a par floater. Hence, the new asset package should also be sold at par.



The buyer is guaranteed to receive LIBOR floating rate interests plus par. We have

$$C(0) = \overbrace{1 - A(0)s(0)}^{\text{PV \$1 at } t_n} + A(0)\overline{c}$$

while $A(0)s(0) - A(0)\overline{c}$ gives the value of the interest rate swap.

- The two interest swaps with floating leg at LIBOR $+ s^A(0)$ and LIBOR, respectively, differ in values by $s^A(0)A(0)$.

- Let $V_{swap\_L+s^A}$ denote the value of the swap at $t = 0$ whose floating rate is set at LIBOR $+ s^A(0)$. Both asset swap packages are sold at par. We then have

$$1 = \overline{C}(0) + V_{swap\_L+s^A} = C(0) + V_{swap\_L}.$$

Hence, the difference in $C(0)$ and $\overline{C}(0)$ is the present value of the annuity stream at the rate $s^A(0)$, that is,

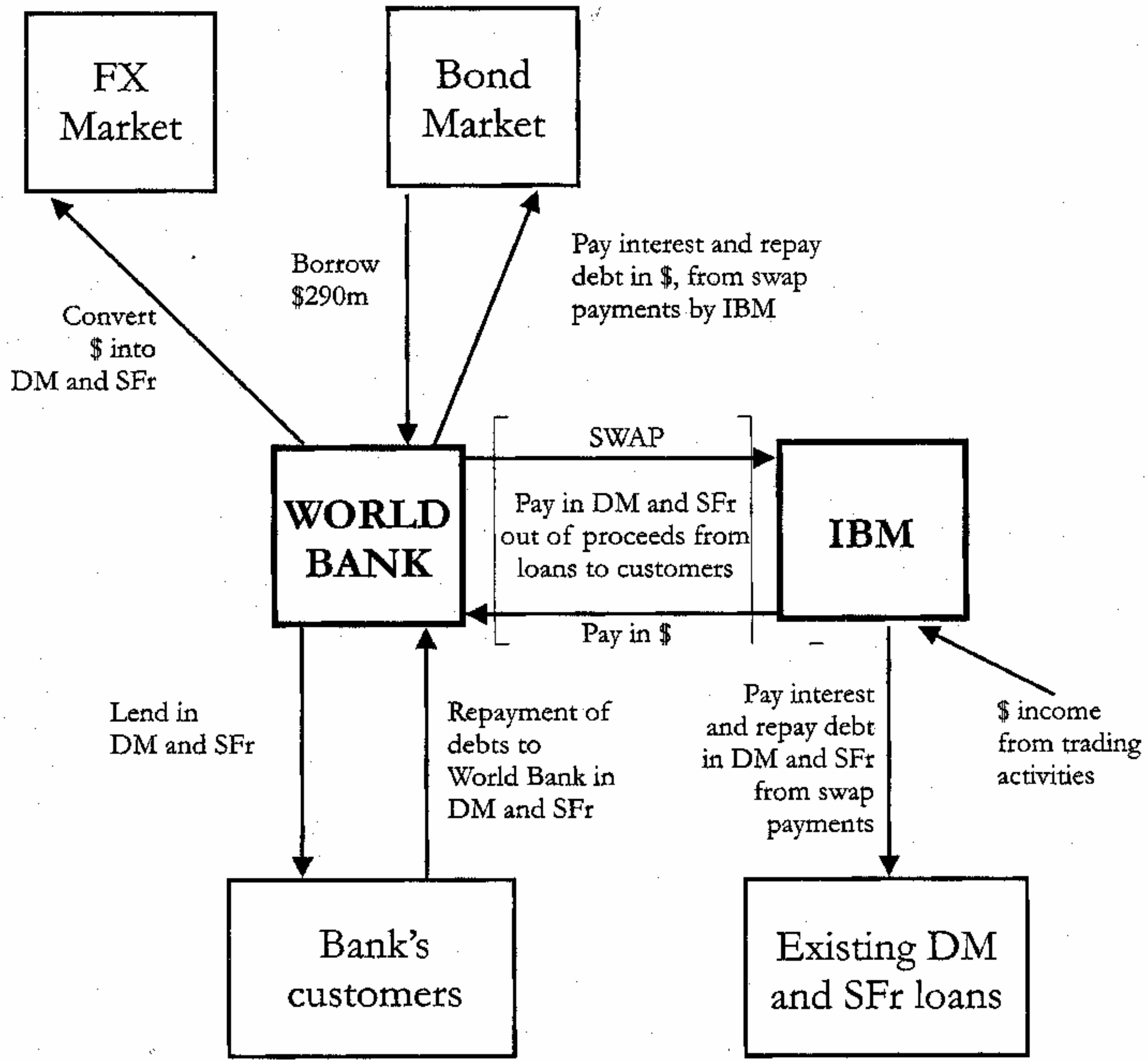
$$C(0) - \overline{C}(0) = V_{swap\_L+s^A} - V_{swap\_L} = s^A(0)A(0).$$

IBM/World Bank – first currency swap structured in 1981

- IBM had existing debts in DM and Swiss francs. This had created a FX exposure since IBM had to convert USD into DM and Swiss Francs regularly to make the coupon payments. Due to a depreciation of the DM and Swiss franc against the dollar, IBM could realize a large foreign exchange gain, but only if it could eliminate its DM and Swiss franc liabilities and "lock in" the gain and remove any future exposure.

- The World Bank was raising most of its funds in DM (interest rate = 12%) and Swiss francs (interest rate = 8%). It did not borrow in dollars, for which the interest rate cost was about 17%. Though it wanted to lend out in DM and Swiss francs, the bank was concerned that saturation in the bond markets could make it difficult to borrow more in these two currencies at a favorable rate. Its objective, as always, was to raise cheap funds.

World Bank/IBM Currency Swap, 1981



- IBM was willing to take on dollar liabilities and made dollar payments (periodic coupons and principal at maturity) to the World Bank since it could generate dollar income from normal trading activities.

- The World Bank could borrow dollars, convert them into DM and SFr in FX market, and through the swap take on payment obligations in DM and SFr.

1. The foreign exchange gain on dollar appreciation is realized by IBM through the negotiation of a favorable swap rate in the swap contract.

2. The swap payments by the World Bank to IBM were scheduled so as to allow IBM to meet its debt obligations in DM and SFr.

Under the currency swap

- IBM pays regular US coupons and US principal at maturity.

- World Bank pays regular DM and SFr coupons together with DM and SFr principal at maturity.

Now IBM converted its DM and SFr liabilities into USD, and the World Bank effectively raised hard currencies at a cheap rate. Both parties achieved their objectives!

Financial options

- A *call* (or *put*) option is a contract which gives the holder the *right* to buy (or sell) a prescribed asset, known as the *underlying asset*, by a certain date (*expiration date*) for a predetermined price (commonly called the *strike price* or *exercise price*).

- The option is said to be *exercised* when the holder chooses to buy or sell the asset.

- If the option can only be exercised on the expiration date, then the option is called a *European* option.

- If the exercise is allowed at any time prior to the expiration date, then it is called an *American* option

Terminal payoff

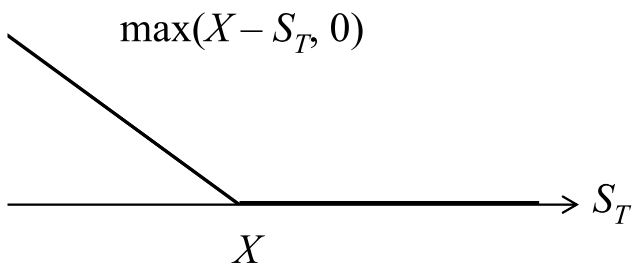
- The terminal payoff from the long position (holder's position) of a European call is then

$$\max(S_T - X, 0).$$

- The terminal payoff from the long position in a European put can be shown to be

$$\max(X - S_T, 0),$$

since the put will be exercised at expiry only if $S_T < X$, whereby the asset worth S_T is sold at a higher price of X.


max(X – S_T, 0)
S_T
X

Questions and observations

What should be the fair option premium (usually called option price or option value) so that the deal is fair to both writer and holder?

What should be the optimal strategy to exercise prior the expiration date for an American option?

At least, the option price is easily seen to depend on the strike price, time to expiry and current asset price. The less obvious factors for the pricing models are the prevailing *interest rate* and the degree of randomness of the asset price, commonly called the *volatility*.

Hedging

- If the writer of a call does not simultaneously own any amount of the underlying asset, then he is said to be in a *naked position*.

- Suppose the call writer owns some amount of the underlying asset, the loss in the short position of the call when asset price rises can be compensated by the gain in the long position of the underlying asset.

- This strategy is called *hedging*, where the risk in a portfolio is monitored by taking opposite directions in two securities which are highly negatively correlated.

- In a *perfect hedge* situation, the *hedger* combines a risky option and the corresponding underlying asset in an appropriate proportion to form a riskless portfolio.

Swaptions – Product nature

- The buyer of a swaption has the right to enter into an interest rate swap by some specified date. The swaption also specifies the maturity date of the swap.

- The buyer can be the fixed-rate receiver (put swaption) or the fixed-rate payer (call swaption).

- The writer becomes the counterparty to the swap if the buyer exercises.

- The strike rate indicates the fixed rate that will be swapped versus the floating rate.

- The buyer of the swaption either pays the premium upfront or can be structured into the swap rate.

Uses of swaptions

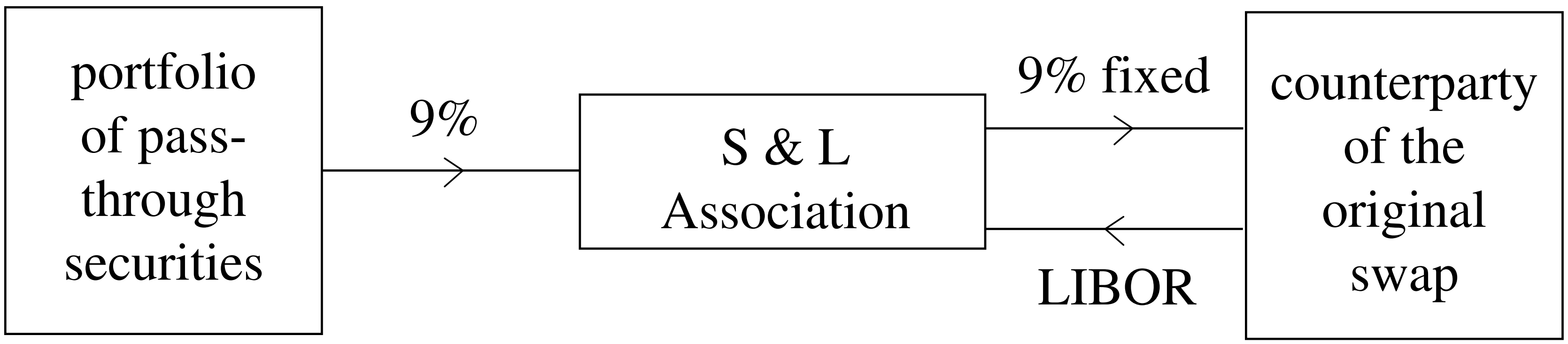
Used to hedge a portfolio strategy that uses an interest rate swap but where the cash flow of the underlying asset or liability is uncertain.

Uncertainties come from (i) callability, eg, a callable bond or mortgage loan, (ii) exposure to default risk.

Example

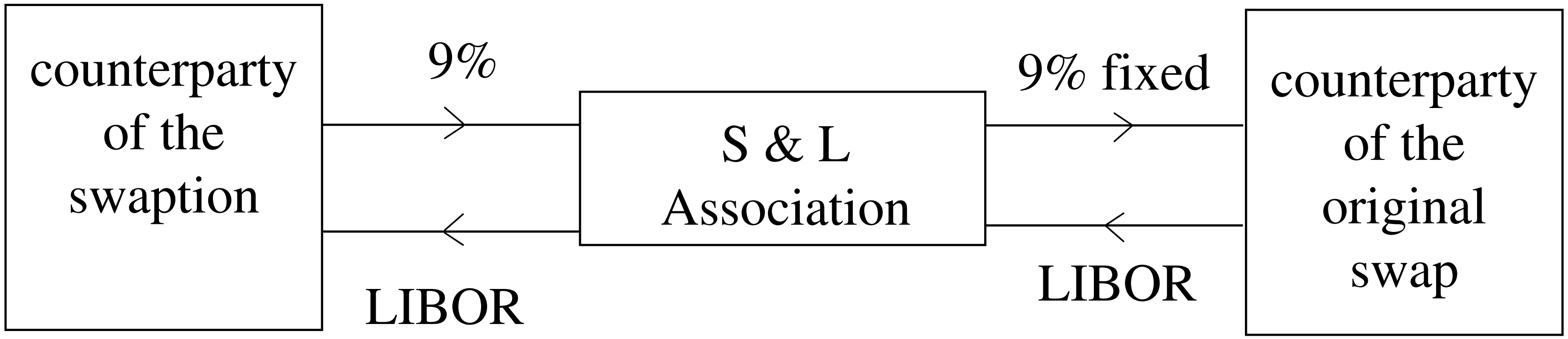
Consider a S & L Association entering into a 4-year swap in which it agrees to pay 9% fixed and receive LIBOR.

- The fixed rate payments come from a portfolio of mortgage pass-through securities with a coupon rate of 9%. One year later, mortgage rates decline, resulting in large prepayments.
- The purchase of a put swaption with a strike rate of 9% would be useful to offset the original swap position.



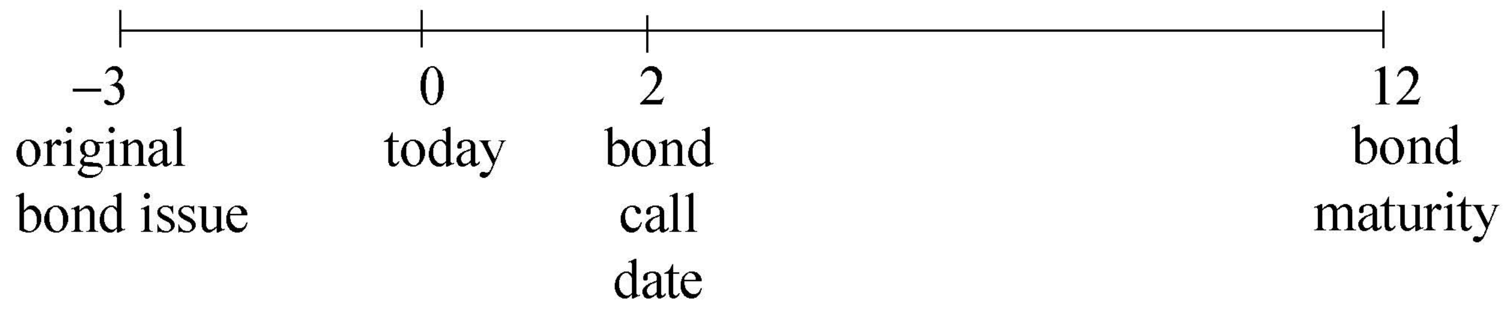
Due to decline in the interest rate, large prepayments are resulted in the mortgage pass-through securities. The source of 9% fixed payment dissipates. The swaption is in-the-money since the interest rate declines, so does the swap rate.

By exercising the put swaption, the S & L Association receives a fixed rate of 9%



Management of callable debts

Three years ago, XYZ issued 15-year fixed rate callable debt with a coupon rate of 12%.



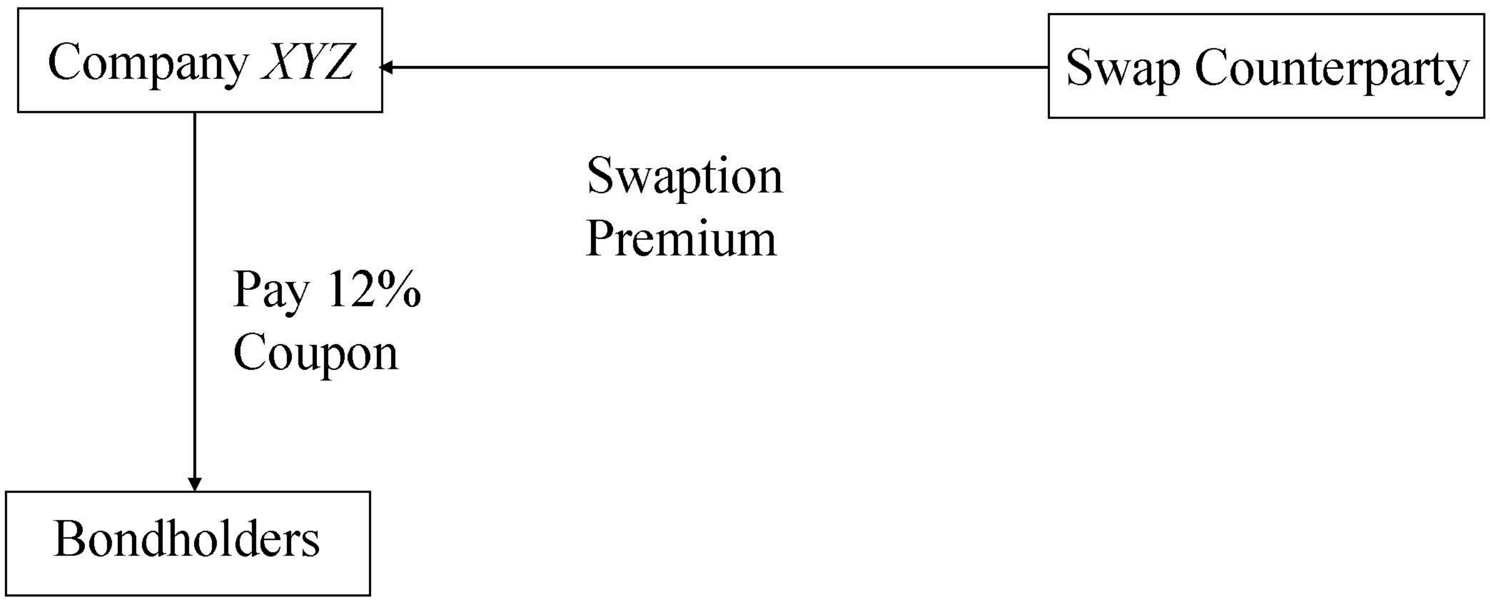
Strategy

The issuer sells a two-year receiver option on a 10-year swap, that gives the holder the right, but not the obligation, to receive the fixed rate of 12%.

Call monetization

By selling the swaption today, the company has committed itself to paying a 12% coupon for the remaining life of the original bond.

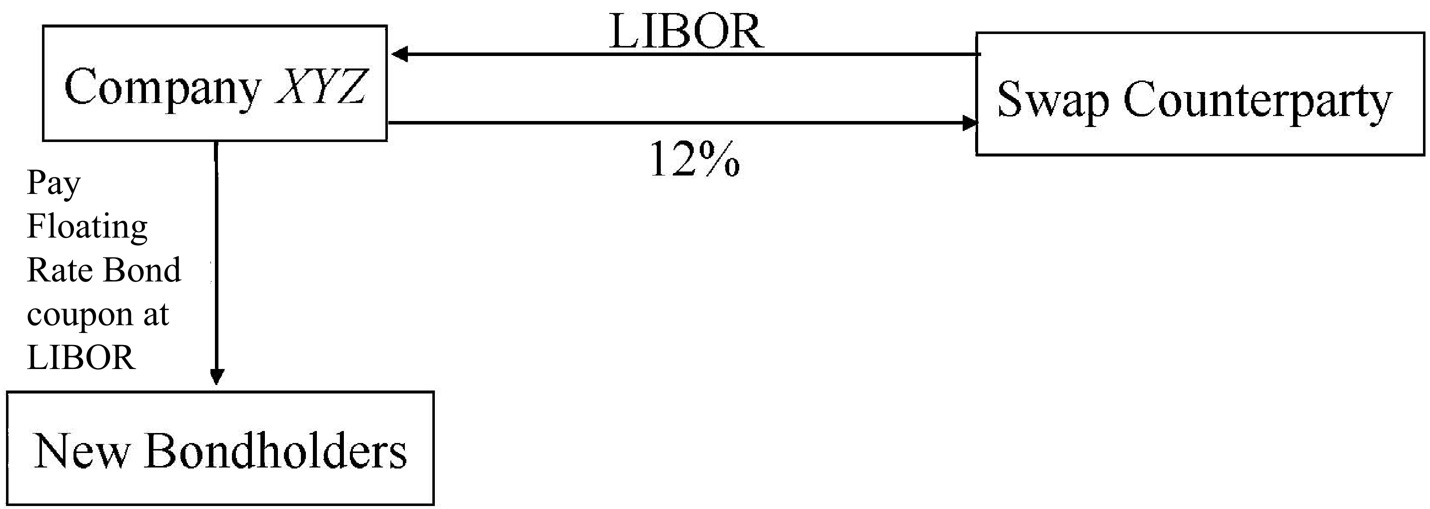
- The swaption was sold in exchange for an upfront swaption premium received at date 0. The monetization of the callable right is realized by the swaption premium received.



Call-Monetization cash flow: Swaption expiration date

swap rate $\geq$ 12% (swap counterparty does not exercise)

swap rate $<$ 12% (swap counterparty exercises)



Disasters for Company XYZ

- The fixed rate on the 10-year swap is below 12% in two years but its debt refunding rate in the capital market is above 12% (due to credit deterioration)

- The company would be forced to enter into a swap that it does not want while it may liquidate the original debt liabilities position at a disadvantage and not be able to refinance its borrowing profitably.

1.2 Rational boundaries for option values

- We do not specify the probability distribution of the movement of the asset price so that the *fair* option value cannot be derived.

- Mathematical properties of the option values as functions of the strike price X, asset price S and time to expiry τ are derived.

- We study the impact of dividends on these rational boundaries.

- The optimal early exercise policy of American options on a non-dividend paying asset can be inferred from the analysis of these bounds on option values.

- The relations between put and call prices (called the *put-call parity relations*) are also deduced.

Non-negativity of option prices

All option prices are non-negative, that is,

$$C \geq 0, \quad P \geq 0, \quad c \geq 0, \quad p \geq 0,$$

as derived from the non-negativity of the payoff structure of option contracts.

If the price of an option were negative, this would mean an option buyer receives cash up-front. He is guaranteed to have a non-negative terminal payoff. In this way, he can always lock in a riskless profit.

Intrinsic values of American options

- $\max(S-X, 0)$ and $\max(X-S, 0)$ are commonly called the *intrinsic value* of a call and a put, respectively.

- Since American options can be exercised at any time before expiration, their values must be worth at least their intrinsic values, that is,

$$\begin{aligned} C(S, \tau; X) &\geq \max(S-X, 0) \\ P(S, \tau; X) &\geq \max(X-S, 0). \end{aligned}$$

- Suppose C is less than $S-X$ when $S \geq X$, then an arbitrageur can lock in a riskless profit by borrowing $C+X$ dollars to purchase the call and exercise it immediately to receive the asset worth S. The riskless profit would be $S-X-C > 0$.

American options are worth at least their European counterparts

An American option confers all the rights of its European counterpart plus the privilege of early exercise. The additional privilege cannot have negative value.

$$
\begin{aligned}
C(S, \tau; X) &\geq c(S, \tau; X) \\
P(S, \tau; X) &\geq p(S, \tau; X).
\end{aligned}
$$

- The European put value can be below the intrinsic value $X - S$ at sufficiently low asset value and the value of a European call on a dividend paying asset can be below the intrinsic value $S - X$ at sufficiently high asset value.

Values of options with different dates of expiration

Consider two American options with different times to expiration τ_2 and τ_1 ($\tau_2 > \tau_1$), the one with the longer time to expiration must be worth at least that of the shorter-lived counterpart since the longer-lived option has the additional right to exercise between the two expiration dates.

$$
\begin{aligned}
C(S, \tau_2; X) &> C(S, \tau_1; X), \quad \tau_2 > \tau_1, \\
P(S, \tau_2; X) &> P(S, \tau_1; X), \quad \tau_2 > \tau_1.
\end{aligned}
$$

The above argument cannot be applied to European options due to lack of the early exercise privilege.

Values of options with different strike prices

$$\begin{aligned} c(S,\tau;X_2) &< c(S,\tau;X_1), \quad X_1 < X_2, \\ C(S,\tau;X_2) &< C(S,\tau;X_1), \quad X_1 < X_2. \end{aligned}$$

and

$$\begin{aligned} p(S,\tau;X_2) &> p(S,\tau;X_1), \quad X_1 < X_2, \\ P(S,\tau;X_2) &> P(S,\tau;X_1), \quad X_1 < X_2. \end{aligned}$$

Values of options at varying asset value levels

$$\begin{aligned} c(S_2,\tau;X) &> c(S_1,\tau;X), \quad S_2 > S_1, \\ C(S_2,\tau;X) &> C(S_1,\tau;X), \quad S_2 > S_1; \end{aligned}$$

and

$$\begin{aligned} p(S_2,\tau;X) &< p(S_1,\tau;X), \quad S_2 > S_1, \\ P(S_2,\tau;X) &< P(S_1,\tau;X), \quad S_2 > S_1. \end{aligned}$$

Upper bounds on call and put values

- A call option is said to be a *perpetual call* if its date of expiration is infinitely far away. The asset itself can be considered as an American perpetual call with zero strike price plus additional privileges such as voting rights and receipt of dividends, so we deduce that $S \geq C(S, \infty; 0)$.

$$S \geq C(S, \infty; 0) \geq C(S, \tau; X) \geq c(S, \tau; X).$$

- The price of an American put equals the strike value when the asset value is zero; otherwise, it is bounded above by the strike price.

$$X \geq P(S, \tau; X) \geq p(S, \tau; X).$$

Lower bounds on the values of call options on a non-dividend paying asset

Portfolio A consists of a European call on a non-dividend paying asset plus a discount bond with a par value of X whose date of maturity coincides with the expiration date of the call. Portfolio B contains one unit of the underlying asset.

Asset value at expiry	$S_T < X$	$S_T \geq X$
Portfolio A	X	$(S_T - X) + X = S_T$
Portfolio B	S_T	S_T
Result of comparison	$V_A > V_B$	$V_A = V_B$

The present value of Portfolio A (dominant portfolio) must be equal to or greater than that of Portfolio B (dominated portfolio). If otherwise, an arbitrage opportunity can be secured by buying Portfolio A and selling Portfolio B.

Write $B(\tau)$ as the price of the unit-par discount bond with time to expiry τ. Then

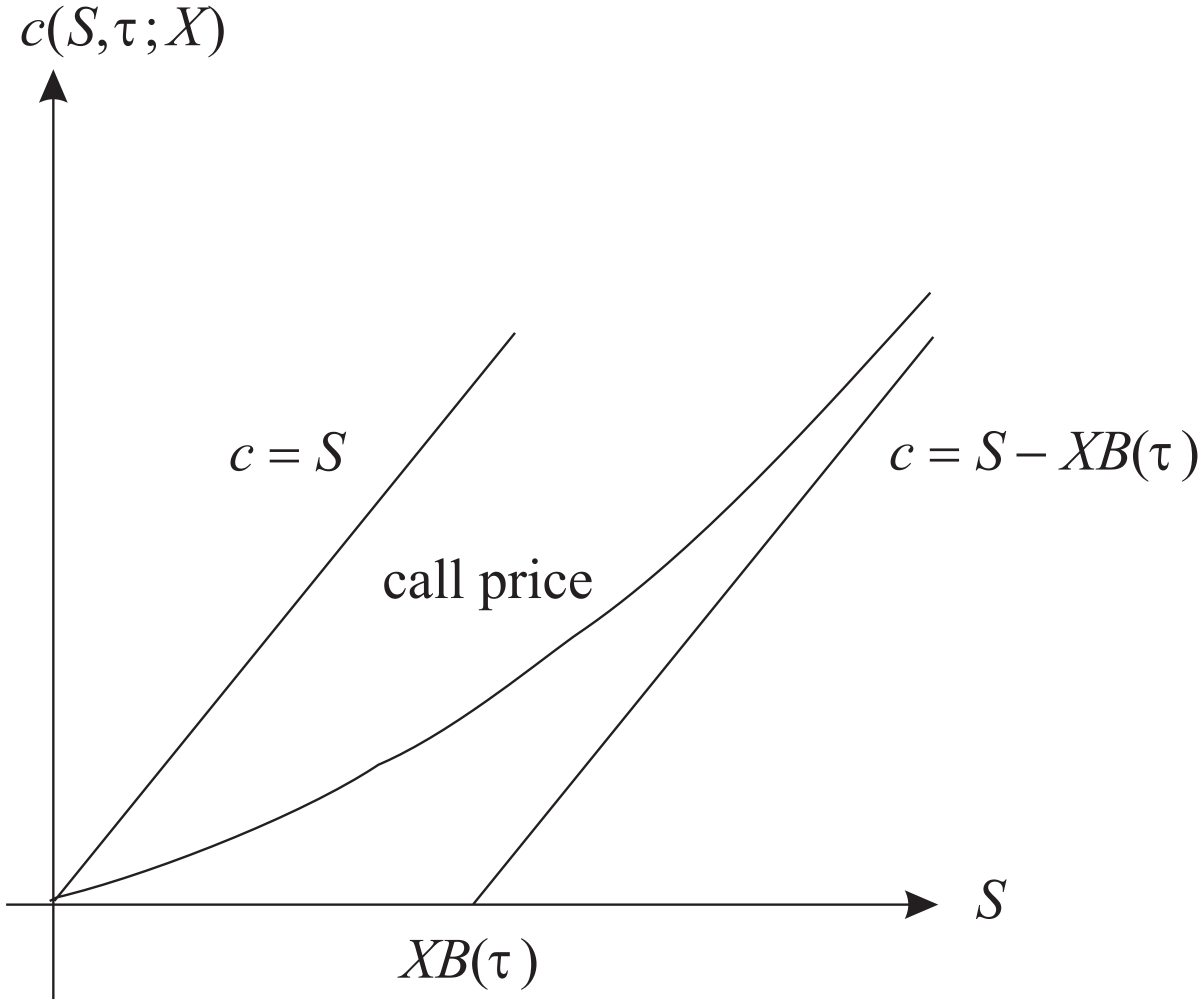
$$c(S, \tau; X) + XB(\tau) \geq S.$$

Together with the non-negativity property of option value.

$$c(S, \tau; X) \geq \max(S - XB(\tau), 0).$$

The upper and lower bounds of the value of a European call on a non-dividend paying asset are given by (see Figure)

$$S \geq c(S, \tau; X) \geq \max(S - XB(\tau), 0).$$

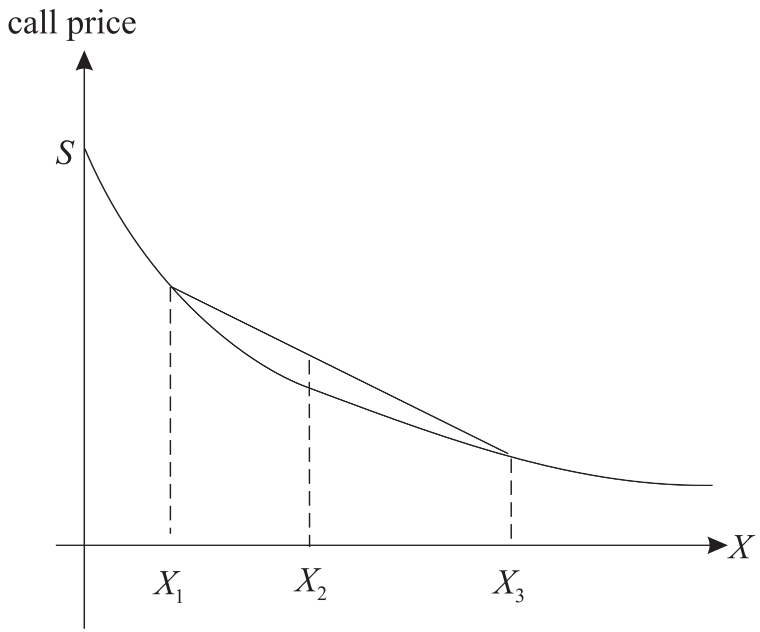


$c(S,\tau;X)$
$c = S$
call price
$c = S - XB(\tau)$
S
$XB(\tau)$

Convexity properties of the option price functions

The call prices are convex functions of the strike price. Write $X_2 = \lambda X_3 + (1-\lambda)X_1$ where $0 \leq \lambda \leq 1, X_1 \leq X_2 \leq X_3$.

$$\begin{aligned} c(S,\tau;X_2) &\leq \lambda c(S,\tau;X_3) + (1-\lambda)c(S,\tau;X_1) \\ C(S,\tau;X_2) &\leq \lambda C(S,\tau;X_3) + (1-\lambda)C(S,\tau;X_1). \end{aligned}$$



Consider the payoffs of the following two portfolios at expiry. Portfolio C contains λ units of call with strike price X_3 and $(1-\lambda)$ units of call with strike price X_1, and Portfolio D contains one call with strike price X_2.

Since $V_C \geq V_D$ for all possible values of S_T, Portfolio C is dominant over Portfolio D. Therefore, the present value of Portfolio C must be equal to or greater than that of Portfolio D.

- The drop in call value for one dollar increase in the strike price should be less than one dollar. The loss in the terminal payoff of the call due to the increase in the strike price is realized only when the call expires in-the-money. Indeed, $\left|\frac{\partial c}{\partial X}\right| \leq B(\tau)$. The factor $B(\tau)$ appears since the potential loss of paying extra one dollar in the strike price occurs at maturity so its present value is $B(\tau)$.

Payoff at expiry of Portfolios C and D.

Asset value at expiry	$S_T \le X_1$	$X_1 \le S_T \le X_2$	$X_2 \le S_T \le X_3$	$X_3 \le S_T$
Portfolio C	0	$(1-\lambda)(S_T - X_1)$	$(1-\lambda)(S_T - X_1)$	$\lambda(S_T - X_3) +$ $(1-\lambda)(S_T - X_1)$
Portfolio D	0	0	$S_T - X_2$	$S_T - X_2$
Result of comparison	$V_C = V_D$	$V_C \ge V_D$	$V_C \ge V_D^*$	$V_C = V_D$

* Recall $X_2 = \lambda X_3 + (1-\lambda)X_1$, and observe

$$
\begin{aligned}
& (1-\lambda)(S_T - X_1) \ge S_T - X_2 \\
\Leftrightarrow\ & X_2 - (1-\lambda)X_1 \ge \lambda S_T \\
\Leftrightarrow\ & X_3 \ge S_T.
\end{aligned}
$$

- There is no factor involving τ, so the result also holds even when the calls in the two portfolios are exercised prematurely. Hence, the convexity property also holds for American calls.

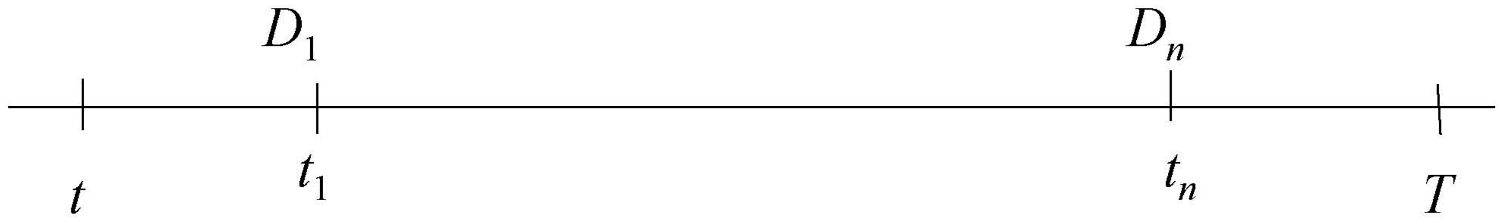
- By changing the call options in the above two portfolios to the corresponding put options, it can be shown that European and American put prices are also convex functions of the strike price.

- By using the linear homogeneity property of the call and put option functions with respect to the asset price and strike price, one can show that the call and put prices (both European and American) are convex functions of the asset price.

Impact of dividends on the asset price

- When an asset pays a certain amount of dividend, we can use no arbitrage argument to show that the asset price is expected to fall by the same amount (assuming there exist no other factors affecting the income proceeds, like taxation and transaction costs).

- Suppose the asset price falls by an amount less than the dividend, an arbitrageur can lock in a riskless profit by borrowing money to buy the asset right before the dividend date, selling the asset right after the dividend payment and returning the loan.



It is assumed that the deterministic dividend amount D_i is paid at time t_i, $i = 1, 2, \cdots, n$. The current time is t and write $\tau_i = t_i - t, i = 1, 2, \cdots, n$. The sum of the present value of the dividends is

$$D = D_1 e^{-r\tau_1} + \cdots + D_n e^{-r\tau_n}.$$

The dividend stream may be visualized as a portfolio of bonds with par value D_i maturing at $t_i, i = 1, 2, \cdots, n$.

Weakness in the assumption

One may query whether the asset can honor the deterministic dividend payments when the asset value becomes very low.

Impact of dividends on the lower bound on a European call value and the early exercise policy of an American call option.

- Portfolio B is modified to contain one unit of the underlying asset and a loan of D dollars of cash (in the form of a portfolio of bonds as specified earlier). At expiry, the value of Portfolio B will always become S_T since the loan of D will be paid back during the life of the option using the dividends received.

- Since $V_A \geq V_B$ at expiry, hence the present value of Portfolio A must be at least as much as that of Portfolio B. Together with the non-negativity property of option values, we obtain

$$c(S, \tau; X, D) \geq \max(S - XB(\tau) - D, 0).$$

- When S is sufficiently high, the European call almost behaves like a forward whose value is $S - D - XB(\tau)$. Recall the put-call parity relation: $c = p + S - D - XB(\tau)$, so

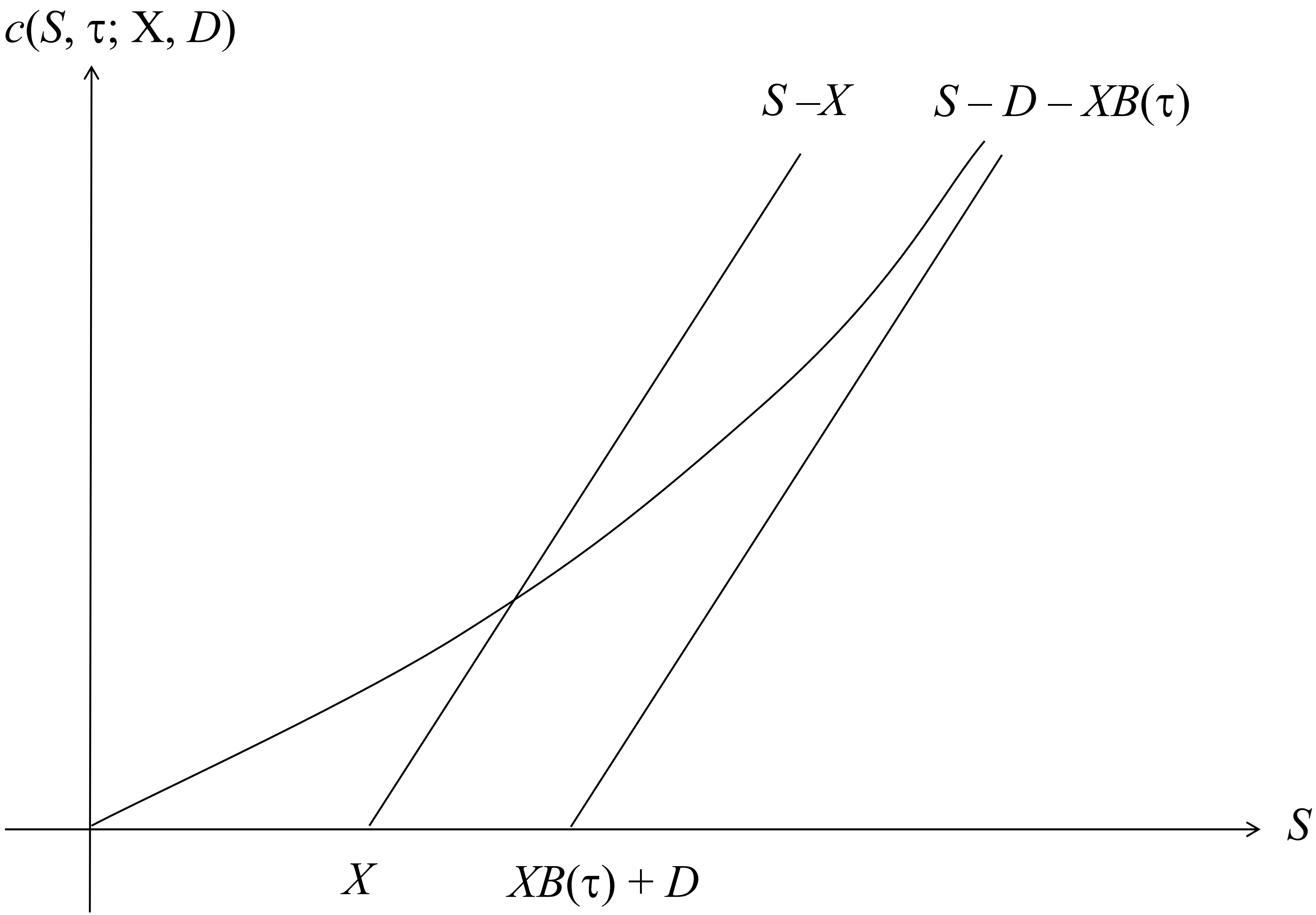
$$p(S, \tau; X, D) \to 0 \text{ as } S \to \infty.$$

- Since the call price is lowered due to the dividends of the underlying asset, it may be possible that the call price becomes less than the intrinsic value $S - X$ when the lumped dividend D is deep enough.

- The necessary condition on D such that $c(S, \tau; X, D)$ may fall below the intrinsic value $S - X$ is given by

$$S - X > S - XB(\tau) - D \text{ or } D > X[1 - B(\tau)].$$

  If D does not satisfy the above condition, it is never optimal to exercise the American call prematurely.



When $D > X[1 - B(\tau)]$, the lower bound $S - D - XB(\tau)$ becomes less than the intrinsic value $S - X$. As $S \rightarrow \infty$, the European call price curve falls below the intrinsic value line.

- The American call must be sufficiently deep in-the-money so that the chance of regret on early exercise is low.

- Since there will be an expected decline in asset price right after a discrete dividend payment, the optimal strategy is to exercise right before the dividend payment so as to capture the dividend paid by the asset.

- The holder of a put option gains when the asset price drops after a discrete dividend is paid since the put value is a decreasing function of the asset price. The American put holder would never optimally early exercise right before the dividend date.

Bounds on puts

The bounds for American and European puts can be shown to be

$$P(S,\tau;X,D) \geq p(S,\tau;X,D) \geq \max(XB(\tau)+D-S,0).$$

- When $XB(\tau)+D < X \Leftrightarrow D < X[1-B(\tau)]$, the lower bound $XB(\tau)-S$ may become less than the intrinsic value $X-S$ when the put is sufficiently deep in-the-money (corresponding to low value for S). Since it is sub-optimal for the holder of an American put option when the put value falls below the intrinsic value, the American put should be exercised prematurely.

- The presence of dividends makes the early exercise of an American put option less likely since the holder loses the future dividends when the asset is sold upon exercising the put.

Put-call parity relations

For a pair of European put and call options on the same underlying asset and with the same expiration date and strike price, we have

$$p = c - S + D + XB(\tau).$$

When the underlying asset is non-dividend paying, we set $D = 0$.

- The first portfolio involves long holding of a European call, a cash amount of $D + XB(\tau)$ and short selling of one unit of the asset.

- The second portfolio contains only one European put.

- The cash amount D in the first portfolio is used to compensate the dividends due to the short position of the asset.

- At expiry, both portfolios are worth $\max(X - S_T, 0)$.

Since both options are European, they cannot be exercised prior to expiry. Hence, both portfolios have the same value throughout the life of the options.

- The parity relation cannot be applied to American options due to their early exercise feature.

Lower and upper bounds on the difference of the prices of American call and put options

First, we assume the underlying asset to be non-dividend paying. Since $P > p$ and $C = c$, and putting $D = 0$,

$$C - P < S - XB(\tau),$$

giving the upper bound on $C - P$.

- Consider the following two portfolios: one contains a European call plus cash of amount X, and the other contains an American put together with one unit of underlying asset.

If there were no early exercise of the American put prior to maturity, the terminal value of the first portfolio is always higher than that of the second portfolio. If the American put is exercised prior to maturity, the second portfolio's value becomes X, which is always less than $c + X$. The first portfolio dominates over the second portfolio, so we have

$$c + X > P + S.$$

Further, since $c = C$ when the asset does not pay dividends, the lower bound on $C - P$ is given by

$$S - X < C - P.$$

Combining the two bounds, the difference of the American call and put option values on a non-dividend paying asset is bounded by

$$S - X < C - P < S - XB(\tau).$$

- The right side inequality: $C - P < S - XB(\tau)$ also holds for options on a dividend paying asset since dividends decrease call value and increase put value.

- The left side inequality has to be modified as: $S - D - X < C - P$.

- Combining the results, the difference of the American call and put option values on a dividend paying asset is bounded by

$$S - D - X < C - P < S - XB(\tau).$$

1.3 Early exercise policies of American options

Non-dividend paying asset

- At any moment when an American call is exercised, its value immediately becomes max$(S - X, 0)$. The exercise value is less than max$(S - XB(\tau), 0)$, the lower bound of the call value if the call remains alive.

- This implies that the act of exercising prior to expiry causes a decline in value of the American call. To the benefit of the holder, an American call on a non-dividend paying asset will not be exercised prior to expiry. Since the early exercise privilege is forfeited, the American and European call values should be the same.

- For an American put, it may become optimal to exercise prematurely when S falls to sufficiently low value, provided that the dividends are not sufficiently deep.

Dividend paying asset

- When the underlying asset pays dividends, the early exercise of an American call prior to expiry may become optimal when (i) S is very high and (ii) the dividends are sizable. Under these circumstances, it then becomes more attractive for the investor to acquire the asset rather than holding the option.
  - When S is high, the chance of regret of early exercise is low; equivalently, the insurance value of holding the call is lower.
  - When the dividends are sizable, it is more attractive to hold the asset directly instead of holding the call.

- For an American put, when D is sufficiently high, it may become non-optimal to exercise prematurely even at very low value of S (even when the put is very deep-in-the-money).

American call on an asset with discrete dividends

- Since the holder of an American call on an asset with discrete dividends will not receive any dividend in between dividend times, so within these periods, it is never optimal to exercise the American call.

- It may be optimal to exercise the American call *immediately before* the asset goes ex-dividend. What are the necessary and sufficient conditions?

One-dividend model – Amount of D is paid out at t_d

- If the American call is exercised at t_d^-, the call value becomes $S_d^- - X$. If there is no early exercise, then the asset price drops to $S_d^+ = S_d^- - D$ right after the dividend payout.

- It behaves like an ordinary European option for $t > t_d^+$. This is because when there is no further dividend, it becomes always non-optimal to exercise the American call.

- The lower bound of the one-dividend American call value at t_d^+ is the same lower bond for a European call, which is given by $S_d^+ - Xe^{-r(T-t_d^+)}$, where $T - t_d^+$ is the time to expiry.

- By virtue of the continuity of the call value across the dividend date, the lower bound B for the call value at time t_d^- should also be equal to $B = S_d^+ - Xe^{-r(T-t_d)} = (S_d^- - D) - Xe^{-r(T-t_d)}$.

By comparing the call value (continuation) with the early exercise proceed: $S_d^- - X$, we deduce

(i) If $S_d^- - X \leq B$,

$$S_d^- - X \leq (S_d^- - D) - Xe^{-r(T-t_d)}$$
$$\text{or} \qquad D \leq X[1 - e^{-r(T-t_d)}]$$

it is never optimal to exercise since exercising leads to a drop in call value.

(ii) When the discrete dividend D is sufficiently deep such that

$$D > X[1 - e^{-r(T-t_d)}],$$

it may become optimal to exercise at t_d^- when the asset price S_d^- is above some threshold value S_d^*.

Let $C_d(S_d^-, T - t_d^-)$ denote the American call price at t_d^-, where time to expiry is $T - t_d^-$.

If the American call stays alive, then

$$C_d(S_d^-, T - t_d^-) = c(S_d^- - D, T - t_d^+) = c(S_d^+, T - t_d^+).$$

Optimal exercise price S_d^* is the solution to

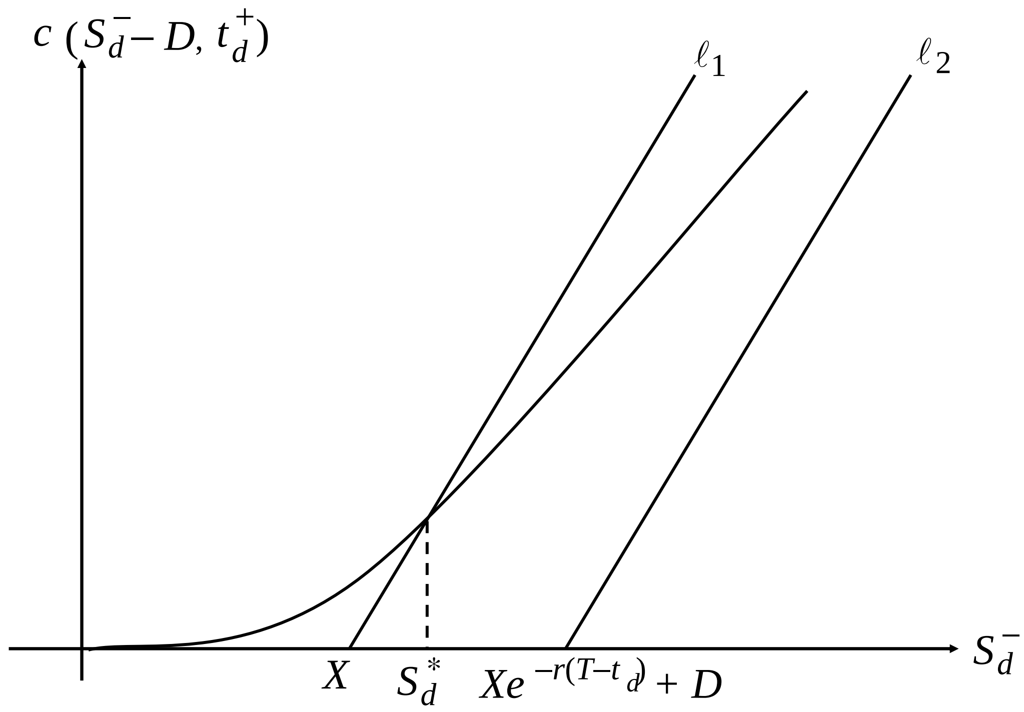
$$c(S_d^- - D, T - t_d^+) = S_d^- - X.$$

When $D > X[1 - e^{-r(T - t_d)}]$, then

$$C_d(S_d^-, T - t_d^-) = \begin{cases} c(S_d^- - D, T - t_d^+) & \text{when } S_d^- < S_d^* \\ S_d^- - X & \text{when } S_d^- \geq S_d^* \end{cases}.$$

Thus, S_d^* depends on D, which decreases when D increases. This is because the price curve of $c(S_d^- - D, T - t_d^+)$ is lowered and it cuts the intrinsic value line ℓ_1 at a lower value of S_d^*.

Determination of S_d^* (potential early exercise at t_d^- when D is sufficiently deep)



The European call price function $V = c(S_d^- - D, t_d^+)$ falls below the exercise payoff line $\ell_1 : E = S_d^- - X$ when ℓ_1 lies to the left of the lower bound value line $\ell_2 : B = S_d^- - D - Xe^{-r(T-t_d)}$. Here, S_d^* is the value of S_d^- at which the European call price curve cuts the exercise payoff line ℓ_1.

Summary of early exercise policies of American calls

- With no dividends, the decision of early exercise of an American option (call or put) depends on the competition between the time value of X and the loss of insurance value associated with the holding of the option.
- Early exercise of non-dividend paying American call is non-optimal since this leads to the loss of insurance value of the call plus the loss of time value of X.
- For an American call on a discrete dividend paying asset, it may become optimal to exercise at time right before the ex-dividend time, provided that the dividend amount is sizable and the call is sufficiently deep in-the-money. The critical asset price is a decreasing function of the size of dividend. Early exercise at a lower asset price level leads to a greater loss of insurance value but the loss is offset by the more sizable dividend received.

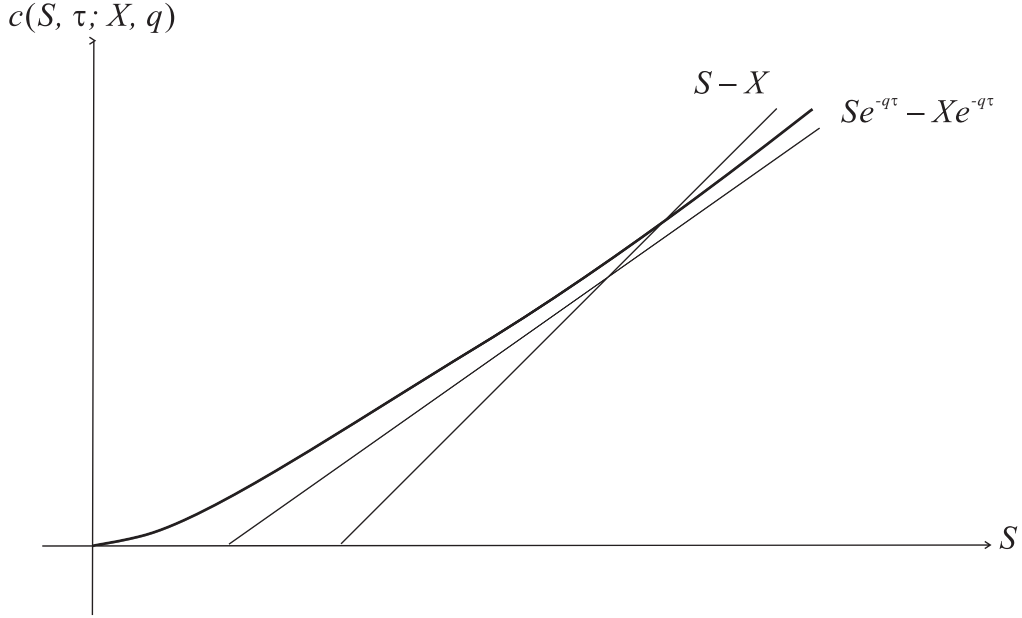
Continuous dividend model

Under constant dividend yield q, the dividend amount received during $(t, t + dt)$ from holding one unit of asset is $qS_t\, dt$. Also, $e^{-q(T-t)}$ unit of the asset at time t will become one unit at time T through accumulation of the dividends into purchase of the asset.

Why do we consider dividend yield model?

- It is considered as a continuous approximation to the discrete dividends model. Otherwise, pricing under the discrete n-dividend model requires the joint distribution of asset prices at all dividend dates: $S_{t_{d_1}}, S_{t_{d_2}}, \cdots, S_{t_{d_n}}$.

- The foreign money market account, which serves as the underlying asset in exchange options, earns the foreign interest rate r_f as dividend yield.



When the underlying asset pays dividend yield q, the lower bound of a European call $c(S,\tau;X,q)$ becomes $\max\left(Se^{-q\tau} - Xe^{-r\tau}, 0\right)$. As S becomes sufficiently high, $Se^{-q\tau} - Xe^{-r\tau}$ becomes less than $S - X$. Actually, as $S \to \infty$,

$$c(S,\tau;X,q) \to Se^{-q\tau} - Xe^{-r\tau}.$$

Smooth pasting condition at $S^*(\tau)$ under continuous dividend yield model

Value matching at $S^*(\tau) : C(S^*(\tau), \tau) = S^*(\tau) - X$.

Smooth pasting at $S^*(\tau) : \dfrac{\partial C}{\partial S}(S^*(\tau), \tau) = 1$.

$S^*(\tau)$ can be visualized as the lowest asset price at which the American call does not depend on the time to expiry. That is,

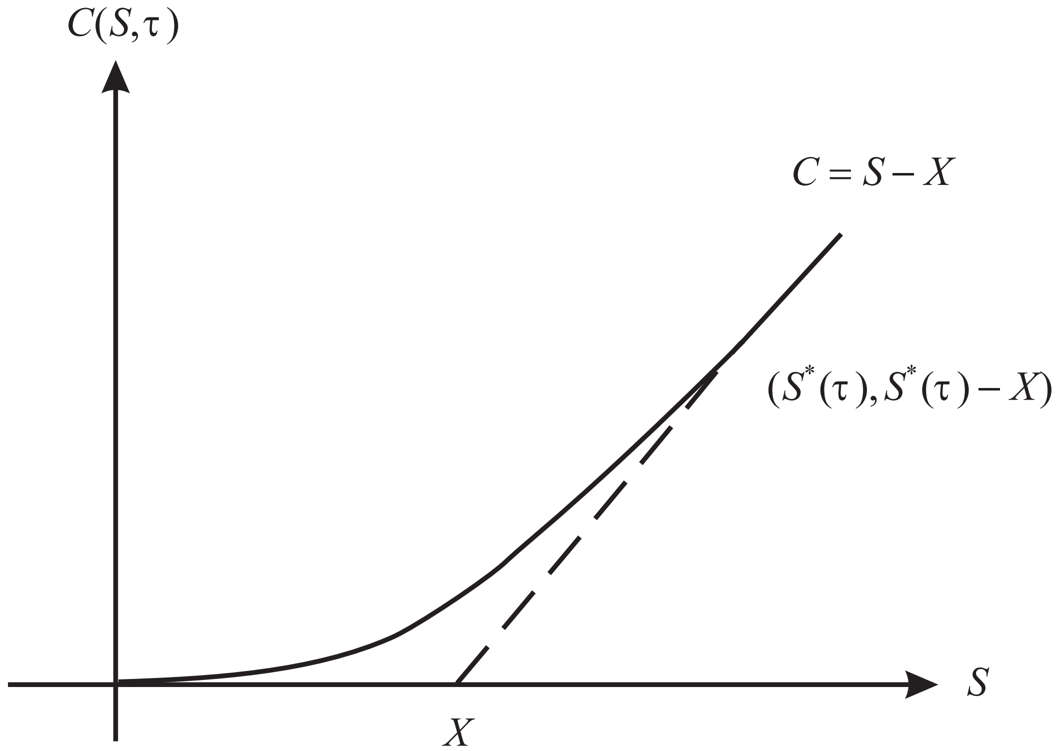
$$\frac{\partial C}{\partial \tau} = 0 \quad \text{at} \quad S = S^*(\tau).$$

Find the total derivative of the value matching condition with respect to τ:

$$\frac{d}{d\tau}\left[C(S^*(\tau), \tau)\right] = \frac{\partial C}{\partial \tau}(S^*(\tau), \tau) + \frac{\partial C}{\partial S}(S^*(\tau), \tau)\frac{dS^*(\tau)}{d\tau} = \frac{dS^*(\tau)}{d\tau}$$

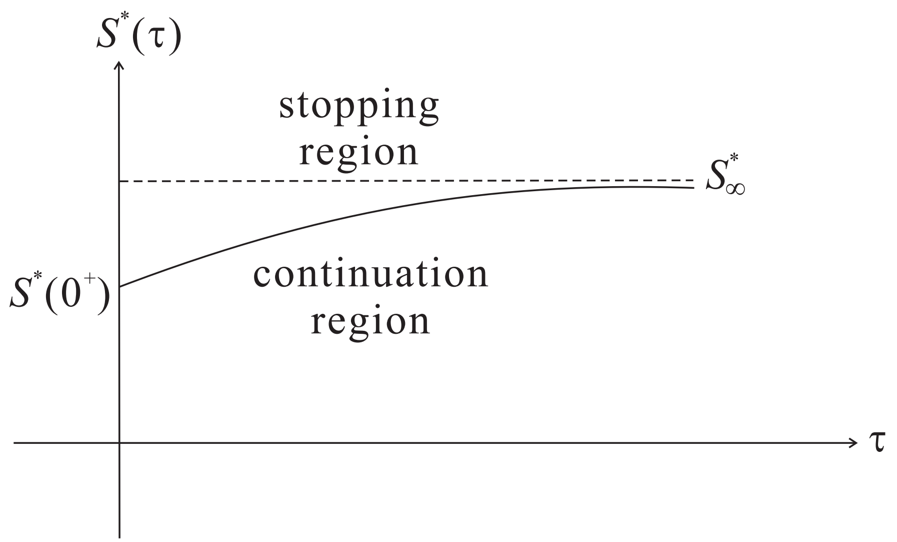
so that $\dfrac{\partial C}{\partial S}(S^*(\tau), \tau) = 1$. The smooth pasting condition can also be derived from the optimality of early exercise (first order derivative condition).

American call on a continuous dividend yield paying asset



The option price curve of a longer-lived American call will be above that of its shorter-lived counterpart for all values of S. The upper price curve cuts the intrinsic value tangentially at a higher critical asset value $S^*(\tau)$. Hence, $S^*(\tau)$ for an American call is an increasing function of τ.

Properties of optimal early exercise boundary $S^*(\tau)$ of an American call under continuous dividend yield model



X = strike price, r = riskfree interest rate, q = constant dividend yield, σ = volatility of asset price

$$S^*(0) = X\max\left(1, \frac{r}{q}\right), S^*_\infty = \frac{\mu_+}{\mu_+ - 1}X,$$

$$0 < \mu_+ = \frac{-\left(r - q - \frac{\sigma^2}{2}\right) + \sqrt{\left(r - q - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r}}{\sigma^2}$$

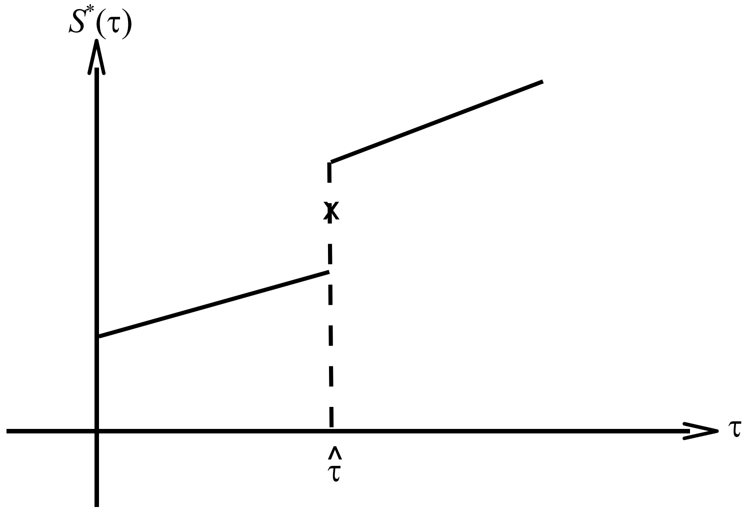
Stopping region = $\{(S, \tau) : S \geq S^*(\tau)\}$, inside which the American call should be optimally exercised. When $S < S^*(\tau)$, it is optimal for the holder to continue holding the American call option.

1. $S^*(\tau)$ is monotonically increasing with respect to τ with $S^*(0^+) = X \max\left(1, \frac{r}{q}\right)$ and $S^*_\infty = \frac{\mu_+}{\mu_+ - 1}X$. The determination of S^*_∞ requires a pricing model of the perpetual American call option.

2. $S^*(\tau)$ is a continuous function of τ when the asset price process is continuous.

3. $S^*(\tau) \geq X$ for $\tau \geq 0$

   Suppose $S^* < X$, then the early exercise proceed $S^*(\tau) - X$ becomes negative. This must be ruled out.



Suppose $S^*(\tau)$ has a downward jump as τ decreases across $\tilde{\tau}$. Let the stock price at $\tilde{\tau}$ satisfy $S^*(\tilde{\tau}^-) < S_{\tilde{\tau}} < S^*(\tilde{\tau}^+)$. Assume that there is no jump in the asset price.

- At $\tilde{\tau}^+, C(S, \tilde{\tau}^+) > S - X$ since $S < S^*(\tilde{\tau}^+)$ (continuation region).

- At $\tilde{\tau}^-, C(S, \tilde{\tau}^-) = S - X$ since $S > S^*(\tilde{\tau}^-)$ (stopping region).

The discrete downward jump in option value across $\tilde{\tau}$ would lead to an arbitrage opportunity.

Asymptotic behavior of $S^*(0^+)$ of an American call

At time close to expiry, the insurance value is negligible. It then suffices to determine the optimal early exercise policy by considering only the tradeoff between the gain in dividend and loss in the time value of the strike price.

(i) $q < r$

The American call is kept alive when $X < S < \frac{r}{q}X$. This is derived from the financial intuition that within a short time interval δt prior to expiry, the dividend $qS\delta t$ earned from holding the asset is less than the interest $rX\delta t$ earned from depositing the amount X in a bank to earn interest at the rate r.

Hence, for $q < r$, $\lim_{\tau \to 0^+} S^*(\tau) = \frac{r}{q}X$.

American call on non-dividend paying asset

In particular, when $q = 0, S^*(\tau) \to \infty$ as $\tau \to 0^+$. Since $S^*(\tau)$ is monotonically increasing with respect to τ, so

$$S^*(\tau) \to \infty \quad \text{for all values of } \tau.$$

This agrees with the earlier conclusion that it is always non-optimal to exercise an American call prematurely when $q = 0$.

(ii) $q \geq r$

In this case, $\frac{r}{q}X$ becomes less than X. We argue that $S(0^+) \leq X$. Assume the contrary, suppose $S^*(0^+) > X$ so that the American call is still alive when $X < S < S^*(0^+)$ at time close to expiry. Now, since $q \geq r$ and $S > X$, the loss in dividend amount $qS\delta t$ not earned is more than the interest amount $rX\delta t$ earned. This represents a non-optimal early exercise policy.

Together with the condition: $S^*(0^+) \geq X$, we must have

$$\lim_{\tau \to 0^+} S^*(\tau) = X \quad \text{for} \quad q \geq r.$$

In summary,

$$\lim_{\tau \to 0^+} S^*(\tau) = \begin{cases} \frac{r}{q}X & q < r \\ X & q \geq r \end{cases}$$
$$= X \max\left(1, \frac{r}{q}\right).$$

- When $q \geq r$, an American call which is optimally held to expiration will have zero value at expiration.
- At expiry $\tau = 0$, the American call option will be exercised whenever $S \geq X$ and so $S^*(0) = X$. Hence, for $q < r$, there is a jump of discontinuity of $S^*(\tau)$ at $\tau = 0$.

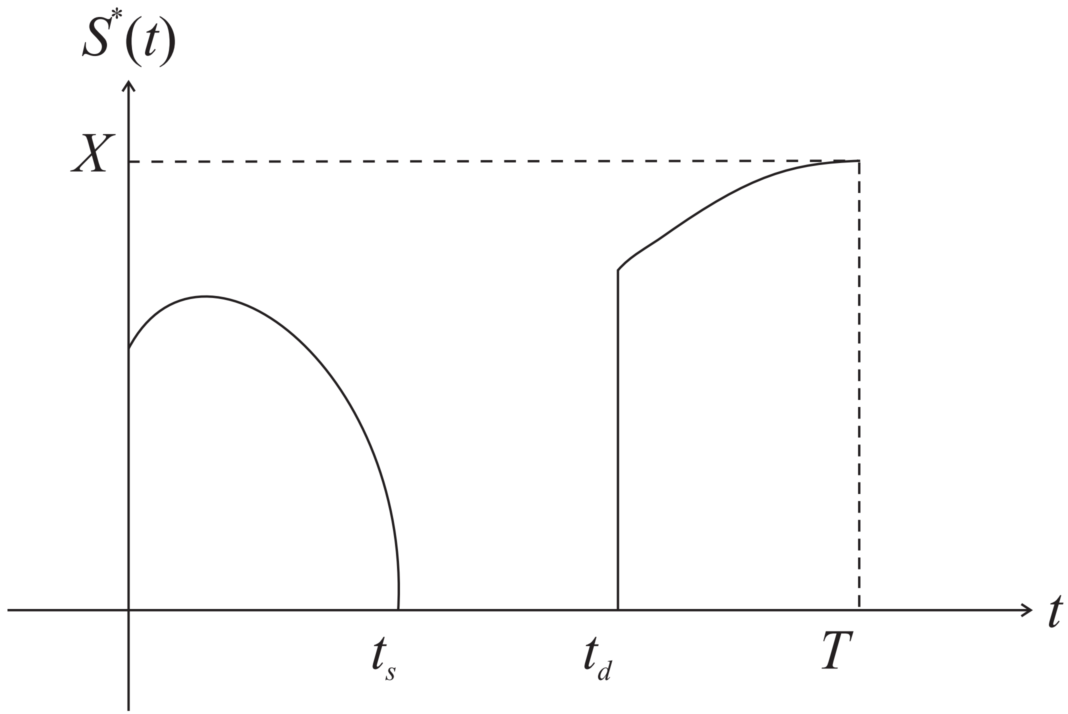
One-dividend paying model for an American put

- A single dividend D is paid at t_d.

- Never exercise immediately prior to the dividend payment for time $t < t_d$, interest income $= X[e^{r(t_d - t)} - 1]$; comparing interest income with dividend D at some time, where

$$X[e^{r(t_d - t_s)} - 1] = D \text{ giving } t_s = t_d - \frac{\ln\left(1 + \frac{D}{X}\right)}{r}.$$

- When $t < t_s$, early exercise is optimal when S falls below some critical asset price $S^*(t)$.



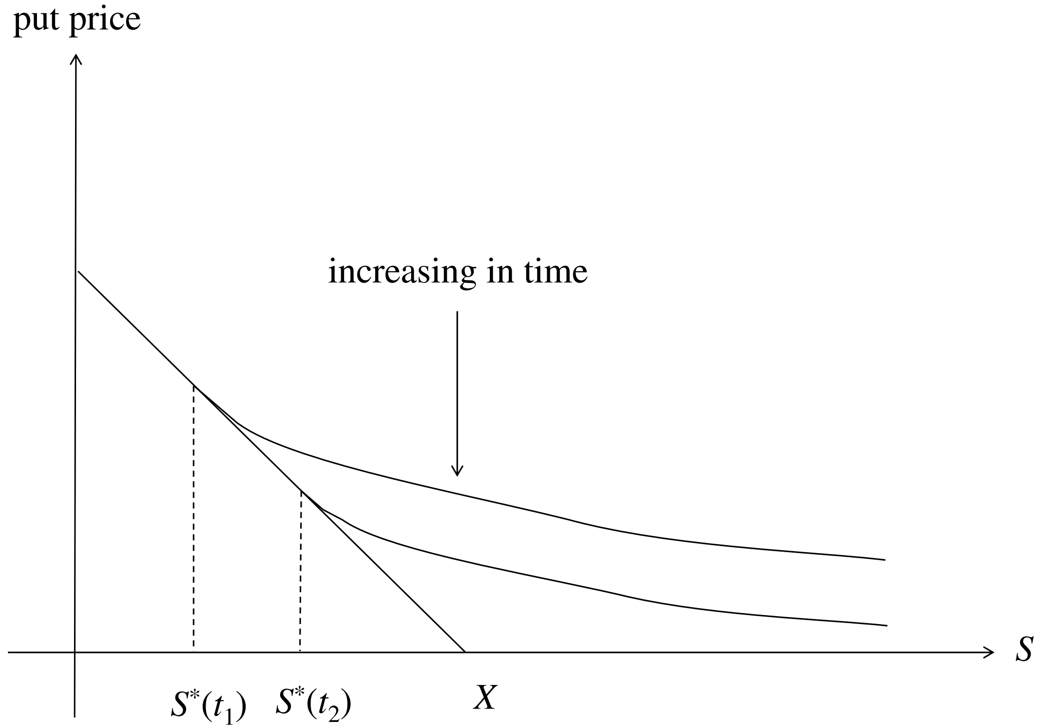
The behavior of the optimal exercise boundary $S^*(t)$ as a function of t for a one-dividend American put option. Note that $S(T) = X$ since the underlying asset is non-dividend paying after t_d.

In summary, the optimal exercise boundary $S^*(t)$ of the one-dividend American put model exhibits the following behavior.

(i) When $t < t_s$, $S^*(t)$ first increases then decreases smoothly with increasing t until it drops to the zero value at t_s.

(ii) $S^*(t)$ stays at the zero value in the interval $[t_s, t_d]$.

(iii) When $t \in (t_d, T], S^*(t)$ is a montonically increasing function of t with $S^*(T) = X$.



Here, $t_d < t_1 < t_2 < T$. The put price curve at time t_1 intersects the intrinsic value line tangentially at $S^*(t_1)$. We observe: $S^*(t_1) < S^*(t_1) < X$.