MATH 571 — Mathematical Models of Financial Derivatives

Topic 3 – Filtrations, martingales and multi-period models

3.1 Information structures and filtrations

3.2 Notion of martingales

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3.4 No-arbitrage principle and martingale pricing measure

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Setup of multiperiod securities model

- Securities models are of multiperiod, where there are $T+1$ trading dates: $t = 0, 1, \cdots, T, T > 1$.
- A finite sample space Ω of K elements, $\Omega = \{\omega_1, \omega_2, \cdots, \omega_K\}$, which represents the possible states of the world.
- A probability measure P defined on the sample space with $P(\omega) > 0$ for all $\omega \in \Omega$.
- M risky securities whose price processes are non-negative stochastic processes, as denoted by $S_m = \{S_m(t); t = 0, 1, \cdots, T\}, m = 1, \cdots, M$.

- Riskfree security whose price process $S_0(t)$ is deterministic, with $S_0(t)$ strictly positive and possibly non-decreasing.

- We may consider $S_0(t)$ as the money market account, and the quantity $r_t = \dfrac{S_0(t) - S_0(t-1)}{S_0(t-1)}, t = 1, \cdots, T$, is visualized as the interest rate over the time interval $(t-1, t)$.

- Specify how the investors learn about the true state of the world on intermediate trading dates in a multi-period model.

- Construct some information structure that models how information is revealed to investors in terms of the partitions of the sample space Ω.

- We form a partition $\mathcal{P}$ of Ω, which is a collection of disjoint subsets (events) of Ω. The information $\mathcal{P}$ is revealed to us if we have learned to which the atoms of $\mathcal{P}$ the true outcome of the experiment belongs. In the dice tossing experiment, $\mathcal{P}_{coarse} = \{\{1,4\},\{2,3,5,6\}\}$ and $\mathcal{P}_{fine} = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}$.

Outline

- How can an information structure be described by a filtration (nested sequence of algebras)? Understand how the security price processes can be adapted to a given filtration $\{\mathcal{F}_t\}_{t=0,1,\cdots,T}$ [$S_m(t)$ is measurable with respect to the algebra $\mathcal{F}_t$].

- We introduce martingales, which are defined with reference to conditional expectations. In the multi-period setting, the risk neutral probability measures are defined in terms of martingales. All discounted price processes of risky securities are martingales under a risk neutral measure.

- Derivation of the Fundamental Theorem of Asset Pricing.

- Multiperiod binomial models.

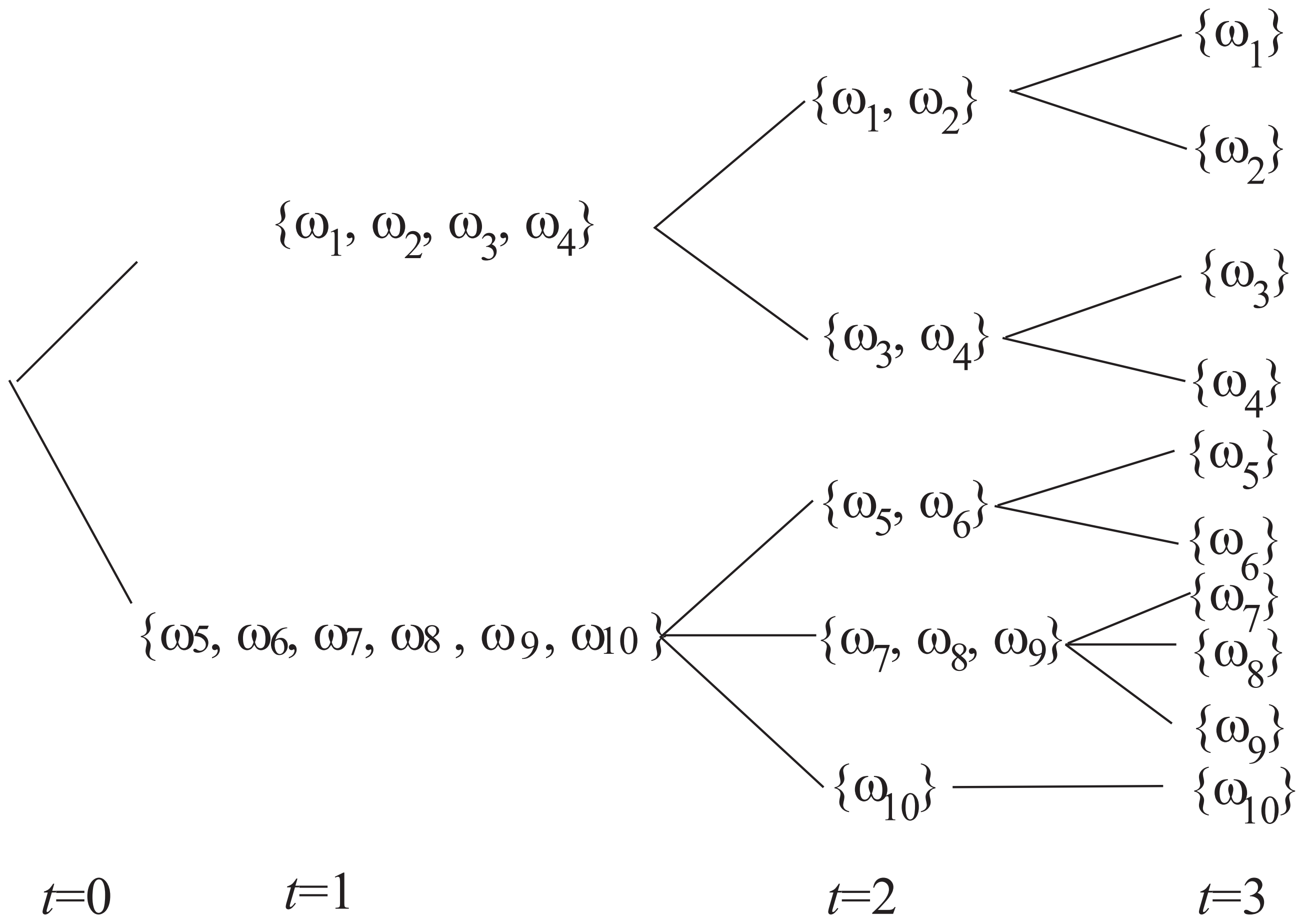
3.1 Information structures and filtrations

Consider the sample space $\Omega = \{\omega_1, \omega_2, \cdots, \omega_{10}\}$ with 10 elements. We can construct various partitions of the set Ω.

A *partition* of Ω is a collection $\mathcal{P} = \{B_1, B_2, \cdots, B_n\}$ such that $B_j, j = 1, \cdots, n$, are subsets of Ω and $B_i \cap B_j = \phi, i \neq j$, and $\bigcup_{j=1}^{n} B_j = \Omega$.

Each of the sets $B_1, \cdots, B_n$ is called an *atom* of the partition. For example, we may form the partitions as

$$
\begin{aligned}
\mathcal{P}_0 &= \{\Omega\} \\
\mathcal{P}_1 &= \{\{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8, \omega_9, \omega_{10}\}\} \\
\mathcal{P}_2 &= \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}, \{\omega_7, \omega_8, \omega_9\}, \{\omega_{10}\}\} \\
\mathcal{P}_3 &= \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}, \{\omega_9\}, \{\omega_{10}\}\}.
\end{aligned}
$$



Information tree of a three-period securities model with 10 possible states. The partitions form a sequence of successively finer partitions. The information structure describes the arrival of information as time lapses.

- Consider a three-period securities model that consists of a sequence of successively finer partitions: $\{\mathcal{P}_k : k = 0, 1, 2, 3\}$. The pair $(\Omega, \mathcal{P}_k)$ is called a *filtered space*, which consists of a sample space Ω and a sequence of partitions of Ω. The filtered space is used to model the unfolding of information through time.

- At time $t = 0$, the investors know only the set of all possible outcomes, so $\mathcal{P}_0 = \{\Omega\}$.

- At time $t = 1$, the investors get a bit more information: the actual state ω is in either $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ or $\{\omega_5, \omega_5, \omega_7, \omega_8, \omega_9, \omega_{10}\}$.

Algebra

Let Ω be a finite set and $\mathcal{F}$ be a collection of subsets of Ω. The collection $\mathcal{F}$ is an *algebra* on Ω if

(i) $\Omega \in \mathcal{F}$

(ii) $B \in \mathcal{F} \Rightarrow B^c \in \mathcal{F}$

(iii) B_1 and $B_2 \in \mathcal{F} \Rightarrow B_1 \cup B_2 \in \mathcal{F}$.

An algebra on Ω is a family of subsets of Ω closed under finitely many set operations.

- Given an algebra $\mathcal{F}$ on Ω, one can always find a unique collection of disjoint subsets B_n such that each $B_n \in \mathcal{F}$ and the union of the subsets equals Ω.

- The algebra $\mathcal{F}$ generated by a partition $\mathcal{P} = \{B_1, \cdots, B_n\}$ is a set of subsets of Ω. Actually, when Ω is a finite sample space, there is a one-to-one correspondence between partitions of Ω and algebras on Ω.

- The information structure defined by a sequence of partitions can be visualized as a sequence of algebras. We define a *filtration* $\mathbb{F} = \{\mathcal{F}_k; k = 0, 1, \cdots, T\}$ to be a nested sequence of algebras satisfying $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$.

- Given the algebra $\mathcal{F} = \{\phi, \{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}, \{\omega_1, \omega_2, \omega_3\}, \{\omega_2, \omega_3, \omega_4\}, \{\omega_1, \omega_4\}, \{\omega_1, \omega_2, \omega_3, \omega_4\}\}$, the corresponding partition $\mathcal{P}$ is found to be $\{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}\}$.

- The atoms of $\mathcal{P}$ are $B_1 = \{\omega_1\}, B_2 = \{\omega_2, \omega_3\}$ and $B_3 = \{\omega_4\}$. A non-empty event whose occurrence to be revealed through revelation of $\mathcal{P}$ would be an union of atoms in $\mathcal{P}$.

- Take the event $A = \{\omega_1, \omega_2, \omega_3\}$ in the algebra $\mathcal{F}$, which is the union of B_1 and B_2. Given that $B_2 = \{\omega_2, \omega_3\}$ of $\mathcal{P}$ has occurred, we can decide whether A or its complement A^c has occurred. However, for another event $\widetilde{A} = \{\omega_1, \omega_2\}$, even though we know that B_2 has occurred, we cannot determine whether $\widetilde{A}$ or $\widetilde{A}^c$ has occurred. The event $\widetilde{A}$ whose occurrence cannot be revealed through revelation of $\mathcal{P}$.

Consider a probability measure P defined on an algebra $\mathcal{F}$. The probability measure P is a function

$$P : \mathcal{F} \rightarrow [0, 1]$$

such that

1. $P(\Omega) = 1$.

2. If $B_1, B_2, \cdots$ are pairwise disjoint sets belonging to $\mathcal{F}$, then

$$P(B_1 \cup B_2 \cup \cdots) = P(B_1) + P(B_2) + \cdots .$$

Equipped with a probability measure, the elements of $\mathcal{F}$ are called measurable events. Given the sample space Ω and a probability measure P defined on Ω, together with the filtration $\mathbb{F}$ associated with $\mathcal{F}$, the triplet $(\Omega, \mathcal{F}, P)$ is called a *filtered probability space*.

Measurability of random variables

- Consider an algebra $\mathcal{F}$ generated by a partition $\mathcal{P} = \{B_1, \cdots, B_n\}$, a random variable X is said to be measurable with respect to $\mathcal{F}$ (denoted by $X \in \mathcal{F}$) if $X(\omega)$ is constant for all $\omega \in B_i, B_i$ is any element in $\mathcal{P}$. For example, consider the algebra $\mathcal{F}_1$ generated by $\mathcal{P}_1 = \{\{\omega_1, \omega_2, \omega_3, \omega_4\},\ \{\omega_5, \omega_6, \omega_7, \omega_8, \omega_9, \omega_{10}\}\}$. If $X(\omega_1) = 3$ and $X(\omega_4) = 5$, then X is not measurable with respect to $\mathcal{F}_1$.

- Consider an example where $\mathcal{P} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5\}\}$ and X is measurable with respect to the algebra $\mathcal{F}$ generated by $\mathcal{P}$. Let $X(\omega_1) = X(\omega_2) = 3, X(\omega_3) = X(\omega_4) = 5$ and $X(\omega_5) = 7$. Suppose the random experiment associated with the random variable X is performed, giving $X = 5$. This tells the information that the event $\{\omega_3, \omega_4\}$ has occurred.

- The information of outcome from the random experiment is revealed through the random variable X. We may say that $\mathcal{F}$ is being generated by X.

- A stochastic process $S_m = \{S_m(t); t = 0, 1, \cdots, T\}$ is said to be *adapted to the filtration* $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \cdots, T\}$ if the random variables $S_m(t)$ is $\mathcal{F}_t$-measurable for each $t = 0, 1, \cdots, T$.

- For the bank account process $S_0(t)$, the interest rate is normally known at the beginning of the period so that $S_0(t)$ is $\mathcal{F}_{t-1}$-measurable, $t = 1, \cdots, T$. In this case, we say that the process $S_0(t)$ is *predictable*.

Conditional expectations

- Consider the filtered probability space defined by the triplet $(\Omega, \mathcal{F}, P)$. Recall that a random variable is a mapping $\omega \to X(\omega)$ that assigns a real number $X(\omega)$ to each $\omega \in \Omega$.

- A random variable is said to be simple if X can be decomposed into the form

$$X(\omega) = \sum_{j=1}^{n} a_j \mathbf{1}_{B_j}(\omega)$$

where $\{B_1, \cdots B_n\}$ is the finite partition of Ω that generates $\mathcal{F}$. The indicator of B_j is defined by

$$\mathbf{1}_{B_j}(\omega) = \begin{cases} 1 & \text{if } \omega \in B_j \\ 0 & \text{if otherwise} \end{cases}.$$

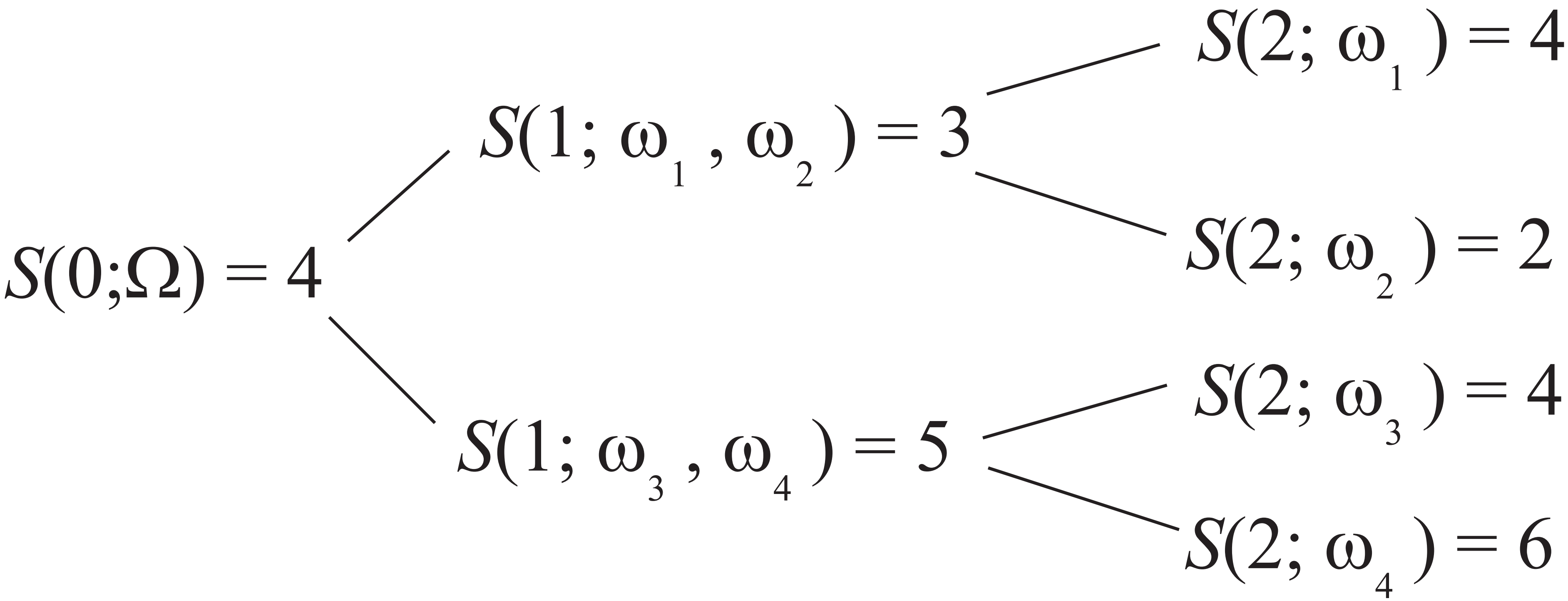
- The expectation of X with respect to the probability measure P is defined as

$$E[X] = \sum_{j=1}^{n} a_j E[\mathbf{1}_{B_j}(\omega)] = \sum_{j=1}^{n} a_j P[B_j],$$

where $P[B_j]$ is the probability that a state ω contained in B_j occurs. The expectation $E[X]$ is a weighted average of values taken by X, weighted according to the various probabilities of occurrence of events. The set of events run through the whole sample space Ω.

- The conditional expectation of X given that event B has occurred is defined to be

$$\begin{aligned} E[X|B] &= \sum_x xP[X = x|B] \\ &= \sum_x xP[X = x, B]/P[B] \\ &= \frac{1}{P[B]} \sum_{\omega \in B} X(\omega)P[\omega]. \end{aligned}$$



The asset price process of a two-period securities model. The filtration $\mathbb{F}$ is revealed through the asset price process that is adapted to $\mathbb{F}$. Here, the partitions are: $\mathcal{P}_2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}, \mathcal{P}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$ and $\mathcal{P}_0 = \{\Omega\}$.

- Consider the sample space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. The probabilities of occurrence of the states are given by $P[\omega_1] = 0.2, P[\omega_2] = 0.3, P[\omega_3] = 0.35$ and $P[\omega_4] = 0.15$.

  Consider the two-period price process S whose values are given by

$$
\begin{array}{llll}
S(1;\omega_1) = 3, & S(1;\omega_2) = 3, & S(1;\omega_3) = 5, & S(1;\omega_4) = 5, \\
S(2;\omega_1) = 4, & S(2;\omega_2) = 2, & S(2;\omega_3) = 4, & S(2;\omega_4) = 6.
\end{array}
$$

The conditional expectations

$$E[S(2)|S(1) = 3] \quad \text{and} \quad E[S(2)|S(1) = 5]$$

are calculated by

$$\begin{aligned}
E[S(2)|S(1) = 3] &= \frac{S(2;\omega_1)P[\omega_1] + S(2;\omega_2)P[\omega_2]}{P[\omega_1] + P[\omega_2]} \\
&= (4 \times 0.2 + 2 \times 0.3)/0.5 = 2.8;
\end{aligned}$$

$$\begin{aligned}
E[S(2)|S(1) = 5] &= \frac{S(2;\omega_3)P[\omega_3] + S(2;\omega_4)P[\omega_4]}{P[\omega_3] + P[\omega_4]} \\
&= (4 \times 0.35 + 6 \times 0.15)/0.5 = 4.6.
\end{aligned}$$

Note that "$S(1) = 3$" is equivalent to the occurrence of either ω_1 or ω_2.

$E[X|\mathcal{F}]$ as a random variable measurable on $\mathcal{F}$

We consider all conditional expectations of the form $E[X|B]$ where the event B runs through the algebra $\mathcal{F}$. We define the quantity $E[X|\mathcal{F}]$ by

$$E[X|\mathcal{F}] = \sum_{j=1}^{n} E[X|B_j]\mathbf{1}_{B_j}.$$

We see that $E[X|\mathcal{F}]$ is actually a random variable that is measurable with respect to the algebra $\mathcal{F}$. In the above numerical example, suppose we write $\mathcal{F}_1 = \{\phi, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega\}$, and the atoms of the partition associated with $\mathcal{F}_1$ are $B_1 = \{\omega_1, \omega_2\}$ and $B_2 = \{\omega_3, \omega_4\}$. Since we have

$$E[S(2)|S(1) = 3] = 2.8 \quad \text{and} \quad E[S(2)|S(1) = 5] = 4.6,$$

so

$$E[S(2)|\mathcal{F}_1] = 2.8\mathbf{1}_{B_1} + 4.6\mathbf{1}_{B_2}.$$

- Suppose that the random variable X is $\mathcal{F}$-measurable, we would like to show $E[XY|\mathcal{F}] = XE[Y|\mathcal{F}]$ for any random variable Y.

- Recall that $X = \sum_{B_j \in \mathcal{P}} a_j \mathbf{1}_{B_j}$, where $X(\omega) = a_j$ when $\omega \in B_j$ and $\mathcal{P}$ is the partition corresponding to the algebra $\mathcal{F}$. We obtain

$$\begin{aligned} E[XY|\mathcal{F}] &= \sum_{B_j \in \mathcal{P}} E[XY|B_j]\mathbf{1}_{B_j} = \sum_{B_j \in \mathcal{P}} E[a_j Y|B_j]\mathbf{1}_{B_j} \\ &= \sum_{B_j \in \mathcal{P}} a_j E[Y|B_j]\mathbf{1}_{B_j} = XE[Y|\mathcal{F}]. \end{aligned}$$

Note that X is known with regard to the information provided by $\mathcal{F}$.

For example, in the above two period security model,

$$E[S(1)S(2)|\mathcal{F}_1] = \begin{cases} 3 \times 2.8 & \text{if } \omega_1 \text{ or } \omega_2 \text{ occurs} \\ 5 \times 4.6 & \text{if } \omega_3 \text{ or } \omega_4 \text{ occurs} \end{cases}.$$

Tower property of conditional expectation

Since $E[X|\mathcal{F}]$ is a random variable, we may compute its expectation. Recall $E[X|\mathcal{F}] = \sum_{j=1}^{n} E[X|B_j]\mathbf{1}_{B_j}$, and $E[\mathbf{1}_{B_j}] = P[B_j]$, so

$$\begin{aligned} E[E[X|\mathcal{F}]] &= \sum_{j=1}^{n} E[X|B_j]P[B_j] \\ &= \sum_{j=1}^{n} \left(\sum_{\omega\in B_j} \frac{X[\omega]P[\omega]}{P[B_j]} \right) P[B_j] = E[X]. \end{aligned}$$

In general, if $\mathcal{F}_1 \subset \mathcal{F}_2$, then

$$E[E[X|\mathcal{F}_2]|\mathcal{F}_1] = E[X|\mathcal{F}_1].$$

If we condition first on the information up to $\mathcal{F}_2$ and later on the information $\mathcal{F}_1$ at an earlier time, then it is the same as conditioning originally on $\mathcal{F}_1$. This is called the *tower property* of conditional expectations.

3.2 Notions of martingales

Martingales are related to models of fair gambling. For example, let X_n represent the amount of money a player possesses at stage n of the game. The martingale property means that the expected amount of the player would have at stage $n+1$ given that $X_n = \alpha_n$, is equal to α_n, regardless of his past history of fortune.

Consider a filtered probability space with filtration $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \cdots, T\}$. An adapted stochastic process $S = \{S(t); t = 0, 1 \cdots, T\}$ is said to be martingale if it observes

$$E[S(t+s)|\mathcal{F}_t] = S(t) \quad \text{for all } t \geq 0 \text{ and } s \geq 0.$$

We define an adapted stochastic process S to be a supermartingale if

$$E[S(t+s)|\mathcal{F}_t] \leq S(t) \quad \text{for all } t \geq 0 \text{ and } s \geq 0;$$

and a submartingale if

$$E[S(t+s)|\mathcal{F}_t] \geq S(t) \quad \text{for all } t \geq 0 \text{ and } s \geq 0.$$

1. All martingales are supermartingales, but not vice versa. The same observation is applied to submartingales.

2. An adapted stochastic process S is a submartingale if and only if $-S$ is a supermartingale; S is a martingales if and only if it is both a supermartingale and a submartingale.

Example

Recall in the earlier two-period security model

$$
\begin{aligned}
E[S(2)|\mathcal{F}_1] &= \begin{cases} 2.8 & \text{if } \omega_1 \text{ or } \omega_2 \text{ occurs} \\ 4.6 & \text{if } \omega_3 \text{ or } \omega_4 \text{ occurs} \end{cases} \\
&\leq S(1) = \begin{cases} 3 & \text{if } \omega_1 \text{ or } \omega_2 \text{ occurs} \\ 5 & \text{if } \omega_3 \text{ or } \omega_4 \text{ occurs} \end{cases} .
\end{aligned}
$$

Also

$$E[S(1)|\mathcal{F}_0] = 0.5 \times 3 + 0.5 \times 5 = 4 = S(0).$$

Hence $S(t)$ is a supermartingale.

- If the price process of a security is a supermartingale, after the arrival of new information, we expect a price decrease. Supermartingales are thus assocated with "unfavorable" games, that is, games where wealth is expected to decrease.

Martingale transforms

Suppose S is a martingale and H is a predictable process with respect to the filtration $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \cdots, T\}$, we define the process

$$G_t = \sum_{u=1}^{t} H_u \Delta S_u,$$

where $\Delta S_u = S_u - S_{u-1}$. One then deduces that $\Delta G_u = G_u - G_{u-1} = H_u \Delta S_u$. If S and H represent the asset price process and trading strategy, respectively, then G can be visualized as the gain process.

Note that trading strategy is a predictable process, that is, H_t is $\mathcal{F}_{t-1}$-measurable. This is because the number of units held for each security is determined at the beginning of the trading period by taking into account all the information available up to that time.

We call G to be the martingale transform of S by H, as G itself is also a martingale.

To show the claim, it suffices to show that $E[G_{t+s}|\mathcal{F}_t] = G_t, t \geq 0, s \geq 0$. We consider

$$\begin{aligned} E[G_{t+s}|\mathcal{F}_t] &= E[G_{t+s} - G_t + G_t|\mathcal{F}_t] \\ &= E[H_{t+1}\Delta S_{t+1} + \cdots + H_{t+s}\Delta S_{t+s}|\mathcal{F}_t] + E[G_t|\mathcal{F}_t] \\ &= E[H_{t+1}\Delta S_{t+1}|\mathcal{F}_t] + \cdots + E[H_{t+s}\Delta S_{t+s}|\mathcal{F}_t] + G_t. \end{aligned}$$

Consider the typical term $E[H_{t+u}\Delta S_{t+u}|\mathcal{F}_t]$, we can express it as $E[E[H_{t+u}\ \Delta S_{t+u}|\mathcal{F}_{t+u-1}]|\mathcal{F}_t]$.

Further, since H_{t+u} is $\mathcal{F}_{t+u-1}$-measurable and S is a martingale, we have

$$E[H_{t+u}\Delta S_{t+u}|\mathcal{F}_{t+u-1}] = H_{t+u}E[\Delta S_{t+u}|\mathcal{F}_{t+u-1}] = 0.$$

Collecting all the calculations, we obtain the desired result.

3.3 Discounted gain process and self-financing strategy under multiperiod securities models

- There is a sample space $\Omega = \{\omega_1, \omega_2, \cdots, \omega_K\}$ of K possible states of the world.

- Let S denote the asset price process $\{S(t); t = 0, 1, \cdots, n\}$, where $S(t)$ is the row vector $S(t) = (S_1(t)\ S_2(t) \cdots S_M(t))$ and whose components are security prices. Also, there is a bank account process $S_0(t)$, whose value is given by

$$S_0(t) = (1 + r_1)(1 + r_2) \cdots (1 + r_t),$$

where r_u is the interest rate applied over one time period starting at time $u, u = 0, 1, \cdots, t - 1$.

- A trading strategy is the rule taken by an investor that specifies the investor's position in each security at each time and in each state of the world based on the available information as prescribed by a filtration. Hence, one can visualize a trading strategy as an adapted stochastic process.

- We prescribe a trading strategy by a vector stochastic process $H(t) = (h_1(t)\ h_2(t) \cdots h_M(t))^T, t = 1, 2, \cdots, T$ (represented as a column vector), where $h_m(t)$ is the number of units held in the portfolio for the m^{th} security from time $t-1$ to time t.

The amount of bank account held at time $t-1$ is given by $h_0(t)S_0(t)$. Note that $h_m(t)$ should be $\mathcal{F}_{t-1}$-measurable, $m = 0, 1, \cdots, M$.

The value of the portfolio is a stochastic process given by

$$V(t) = h_0(t)S_0(t) + \sum_{m=1}^{M} h_m(t)S_m(t), \quad t = 1, 2, \cdots, T,$$

which gives the portfolio value at the moment right after the asset prices are observed but before changes in portfolio weights are made.

- $h(t)$ is held constant from $(t-1)^+$ to t^-; $S(t)$ is revealed exactly at time t; portfolio weight is then adjusted to $h(t+1)$ at t^+.

We write $\Delta S_m(t) = S_m(t) - S_m(t-1)$ as the change in value of one unit of the m^{th} security between times $t-1$ and t. The cumulative gain associated with investing in the m^{th} security from time zero to time t is given by

$$\sum_{u=1}^{t} h_m(u)\Delta S_m(u), \quad m = 0, 1, \cdots, M.$$

We define the gain process $G(t)$ to be the total cumulative gain in holding the portfolio consisting of the M risky securities and the bank account up to time t. The value of $G(t)$ is found to be

$$G(t) = \sum_{u=1}^{t} h_0(u)\Delta S_0(u) + \sum_{m=1}^{M}\sum_{u=1}^{t} h_m(u)\Delta S_m(u), \quad t = 1, 2, \cdots, T.$$

If we define the discounted price process $S_m^*(t)$ by

$$S_m^*(t) = S_m(t)/S_0(t), \quad t = 0, 1, \cdots, T, \quad m = 1, 2, \cdots, M,$$

and write $\Delta S_m^*(t) = S_m^*(t) - S_m^*(t-1)$, then the discounted value process $V^*(t)$ and discounted gain process $G^*(t)$ are given by

$$\begin{aligned} V^*(t) &= h_0(t) + \sum_{m=1}^{M} h_m(t) S_m^*(t), \quad t = 1, 2, \cdots, T, \\ G^*(t) &= \sum_{m=1}^{M} \sum_{u=1}^{t} h_m(u) \Delta S_m^*(u), \quad t = 1, 2, \cdots, T. \end{aligned}$$

Once the asset prices, $S_m(t), m = 1, 2, \cdots, M$, are revealed to the investor, he changes the trading strategy from $H(t)$ to $H(t+1)$ as a response to the arrival of the new information at time t.

Self-financing strategy

$$V(t) = h_0(t+1)S_0(t) + \sum_{m=1}^{M} h_m(t+1)S_m(t).$$

The purchase of additional units of one particular security is financed by the sales of other securities. In this case, the trading strategy is said to be *self-financing*.

If there were no addition or withdrawal of funds at all trading times, then the cumulative change of portfolio value $V(t)-V(0)$ should be equal to the gain $G(t)$ associated with price changes of the securities on all trading dates. Hence, we expect that a trading strategy H is self-financing if and only if

$$V(t) = V(0) + G(t).$$

To show the claim, we consider the portfolio value at time u right after the asset prices are revealed but adjustment in asset holdings has not been made so that

$$V(u) = h_0(u)S_0(u) + \sum_{m=1}^{M} h_m(u)S_m(u).$$

We also consider the portfolio value at time $u-1$ right after adjustment in asset holdings has been made so that

$$V(u-1) = h_0(u)S_0(u-1) + \sum_{m=1}^{M} h_m(u)S_m(u-1).$$

We then subtract the two equations to obtain

$$V(u) - V(u-1) = h_0(u)\Delta S_0(u) + \sum_{m=1}^{M} h_m(u)\Delta S_m(u).$$

Summing the above equation from $u = 1$ to $u = t$, we obtain the result. It can be shown that H is self-financing if and only if

$$V^*(t) = V^*(0) + G^*(t).$$

3.4 No arbitrage principle and martingale pricing measure

A trading strategy H represents an arbitrage opportunity if and only if the value process $V(t)$ and H satisfy the following properties:

(i) $V(0) = 0$,

(ii) $V(T) \geq 0$ and $EV(T) > 0$, and

(iii) H is self-financing.

The self-financing trading strategy H is an arbitrage opportunity if and only if (i) $G^*(T) \geq 0$ and (ii) $EG^*(T) > 0$. Here, the expectation E is taken with respect to the actual probability measure P, with $P(\omega) > 0$.

Like that in single period models, we expect that arbitrage opportunity does not exist if and only if there exists a risk neutral probability measure. In multi-period models, risk neutral probabilities are defined in terms of martingales.

Martingale measure

The measure Q is called a martingale measure (or called a risk neutral probability measure) if it has the following properties:

1. $Q(\omega) > 0$ for all $\omega \in \Omega$.

2. Every discounted price process S_m^* in the securities model is a martingale under $Q, m = 1, 2, \cdots, M$, that is,

$$E_Q[S_m^*(t+s)|\mathcal{F}_t] = S_m^*(t) \quad \text{for all } t \geq 0 \text{ and } s \geq 0.$$

Recall that the conditional expectation $E_Q[S_m^*(t+s)|\mathcal{F}_t]$ is a $\mathcal{F}_t$-measurable random variable, so does $S_m^*(t)$. We call the discounted price process $S_m^*(t)$ to be a Q-martingale.

As a numerical example, we determine the martingale measure Q associated with the earlier two-period securities model. Let $r \geq 0$ be the constant riskless interest rate over one period, and write $Q(\omega_j)$ as the martingale measure associated with the state $\omega_j, j = 1, 2, 3, 4$.

(i) $t = 0$ and $s = 1$

$$4 = \frac{3}{1+r}[Q(\omega_1) + Q(\omega_2)] + \frac{5}{1+r}[Q(\omega_3) + Q(\omega_4)]$$

(ii) $t = 0$ and $s = 2$

$$\begin{aligned} 4 \; = \; & \frac{4}{(1+r)^2}Q(\omega_1) + \frac{2}{(1+r)^2}Q(\omega_2) \\ & + \frac{4}{(1+r)^2}Q(\omega_3) + \frac{6}{(1+r)^2}Q(\omega_4) \end{aligned}$$

(iii) $t = 1$ and $s = 1$

$$3 = \frac{4}{1+r}\frac{Q(\omega_1)}{Q(\omega_1)+Q(\omega_2)} + \frac{2}{1+r}\frac{Q(\omega_2)}{Q(\omega_1)+Q(\omega_2)}$$
$$5 = \frac{4}{1+r}\frac{Q(\omega_3)}{Q(\omega_3)+Q(\omega_4)} + \frac{6}{1+r}\frac{Q(\omega_4)}{Q(\omega_3)+Q(\omega_4)}.$$

recall that $\frac{Q(\omega_1)}{Q(\omega_1)+Q(\omega_2)}$ is the risk neutral conditional probability of going upstate when the state $\{\omega_1, \omega_2\}$ is reached at $t = 1$. The calculation procedure can be simplified by observing that $Q(\omega_j)$ is given by the product of the conditional probabilities along the path from the node at $t = 0$ to the node ω_j at $t = 2$.

We start with the probability p associated with the upper branch $\{\omega_1, \omega_2\}$. The corresponding probability p is given by

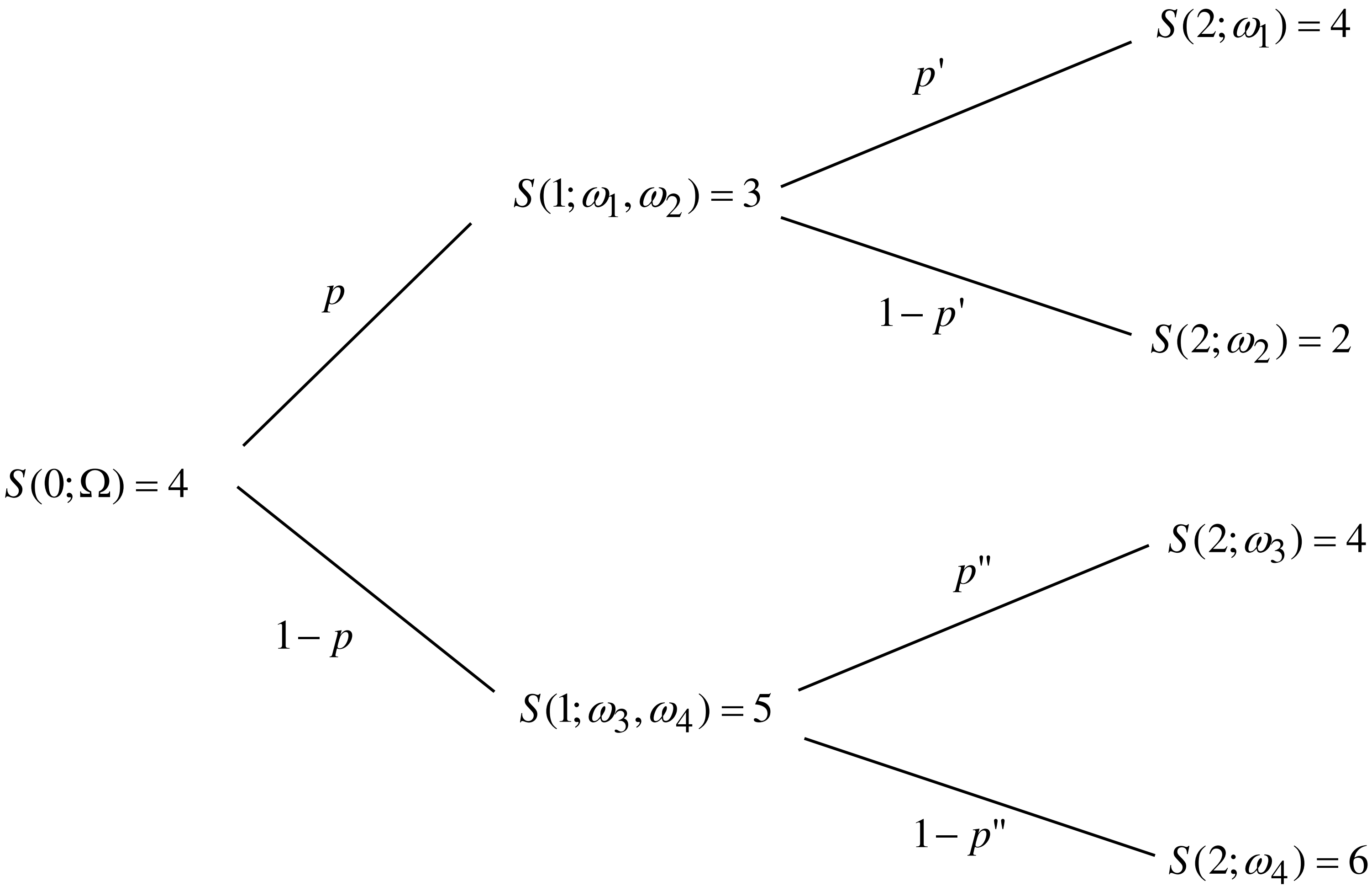
$$4 = \frac{3}{1+r}p + \frac{5}{1+r}(1-p)$$

so that $p = \dfrac{1-4r}{2}$.

Similarly, the conditional probability p' associated with the branch $\{\omega_1\}$ from the node $\{\omega_1, \omega_2\}$ is given by

$$3 = \frac{4}{1+r}p' + \frac{2}{1+r}(1-p')$$

giving $p' = \dfrac{1-3r}{2}$. In a similar manner, the conditional probability p'' associated with $\{\omega_3\}$ from $\{\omega_3, \omega_4\}$ is found to be $\dfrac{1-5r}{2}$.



Determine all the risk neutral conditional probabilities along the path from the node at $t = 0$ to the terminal node (T, ω).

The martingale probabilities are then found to be

$$
\begin{aligned}
Q(\omega_1) &= pp' = \frac{1-4r}{2}\frac{1-3r}{2}, \\
Q(\omega_2) &= p(1-p') = \frac{1-4r}{2}\frac{1+3r}{2}, \\
Q(\omega_3) &= (1-p)p'' = \frac{1+4r}{2}\frac{1-5r}{2}, \\
Q(\omega_4) &= (1-p)(1-p'') = \frac{1+4r}{2}\frac{1+5r}{2}.
\end{aligned}
$$

In order that the martingale probabilities remain positive, we have to impose the restriction: $r < 0.2$.

Martingale property of value processes

Suppose H is a self-financing trading strategy and Q is a martingale measure with respect to a filtration $\mathbb{F}$, then the value process $V(t)$ is a Q-martingale. To show the claim, we use the relation

$$V^*(t) = V^*(0) + G^*(t)$$

since H is self-financing, and deduce that

$$\begin{aligned} V^*(t+1) - V^*(t) &= G^*(t+1) - G^*(t) \\ &= [S^*(t+1) - S^*(t)]H(t+1). \end{aligned}$$

As H is a predictable process, $V^*(t)$ is the martingale transform of the Q-martingale $S^*(t)$. Hence, $V^*(t)$ itself is also a Q-martingale.

existence of $Q \Rightarrow$ non-existence of arbitrage opportunities

- Assume that Q exists. Consider a self-financing trading strategy with $V^*(T) \geq 0$ and $E[V^*(T)] > 0$. Here, E is the expectation under the actual probability measure P, with $P(\omega) > 0$. That is, $V^*(T)$ is strictly positive for some states of the world.

- As $Q(\omega) > 0$, we then have $E_Q[V^*(T)] > 0$. However, since $V^*(T)$ is a Q-martingale so that $V^*(0) = E_Q[V^*(T)]$, and by virtue of $E_Q[V^*(T)] > 0$, we always have $V^*(0) > 0$.

- It is then impossible to have $V^*(T) \geq 0$ and $E[V^*(T)] > 0$ while $V^*(0) = 0$. Hence, the self-financing strategy H cannot be an arbitrage opportunity.

non-existence of arbitrage opportunities $\Rightarrow$ *existence of* Q

- If there are no arbitrage opportunities in the multi-period model, then there will be no arbitrage opportunities in any underlying single period.

- Since each single period does not admit arbitrage opportunities, one can construct the one-period risk neutral conditional probabilities.

- The martingale probability measure $Q(\omega)$ is then obtained by multiplying all the risk neutral conditional probabilities along the path from the node at $t = 0$ to the terminal node (T, ω).

Theorem

A multi-period securities model is arbitrage free if and only if there exists a probability measure Q such that the discounted asset price processes are Q-martingales.

Additional remarks

1. The martingale measure is unique if and only if the multi-period securities model is complete. Here, completeness implies that all contingent claims ($\mathcal{F}_T$-measurable random variables) can be replicated by a self-financing trading strategy.

2. In an arbitrage free complete market, the arbitrage price of a contingent claim is then given by the discounted expected value under the martingale measure of the portfolio that replicates the claim.

Valuation of an attainable contingent claim

Let Y denote an attainable contingent claim at maturity T and $V(t)$ denote the arbitrage price of the contingent claim at time $t, t < T$. We then have

$$V(t) = \frac{S_0(t)}{S_0(T)} E_Q[Y|\mathcal{F}_t],$$

where $S_0(t)$ is the riskless asset and the ratio $S_0(t)/S_0(T)$ is the discount factor over the period from t to T.

3.5 Multi period binomial models

Let c_{uu} denote the call value at two periods beyond the current time with two consecutive upward moves of the asset price and similar notational interpretation for c_{ud} and c_{dd}. The call values c_u and c_d are related to c_{uu}, c_{ud} and c_{dd} as follows:

$$c_u = \frac{pc_{uu} + (1-p)c_{ud}}{R} \text{ and } c_d = \frac{pc_{ud} + (1-p)c_{dd}}{R}.$$

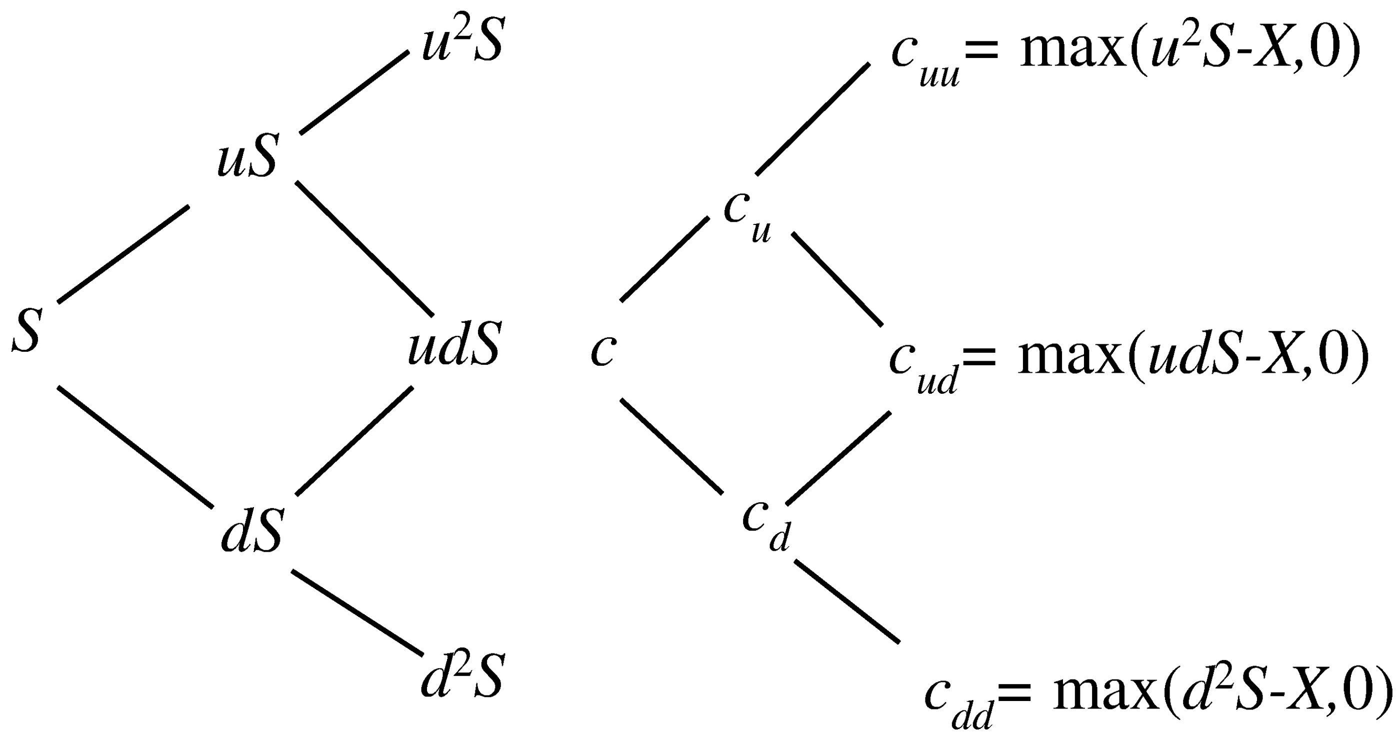
The call value at the current time which is two periods from expiry is found to be

$$c = \frac{p^2 c_{uu} + 2p(1-p)c_{ud} + (1-p)^2 c_{dd}}{R^2},$$

where the corresponding terminal payoff values are given by

$$c_{uu} = \max(u^2 S - X, 0), c_{ud} = \max(udS - X, 0), c_{dd} = \max(d^2 S - X, 0).$$

Note that the coefficients $p^2, 2p(1-p)$ and $(1-p)^2$ represent the respective risk neutral probability of having two up jumps, one up jump and one down jump, and two down jumps in two moves of the binomial process.



Dynamics of asset price and call price in a two-period binomial model.

- With n binomial steps, the risk neutral probability of having j up jumps and $n-j$ down jumps is given by $C_j^n p^j (1-p)^{n-j}$, where $C_j^n = \dfrac{n!}{j!(n-j)!}$ is the binomial coefficient.

- The corresponding terminal payoff when j up jumps and $n-j$ down jumps occur is seen to be $\max(u^j d^{n-j} S - X, 0)$.

- The call value obtained from the n-period binomial model is given by

$$c = \frac{\displaystyle\sum_{j=0}^{n} C_j^n p^j (1-p)^{n-j} \max(u^j d^{n-j} S - X, 0)}{R^n}.$$

We define k to be the smallest non-negative integer such that $u^k d^{n-k} S \geq X$, that is, $k \geq \frac{\ln \frac{X}{S d^n}}{\ln \frac{u}{d}}$. It is seen that

$$\max(u^j d^{n-j} S - X, 0) = \begin{cases} 0 & \text{when} \quad j < k \\ u^j d^{n-j} S - X & \text{when} \quad j \geq k \end{cases}.$$

The integer k gives the minimum number of upward moves required for the asset price in the multiplicative binomial process in order that the call expires in-the-money.

The call price formula is simplified as

$$c = S \sum_{j=k}^{n} C_j^n p^j (1-p)^{n-j} \frac{u^j d^{n-j}}{R^n} - X R^{-n} \sum_{j=k}^{n} C_j^n p^j (1-p)^{n-j}.$$

Interpretation of the call price formula

- The last term in above equation can be interpreted as the expectation value of the payment made by the holder at expiration discounted by the factor R^{-n}, and $\sum_{j=k}^{n} C_j^n p^j (1-p)^{n-j}$ is seen to be the probability (under the risk neutral measure) that the call will expire in-the-money.

- The above probability is related to the *complementary binomial distribution function* defined by

$$\Phi(n,k,p) = \sum_{j=k}^{n} C_j^n p^j (1-p)^{n-j}.$$

Note that $\Phi(n,k,p)$ gives the probability for at least k successes in n trials of a binomial experiment, where p is the probability of success in each trial.

Further, if we write $p' = \dfrac{up}{R}$ so that $1 - p' = \dfrac{d(1-p)}{R}$, then the call price formula for the n-period binomial model can be expressed as

$$c = S\Phi(n,k,p') - XR^{-n}\Phi(n,k,p).$$

Alternatively, from the risk neutral valuation principle, we have

$$c = \frac{1}{R^n}E_Q\left[S_T \mathbf{1}_{\{S_T > X\}}\right] - \frac{X}{R^n}E_Q\left[\mathbf{1}_{\{S_T > X\}}\right].$$

- The first term gives the discounted expectation of the asset price at expiration given that the call expires in-the-money.
- The second term gives the present value of the expected cost incurred by exercising the call.

Futures price

Unlike a call option, the holder of a futures has the obligation to buy the underlying asset at the futures price F at maturity T.

Also, the futures value at initiation is zero. Hence,

$$\text{time-}t \text{ futures value } = 0 = \frac{1}{R^n} E_Q[S_T] - \frac{F}{R^n},$$

so that

$$F = E_Q[S_T] \quad \text{(instead of } E_P[S_T]\text{)}.$$

How to compute $E_Q[S_T]$? Since S_t^* is a Q-martingale,

$$\frac{S_t}{M_t} = E_Q\left[\frac{S_T}{M_T}\right] \text{ so that } E_Q[S_T] = e^{r(T-t)} S_t.$$