MATH 571 — Mathematical Models of Financial Derivatives

Topic 4 – Black-Scholes-Merton framework and Martingale Pricing Theory

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4.1 Review of stochastic processes and Ito calculus

- A *Markovian process* is a stochastic process that, given the value of X_s, the value of $X_t, t > s$, depends only on X_s but not on the values taken by $X_u, u < s$.
- If the asset prices follow a Markovian process, then only the present asset prices are relevant for predicting their future values.
- This Markovian property of asset prices is consistent with the *weak form of market efficiency*, which assumes that the present value of an asset price already impounds all information in past prices and the particular path taken by the asset price to reach the present value is irrelevant.

Market efficiency

Fama's definition (1970)

A market in which prices always "fully reflect" available information is called "efficient".

Malkiel's definition (1992)

A capital market is said to be efficient if it fully and correctly reflects all relevant information in determining security prices. (Repeating Fama's sentence).

Formally, the market is said to be efficient with respect to some information set ... if security prices would be unaffected by revealing that information to all participants. Moreover, efficiency with respect to an information set ... implies that it is impossible to make economic profits by trading on the basis of [that information set].

Weak-form Efficiency: The information set includes only the history of prices or returns themselves.

Semistrong-Form Efficiency: The information set includes all information known to all market participants (*publicly available* information).

Strong-Form Efficiency: The information set includes all information known to any market participant (*private* information).

1. Market efficiency can be tested by revealing information to market participants and measuring the reaction of security prices. If prices do not move when information is revealed, then the market is efficient with respect to that information. Although this is clear conceptually, it is hard to carry out such a test in practice (except perhaps in a laboratory).

2. One can judge the efficiency of a market, by measuring the profits that can be made by trading on information. This idea is the foundation of almost all the empirical work on the market efficiency.

   Many researchers have tried to measure the profits earned by market professionals such as mutual fund managers. If these managers achieve superior returns (after adjustment for risk) then the market is not efficient with respect to the information possessed by the managers.

Brownian process

The *Brownian process with drift* is a stochastic process $\{X(t); t \geq 0\}$ with the following properties:

(i) Every increment $X(t+s) - X(s)$ is normally distributed with mean μt and variance $\sigma^2 t$; μ and σ are fixed parameters.

(ii) For every $t_1 < t_2 < \cdots < t_n$, the increments $X(t_2) - X(t_1), \cdots, X(t_n) - X(t_{n-1})$ are independent random variables with distributions given in (i).

(iii) $X(0) = 0$ and the sample paths of $X(t)$ are continuous.

- Note that $X(t+s) - X(s)$ is independent of the past history of the random path, that is, the knowledge of $X(\tau)$ for $\tau < s$ has no effect on the probability distribution for $X(t+s) - X(s)$. This is precisely the Markovian character of the Brownian motion.

Standard Brownian process

For the particular case $\mu = 0$ and $\sigma^2 = 1$, the Brownian motion is called the *standard Brownian motion* (or *standard Wiener process*). The probability distribution for the standard Wiener process $\{Z(t); t \geq 0\}$ is given by

$$\begin{aligned}
P[Z(t) \leq z | Z(t_0) = z_0] &= P[Z(t) - Z(t_0) \leq z - z_0] \\
&= \frac{1}{\sqrt{2\pi(t - t_0)}} \int_{-\infty}^{z - z_0} \exp\left(-\frac{x^2}{2(t - t_0)}\right) dx \\
&= N\left(\frac{z - z_0}{\sqrt{t - t_0}}\right).
\end{aligned}$$

(a) $E[Z(t)^2] = \text{var}(Z(t)) + E[Z(t)]^2 = t$.

(b) $E[Z(t)Z(s)] = \min(t, s)$.

To show the result in (b), we assume $t > s$ (without loss of generality) and consider

$$
\begin{aligned}
E[Z(t)Z(s)] &= E[\{Z(t) - Z(s)\}Z(s) + Z(s)^2] \\
&= E[\{Z(t) - Z(s)\}Z(s)] + E[Z(s)^2].
\end{aligned}
$$

Since $Z(t) - Z(s)$ and $Z(s)$ are independent and both $Z(t) - Z(s)$ and $Z(s)$ have zero mean, so

$$E[Z(t)Z(s)] = E[Z(s)^2] = s = \min(t, s).$$

Overlapping Brownian increments

When $t > s$, the correlation coefficient ρ between the two overlapping Brownian increments $Z(t)$ and $Z(s)$ is given by

$$\rho = \frac{E[Z(t)Z(s)]}{\sqrt{\text{var}(Z(t))}\sqrt{\text{var}(Z(s))}} = \frac{s}{\sqrt{st}} = \sqrt{\frac{s}{t}}.$$

Joint distribution of $Z(s)$ and $Z(t)$

Since both $Z(t)$ and $Z(s)$ are normally distributed with zero mean and variance t and s, respectively, the probability distribution of the overlapping Brownian increments is given by the bivariate normal distribution function.

If we define $X_1 = Z(t)/\sqrt{t}$ and $X_2 = Z(s)/\sqrt{s}$, then X_1 and X_2 become standard normal random variables. We then have

$$\begin{aligned} P[Z(t) \le z_t, Z(s) \le z_s] &= P[X_1 \le z_t/\sqrt{t}, X_2 \le z_s/\sqrt{s}] \\ &= N_2(z_t/\sqrt{t}, z_s/\sqrt{s}; \sqrt{s/t}) \end{aligned}$$

where the bivariate normal distribution function is given by

$$\begin{aligned} N_2(x_1, x_2; \rho) = & \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} \frac{1}{2\pi\sqrt{1-\rho^2}} \\ & \exp\left(-\frac{\xi_1^2 - 2\rho\xi_1\xi_2 + \xi_2^2}{2(1-\rho^2)}\right) d\xi_1 d\xi_2. \end{aligned}$$

Geometric Brownian process

Let $X(t)$ denote the Brownian process with drift parameter μ and variance parameter σ^2. The stochastic process defined by

$$Y(t) = e^{X(t)}, \quad t \geq 0,$$

is called the *Geometric Brownian process*. The value taken by $Y(t)$ is non-negative.

Since $X(t) = \ln Y(t)$ is a Brownian process, by properties (i) and (ii), we deduce that $\ln Y(t) - \ln Y(0)$ is normally distributed with mean μt and variance $\sigma^2 t$. For common usage, $\dfrac{Y(t)}{Y(0)}$ is said to be lognormally distributed.

The density function of $\dfrac{Y(t)}{Y(0)}$ is deduced to be

$$f_Y(y,t) = \frac{1}{y\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(\ln y - \mu t)^2}{2\sigma^2 t}\right).$$

The mean of $Y(t)$ conditional on $Y(0) = y_0$ is found to be

$$\begin{aligned}
& E[Y(t)|Y(0) = y_0] \\
= \quad & y_0 \int_0^\infty y f_Y(y,t)\ dy \\
= \quad & y_0 \int_{-\infty}^\infty \frac{e^x}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(x-\mu t)^2}{2\sigma^2 t}\right)\ dx, \qquad x = \ln y, \\
= \quad & y_0 \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{[x - (\mu t + \sigma^2 t)]^2 - 2\mu t\sigma^2 t - \sigma^4 t^2}{2\sigma^2 t}\right)\ dx \\
= \quad & y_0 \exp\left(\mu t + \frac{\sigma^2 t}{2}\right).
\end{aligned}$$

The variance of $Y(t)$ conditional on $Y(0) = y_0$ is found to be

$$
\begin{aligned}
& \mathrm{var}(Y(t)|Y(0) = y_0) \\
= \quad & y_0^2 \int_0^\infty y^2 f_Y(y,t)\ dy - \left[y_0 \exp\left(\mu t + \frac{\sigma^2 t}{2} \right) \right]^2 \\
= \quad & y_0^2 \left\{ \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{[x - (\mu t + 2\sigma^2 t)]^2 - 4\mu t \sigma^2 t - 4\sigma^4 t^2}{2\sigma^2 t} \right) dx \right. \\
& \qquad \left. - \left[\exp\left(\mu t + \frac{\sigma^2 t}{2} \right) \right]^2 \right\} \\
= \quad & y_0^2 \exp(2\mu t + \sigma^2 t)[\exp(\sigma^2 t) - 1].
\end{aligned}
$$

Brownian paths are seen to be non-differentiable. The non-differentiability property can be shown easily by proving the finiteness of the quadratic variation of a Brownian motion.

Quadratic variation of Brownian motions

Suppose we form a partition π of the time interval $[0, T]$ by the discrete points

$$0 = t_0 < t_1 < \cdots < t_n = T,$$

and let $\delta t_{max} = \max_k (t_k - t_{k-1})$. We write $\Delta t_k = t_k - t_{k-1}$, and define the corresponding quadratic variation of the standard Brownian motion $Z(t)$ by

$$Q_\pi = \sum_{k=1}^{n} [Z(t_k) - Z(t_{k-1})]^2.$$

The quadratic variation of $Z(t)$ over $[0, T]$ is given by

$$Q_{[0,T]} = \lim_{\delta t_{max} \to 0} Q_\pi = T.$$

First, we consider

$$
\begin{aligned}
& E[Q_\pi] \\
= & \sum_{k=1}^{n} E[\{Z(t_k) - Z(t_{k-1})\}^2] \\
= & \sum_{k=1}^{n} \mathrm{var}(Z(t_k) - Z(t_{k-1})) \\
& \text{since } Z(t_k) - Z(t_{k-1}) \text{ has zero mean} \\
= & \mathrm{var}(Z(t_n) - Z(t_0)) \\
& \text{since } Z(t_k) - Z(t_{k-1}), k = 1, \cdots, n \text{ are independent} \\
= & \ t_n - t_0 = T
\end{aligned}
$$

so that

$$\lim_{\delta t_{max} \to 0} E[Q_\pi] = T.$$

Consider

$$\operatorname{var}(Q_\pi - T) = E\left[\sum_{k=1}^{n}\sum_{\ell=1}^{n}\left\{[Z(t_k)-Z(t_{k-1})]^2-\Delta t_k\right\}\right.$$
$$\left.\{[Z(t_\ell)-Z(t_{\ell-1})]^2-\Delta t_\ell\}\right].$$

Since the increments $[Z(t_k)-Z(t_{k-1})], k=1,\cdots,n$ are independent, only those terms corresponding to $k=\ell$ in the above series survive, so we have

$$\begin{aligned}\operatorname{var}(Q_\pi - T) &= E\left[\sum_{k=1}^{n}\left\{[Z(t_k)-Z(t_{k-1})]^2-\Delta t_k\right\}^2\right]\\ &= \sum_{k=1}^{n} E\left[\{Z(t_k)-Z(t_{k-1})\}^4\right]\\ &\quad - 2\Delta t_k \sum_{k=1}^{n} E\left[\{Z(t_k)-Z(t_{k-1})\}^2\right] + \Delta t_k^2.\end{aligned}$$

Since $Z(t_k) - Z(t_{k-1})$ is normally distributed with zero mean and variance Δt_k, its fourth order moment is known to be

$$E[\{Z(t_k) - Z(t_{k-1})\}^4] = 3\Delta t_k^2,$$

so

$$\text{var}(Q_\pi - T) = \sum_{k=1}^{n} [3\Delta t_k^2 - 2\Delta t_k^2 + \Delta t_k^2] = 2\sum_{k=1}^{n} \Delta t_k^2.$$

In taking the limit $\delta t_{max} \to 0$, we observe that $\text{var}(Q_\pi - T) \to 0$.

By virtue of $\lim_{n\to\infty} \text{var}(Q_\pi - T) = 0$, we say that T is the *mean square limit* of Q_π.

Remarks

1. In general, the quadratic variation of the Brownian motion with variance rate σ^2 over the time interval $[t_1, t_2]$ is given by

$$Q_{[t_1,t_2]} = \sigma^2(t_2 - t_1).$$

2. If we write $dZ(t) = Z(t) - Z(t - dt)$, where $dt \to 0$, then we can deduce from the above calculations that

$$E[dZ(t)^2] = dt \quad \text{and} \quad \text{var}(dZ(t)^2) = 2\ dt^2.$$

   Since dt^2 is a higher order infinitesimally small quantity, we may claim that the random quantity $dZ(t)^2$ converges in the mean square sense to the deterministic quantity dt.

Definition of stochastic integration

Let $f(t)$ be an arbitrary function of t and $Z(t)$ be the standard Brownian motion. First, we consider the definition of the stochastic integral $\int_0^T f(t)\, dZ(t)$ as a limit of the following partial sums (defined in the usual Riemann-Stieltjes sense):

$$\int_0^T f(t)\ dZ(t) = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)[Z(t_k) - Z(t_{k-1})]$$

where the discrete points $0 < t_0 < t_1 < \cdots < t_n = T$ form a partition of the interval $[0, T]$ and ξ_k is some immediate point between t_{k-1} and t_k. The limit is taken in the mean square sense.

Unfortunately, the limit depends on how the immediate points are chosen. For example, suppose we take $f(t) = Z(t)$ and choose $\xi_k = \alpha t_k + (1-\alpha)t_{k-1}, 0 \leq \alpha \leq 1$, for all k. We consider

$$\begin{aligned}
& E\left[\sum_{k=1}^{n} Z(\xi_k)[Z(t_k) - Z(t_{k-1})\right] \\
= & \sum_{k=1}^{n} E\left[Z(\xi_k)Z(t_k) - Z(\xi_k)Z(t_{k-1})\right] \\
= & \sum_{k=1}^{n} [\min(\xi_k, t_k) - \min(\xi_k, t_{k-1})] \\
= & \sum_{k=1}^{n} (\xi_k - t_{k-1}) = \alpha \sum_{k=1}^{n} (t_k - t_{k-1}) = \alpha T,
\end{aligned}$$

so that the expected value of the stochastic integral depends on the choice of the immediate points.

- A function is said to be *non-anticipative* (非預見) with respect to the Brownian motion $Z(t)$ if the value of the function at time t is determined by the path history of $Z(t)$ up to time t.

  Examples

  1. $f_1(t) = \begin{cases} 0 & \text{if } \max\limits_{0 \le s \le t} Z(s) < 5 \\ 1 & \text{if } \max\limits_{0 \le s \le t} Z(s) \ge 5 \end{cases}$ is non-anticipative.

  2. $f_2(t) = \begin{cases} 0 & \text{if } \max\limits_{0 \le s \le 1} Z(s) < 5 \\ 1 & \text{if } \max\limits_{0 \le s \le 1} Z(s) \ge 5 \end{cases}$ is not non-anticipative.

     For $t < 1$, the value of $f_2(t)$ is determined by the realization of the path of Z over $[0, 1]$.

- In finance, the investor's action is non-anticipative in nature since he makes the investment decision before the asset prices move.

- Define the stochastic integration by taking $\xi_k = t_{k-1}$ (left-hand point in each sub-interval) so that integration is taken to be non-anticipatory. The Ito definition of stochastic integral is given by

$$\int_0^T f(t)\ dZ(t) = \lim_{n\to\infty} \sum_{k=1}^{n} f(t_{k-1})[Z(t_k) - Z(t_{k-1})],$$

  where the limit is taken in the mean square sense and $f(t)$ is non-anticipative with respect to $Z(t)$.

  A path is "sliced" into consecutive Gaussian increments, each increment is multiplied by a random variable, and these numbers are added together to give the stochastic integral.

Consider the k^{th} term: $f(t_{k-1})\Delta Z_k = f(t_{k-1})[Z(t_k) - Z(t_{k-1})]$, once the history of the path up to time t_{k-1} is revealed, the value of $f(t_{k-1})$ is known. The increment of the stochastic integral over (t_{k-1}, t_k) conditional on the path history up to t_{k-1} is Gaussian with mean zero and variance $f(t_{k-1})^2(t_k - t_{k-1})$.

Example

Consider the evaluation of the Ito stochastic integral $\int_0^T Z(t)\ dZ(t)$. A naive evaluation according to the usual integration rule gives

$$\int_0^T Z(t)\ dZ(t) = \frac{1}{2}\int_0^T \frac{d}{dt}[Z(t)]^2\ dt = \frac{Z(T)^2 - Z(0)^2}{2},$$

which gives a wrong result.

Consider

$$
\begin{aligned}
\int_0^T Z(t)\ dZ(t) &= \lim_{n\to\infty} \sum_{k=1}^{n} Z(t_{k-1})[Z(t_k) - Z(t_{k-1})] \\
&= \lim_{n\to\infty} \frac{1}{2} \sum_{k=1}^{n} (\{Z(t_{k-1}) + [Z(t_k) - Z(t_{k-1})]\}^2 \\
&\qquad - Z(t_{k-1})^2 - [Z(t_k) - Z(t_{k-1})]^2) \\
&= \frac{1}{2} \lim_{n\to\infty} [Z(t_n)^2 - Z(t_0)^2] \\
&\quad - \frac{1}{2} \lim_{n\to\infty} \sum_{k=1}^{n} [Z(t_k) - Z(t_{k-1})]^2 \\
&= \frac{Z(T)^2 - Z(0)^2}{2} - \frac{T}{2}.
\end{aligned}
$$

Rearranging the terms,

$$2\int_0^T Z(t)\ dZ(t) + \int_0^T\ dt = \int_0^T \frac{d}{dt}[Z(t)]^2\ dt,$$

or in differential form,

$$2Z(t)\ dZ(t) + dt = d[Z(t)]^2.$$

Unlike the usual differential rule, we have the extra term dt.

This comes from the finiteness of the quadratic variation of the Brownian motion, since $|Z(t_k) - Z(t_{k-1})|^2$ is of order Δt_k and $\lim_{n\to\infty} \sum_{k=1}^{n} [Z(t_k) - Z(t_{k-1})]^2$ remains finite on taking the limit.

Stochastic differentials

Let $\mathcal{F}_t$ be the natural filtration generated by the standard Brownian motion $Z(t)$ through the observation of the trajectory of $Z(t)$. Let $\mu(t)$ and $\sigma(t)$ be adapted to $\mathcal{F}_t$ with $\int_0^T |\mu(t)|\ dt < \infty$ and $\int_0^T \sigma^2(t)\ dt < \infty$ (almost surely) for all T, then the process $X(t)$ defined by

$$X(t) = X(0) + \int_0^t \mu(s)\ ds + \int_0^t \sigma(s)\ dZ(s),$$

is called an *Ito process*. The differential form of the above equation is given as

$$dX(t) = \mu(t)\ dt + \sigma(t)\ dZ(t).$$

Ito's Lemma

Suppose $f(x,t)$ is a twice continuously differentiable function and the stochastic process Y is defined by $Y = f(X,t)$. Since $dZ(t)^2$ converges in the mean square sense to dt, the second order term dX^2 also contributes to the differential dY. The Ito formula of computing the differential of the stochastic function $f(X,t)$ is given by

$$\begin{aligned} dY &= \left[\frac{\partial f}{\partial t}(X,t) + \mu(t)\frac{\partial f}{\partial x}(X,t) + \frac{\sigma^2(t)}{2}\frac{\partial^2 f}{\partial x^2}(X,t)\right] dt \\ &+ \sigma(t)\frac{\partial f}{\partial x}(X,t)\ dZ. \end{aligned}$$

Expand ΔY by the Taylor series up to the second order terms:

$$\begin{aligned}\Delta Y &= \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x}\Delta X \\ &+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}\Delta t^2 + 2\frac{\partial^2 f}{\partial x \partial t}\Delta X \Delta t + \frac{\partial^2 f}{\partial x^2}\Delta X^2\right) + O(\Delta X^3, \Delta t^3).\end{aligned}$$

In the limit $\Delta X \to 0$ and $\Delta t \to 0$, we apply the multiplication rules where $dZ^2 = dt, dZdt = 0$ and $dt^2 = 0$ so that

$$dY = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX + \frac{\sigma^2(t)}{2}\frac{\partial^2 f}{\partial x^2}dt.$$

Writing out in full in terms of dZ and dt, we obtain the Ito formula.

Example

Consider the exponential Brownian

$$S(t) = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma Z(t)}.$$

Suppose we write

$$X(t) = \left(r - \frac{\sigma^2}{2}\right) t + \sigma Z(t)$$

so that

$$dX(t) = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma\, dZ(t)$$

$$S(t) = S_0 e^{X(t)}.$$

The respective partial derivatives of S are

$$\frac{\partial S}{\partial t} = 0, \quad \frac{\partial S}{\partial X} = S \quad \text{and} \quad \frac{\partial^2 S}{\partial X^2} = S.$$

By the Ito lemma, we obtain

$$dS = \left(r - \frac{\sigma^2}{2} + \frac{\sigma^2}{2}\right) S\ dt + \sigma S\ dZ$$

or

$$\frac{dS}{S} = r\ dt + \sigma\ dZ.$$

Since $E[X(t)] = \left(r - \frac{\sigma^2}{2}\right)t$ and $\text{var}(X(t)) = \sigma^2 t$, the mean and variance of $\ln \frac{S(t)}{S_0}$ are found to be $\left(r - \frac{\sigma^2}{2}\right)t$ and $\sigma^2 t$, respectively.

Multi-dimensional version of Ito's lemma

Suppose $f(x_1, \cdots, x_n, t)$ is a multi-dimensional twice continuously differentiable function and the stochastic process Y_n is defined by

$$Y_n = f(X_1, \cdots, X_n, t),$$

where the process $X_j(t)$ follows the Ito process

$$dX_j(t) = \mu_j(t)\ dt + \sigma_j(t)\ dZ_j(t), \quad j = 1, 2, \cdots, n.$$

The Brownian motions $Z_j(t)$ and $Z_k(t)$ are assumed to be correlated with correlation coefficient ρ_{jk} so that $dZ_j\ dZ_k = \rho_{jk}\ dt$.

In a similar manner, we expand ΔY_n up to the second order term in ΔX_j:

$$\begin{aligned}\Delta Y_n &= \frac{\partial f}{\partial t}(X_1, \cdots, X_n, t)\ \Delta t + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(X_1, \cdots, X_n, t)\ \Delta X_j \\ &\quad + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k}(X_1, \cdots, X_n, t)\ \Delta X_j\ \Delta X_k \\ &\qquad + O(\Delta t \Delta X_j) + O(\Delta t^2).\end{aligned}$$

In the limits $\Delta X_j \to 0, j = 1, 2, \cdots, n$, and $\Delta t \to 0$, we neglect the higher order terms in $O(\Delta t \Delta X_j)$ and $O(\Delta t^2)$ and observe $dX_j\, dX_k = \sigma_j(t)\sigma_k(t)\rho_{jk}\ dt$. We then obtain the following multi-dimensional version of the Ito lemma:

$$
\begin{aligned}
dY_n &= \left[\frac{\partial f}{\partial t}(X_1, \cdots, X_n, t) + \sum_{j=1}^{n} \mu_j(t) \frac{\partial f}{\partial x_j}(X_1, \cdots, X_n, t) \right. \\
&\quad \left. + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_j(t)\sigma_k(t)\rho_{jk} \frac{\partial^2 f}{\partial x_j \partial x_k}(X_1, \cdots, X_n, t)\right] dt \\
&\quad + \sum_{j=1}^{n} \sigma_j(t) \frac{\partial f}{\partial x_j}(X_1, \cdots, X_n, t)\ dZ_j.
\end{aligned}
$$

4.2 Change of measure – Girsanov's Theorem

Consider an Ito process defined in integral form

$$X(t) = X(0) + \int_0^t \mu(s)\ ds + \int_0^t \sigma(s)\ dZ(s)$$

with non-zero drift term $\mu(t)$. We write $M(t) = \int_0^t \sigma(s)\ dZ(s)$.

Note that

$$M(T) = M(t) + \int_t^T \sigma(s)\ dZ(s), \qquad T > t.$$

Suppose we take the conditional expectation of $M(T)$ given the history of the Brownian path up to the time t (denoted by the operator E_t), we obtain

$$E_t[M(T)] = M(t)$$

since the stochastic integral has zero conditional expectation. Hence, $M(t)$ is a martingale. However, $X(t)$ is not a martingale if $\mu(t)$ is non-zero.

Change of measure

Under the actual probability measure P, the asset price process follows

$$\frac{dS_t}{S_t} = \rho\, dt + \sigma\, dZ_t^P$$

where Z_t^P is P-Brownian. Let $S_t^* = S_t/M_t$ be the discounted asset price process, where $M_t = e^{rt}$ [M_t is the solution to $dM_t = rM_t\, dt$, with $M(0) = 1$] and r is the riskfree interest rate.

Under a risk neutral measure Q, S_t^* is Q-martingale and its dynamics is governed by

$$\frac{dS_t^*}{S_t^*} = \sigma\, dZ_t^Q \quad \text{or} \quad \frac{dS_t}{S_t} = r\, dt + \sigma\, dZ_t^Q,$$

where Z_t^Q is Q-Brownian. How do we relate Z_t^P and Z_t^Q?

$$\boxed{\text{Answer: } dZ_t^Q = dZ_t^P + \frac{\rho - r}{\sigma}\, dt}$$

Transition density function

Let X_t be the unrestricted zero-drift Brownian process with variance rate σ^2. Write $u(x,t)$ as the density function such that X_t falls within the interval $\left(x - \frac{dx}{2}, x + \frac{dx}{2}\right)$ with probability $u(x,t)\,dx$.

Assume that $X_0 = \xi$, that is, the Brownian path starts at the position ξ at $t = 0$. The governing equation for $u(x,t)$ is given by

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, t > 0,$$

with the initial condition: $u(x,0) = \delta(x - \xi)$.

The solution to $u(x,t)$ is known to be

$$u(x,t) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{(x-\xi)^2}{2\sigma^2 t}\right).$$

This is the same as the density function of a normal random variable with mean ξ and variance $\sigma^2 t$.

For a Brownian process with variance rate σ^2 and drift rate μ, the density function is

$$u(x,t) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{(x-\mu t-\xi)^2}{2\sigma^2 t}\right)$$

so that the mean position at time t is $\xi + \mu t$. If we let $x = y + \mu t$, then y gives the spatial position when the frame of reference is moving at the rate μ. Say, a position at η in the x-frame becomes $\eta - \mu t$ in the y-frame.

In terms of y, the density function becomes

$$u(y,t) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{(y-\xi)^2}{2\sigma^2 t}\right),$$

which gives the density function of a zero-drift Brownian process with variance rate σ^2 and starting position ξ under the y-frame.

Apparently, a Brownian process with drift can be transformed into the zero-drift Brownian process by adjusting the frame of reference appropriately. We would like to find the relation between the corresponding density functions.

In subsequent discussion, we consider unit variance Brownian processes so that $\sigma^2 = 1$. Also, the starting position ξ is taken to be zero.

The density function

$$u^0(y,t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right)$$

is transformed into the density function

$$u^\mu(y,t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y+\mu t)^2}{2t}\right)$$

through multiplication by the factor $\exp\left(-\mu y - \dfrac{\mu^2 t}{2}\right)$. That is,

$$u^\mu(y,t) = u^0(y,t) \exp\left(-\mu y - \frac{\mu^2 t}{2}\right).$$

If we set $x = y + \mu t$, then

$$u^\mu(y,t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) = u^0(x,t).$$

Radon-Nikodym derivatives

Consider the standard P-Brownian process $Z_P(t)$, for some time horizon T, $Z_P(T)$ is known to have zero mean and variance T under the measure P. Adding the drift μt to $Z_P(t)$ (here μ is taken to be constant) and writing

$$Z_{\widetilde{P}}(t) = Z_P(t) + \mu t,$$

then $Z_{\widetilde{P}}(t)$ is a Brownian process with drift under the measure P.

Can we modify the probability density through multiplication of dP by a factor such that $Z_{\widetilde{P}}(t)$ becomes a Brownian process (zero drift) under the modified measure $\widetilde{P}$?

The factor is called the Radon-Nikodym derivative $\frac{d\widetilde{P}}{dP}$. This procedure is called the change of measure from the original measure P to the new measure $\widetilde{P}$.

In this case, the corresponding Radon-Nikodym derivative can be found to be

$$\frac{d\widetilde{P}}{dP} = \exp\left(-\mu Z_P(T) - \frac{\mu^2}{2}T\right).$$

To verify the claim, it suffices to show that $Z_{\widetilde{P}}(T)$ is normal with zero mean and variance T under the measure $\widetilde{P}$ by looking at the corresponding moment generating function.

Recall the following well known result in probability theory:

A random variable X is normal with mean m and variance σ^2 under a measure P if and only if

$$E_P[\exp(\alpha X)] = \exp\left(\alpha m + \frac{\alpha^2}{2}\sigma^2\right), \quad \text{for any real } \alpha.$$

Now, we consider

$$\begin{aligned}
& E_{\widetilde{P}}\left[\exp(\alpha Z_{\widetilde{P}}(T))\right] \\
= \;& E_P\left[\frac{d\widetilde{P}}{dP}\exp(\alpha Z_P(T) + \alpha\mu T)\right] \\
= \;& E_P\left[\exp\left((\alpha-\mu)Z_P(T)\right)\exp\left(\alpha\mu T - \frac{\mu^2}{2}T\right)\right] \\
= \;& \exp\left(\frac{(\alpha-\mu)^2}{2}T + \alpha\mu T - \frac{\mu^2}{2}T\right) = \exp\left(\frac{\alpha^2}{2}T\right), \quad \text{for any real } \alpha,
\end{aligned}$$

hence $Z_{\widetilde{P}}(T)$ is normal with zero mean and variance T under $\widetilde{P}$.

Is $\exp\left(-\mu Z_P(t) - \frac{\mu^2 t}{2}\right)$ a martingale under P?

For $s < t$, consider

$$
\begin{aligned}
& E_P\left[\exp\left(-\mu Z_P(t) - \frac{\mu^2 t}{2}\right)\middle|\mathcal{F}_s\right] \\
= \;& E_P\left[\exp\left(-\mu Z_P(s) - \frac{\mu^2 s}{2}\right)\exp\left(-\mu \underbrace{(Z_P(t) - Z_P(s))}_{\text{normal with variance } t-s} - \frac{\mu^2}{2}(t-s)\middle|\mathcal{F}_s\right)\right] \\
= \;& \exp\left(-\mu Z_P(s) - \frac{\mu^2 s}{2}\right)\exp\left(\frac{\mu^2}{2}(t-s)\right)\exp\left(-\frac{\mu^2}{2}(t-s)\right) \\
= \;& \exp\left(-\mu Z_P(s) - \frac{\mu^2 s}{2}\right).
\end{aligned}
$$

Remark Recall that the solution to the SDE: $\frac{dS_t^*}{S_t^*} = \sigma\, dZ_t$ is given by

$$S_t^* = S_0^* e^{\left(-\frac{\sigma^2}{2}t + \sigma Z(t)\right)}$$

so S_t^* is a martingale.

Girsanov Theorem

Consider a stochastic process $\gamma(t)$ which satisfies the Novikov condition:

$$E[e^{\int_0^t \frac{1}{2}\gamma(s)^2\ ds}] < \infty,$$

and consider the Radon-Nikodym derivative:

$$\frac{d\widetilde{P}}{dP} = \rho(t)$$

where

$$\rho(t) = \exp\left(\int_0^t -\gamma(s)\ dZ(s) - \frac{1}{2}\int_0^t \gamma(s)^2\ ds\right).$$

Here, $Z_P(t)$ is a Brownian process under the measure P (called P-Brownian process). Under the measure $\widetilde{P}$, the stochastic process

$$Z_{\widetilde{P}}(t) = Z_P(t) + \int_0^t \gamma(s)\ ds$$

is $\widetilde{P}$-Brownian.

Feynman-Kac representation formula

- Suppose the Ito process $X(t)$ is governed by the stochastic differential equation

$$dX(s) = \mu(X(s), s)\ ds + \sigma(X(s), s)\ dZ(s), \quad t \le s \le T, \qquad (A)$$

  with initial condition: $X(t) = x$.

- Consider a smooth function $F(X(t), t)$, by virtue of the Ito lemma, the differential of which is given by

$$dF = \left[\frac{\partial F}{\partial t} + \mu(X,t)\frac{\partial F}{\partial X} + \frac{\sigma^2(X,t)}{2}\frac{\partial^2 F}{\partial X^2}\right] dt + \sigma \frac{\partial F}{\partial X}\ dZ.$$

- The infinitesimal generator $\mathcal{A}$ associated with the Ito process $X(t)$ is defined by

$$\mathcal{A} = \mu(X,t)\frac{\partial}{\partial X} + \frac{\sigma^2(X,t)}{2}\frac{\partial^2}{\partial X^2}.$$

Suppose F satisfies the parabolic partial differential equation

$$\frac{\partial F}{\partial t} + \mathcal{A}F = 0 \qquad (B)$$

with terminal condition: $F(X(T), T) = h(X(T))$, then dF becomes

$$dF = \sigma \frac{\partial F}{\partial X}\ dZ.$$

Supposing that $\sigma \frac{\partial F}{\partial X}$ is non-anticipative with the Brownian process $Z(t)$, we can express the above stochastic differential form into the following integral form

$$F(X(s), s) = F(X(t), t) + \int_t^s \sigma(X(u), u) \frac{\partial F}{\partial X}(X(u), u)\ dZ(u).$$

- The stochastic integral can be viewed as a sum of inhomogeneous consecutive Gaussian increments with mean zero, hence it has zero conditional expectation.

- By taking the conditional expectation and setting $s = T$ and $F(X(T), T) = h(X(T))$, we then obtain the following Feynman-Kac representation formula

$$F(x, t) = E_{x,t}[h(X(T))], \quad t < T,$$

  where $F(x, t)$ satisfies the partial differential equation and $E_{x,t}$ refers to expectation taken conditional on $X(t) = x$.

- The process $X(t)$ is initialized at the fixed point x at time t and it follows the Ito process defined in Eq. (A).

4.3 Riskless hedging principle and dynamic replicating strategy

Riskless hedging principle

Writer of a call option – hedges his exposure by holding certain units of the underlying asset in order to create a riskless portfolio.

In an efficient market with no riskless arbitrage opportunity, a riskless portfolio must earn its rate of return equals the riskless interest rate.

Let $\Pi(t)$ be the value of a riskless hedged portfolio. By invoking no-arbitrage argument, we must have

$$d\Pi(t) = r\Pi(t)\,dt,$$

where r is the riskfree interest rate.

Dynamic replication strategy

How to replicate an option dynamically by a portfolio of the riskless asset in the form of money market account and the risky underlying asset?

The cost of constructing the replicating portfolio gives the fair price of an option.

Equality of market price of risk

Hedgeable securities should have the same market price of risk. Recall

$$\lambda_S = \frac{\rho_S - r}{\sigma_S} \quad \text{and} \quad \lambda_V = \frac{\rho_V - r}{\sigma_V}$$

and $\lambda_S = \lambda_V$ if the stock and option (*both tradeable*) are hedgeable with each other.

Black-Scholes' assumptions on the financial market

(i) Trading takes place continuously in time.

(ii) The riskless interest rate r is known and constant over time.

(iii) The asset pays no dividend.

(iv) There are no transaction costs in buying or selling the asset or the option, and no taxes.

(v) The assets are perfectly divisible.

(vi) There are no penalties to short selling and the full use of proceeds is permitted.

(vii) There are no arbitrage opportunities.

- The ability to construct a perfectly hedged portfolio relies on the assumption of continuous trading and continuous asset price process.

- The stochastic process of the asset price S_t is assumed to follow the Geometric Brownian process

$$\frac{dS_t}{S_t} = \rho\ dt + \sigma\ dZ_t.$$

- Consider a portfolio which involves short selling of one unit of a European call option and long holding of Δ_t units of the underlying asset. The portfolio value $\Pi(S_t, t)$ at time t is given by

$$\Pi = -c + \Delta_t S_t,$$

where $c = c(S_t, t)$ denotes the call price.

- Note that Δ_t changes with time t, reflecting the dynamic nature of hedging. Since both c and Π are random variables, we apply the Ito Lemma to give

$$dc = \frac{\partial c}{\partial t}\ dt + \frac{\partial c}{\partial S_t}\ dS_t + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 c}{\partial S_t^2}\ dt.$$

Black and Scholes assume that Δ_t is held fixed from t to $t+dt$, so that the differential change in the portfolio value π is given by

$$
\begin{aligned}
& -dc+\Delta_t\ dS_t \\
= & \left(-\frac{\partial c}{\partial t}-\frac{\sigma^2}{2}S_t^2\frac{\partial^2 c}{\partial S_t^2}\right)dt+\left(\Delta_t-\frac{\partial c}{\partial S_t}\right)dS_t \\
= & \left[-\frac{\partial c}{\partial t}-\frac{\sigma^2}{2}S_t^2\frac{\partial^2 c}{\partial S_t^2}+\left(\Delta_t-\frac{\partial c}{\partial S_t}\right)\rho S_t\right]dt+\left(\Delta_t-\frac{\partial c}{\partial S_t}\right)\sigma S_t\ dZ_t.
\end{aligned}
$$

By taking $\Delta_t=\frac{\partial c}{\partial S_t}$, the stochastic term associated with dZ_t vanishes. Also, the term involving ρ also vanishes. The riskless hedged portfolio should earn the riskless rate of return. We then have

$$d\Pi_t = r\Pi_t\,dt$$

so that

$$
\begin{aligned}
& -\frac{\partial c}{\partial t}-\frac{\sigma^2}{2}S_t^2\frac{\partial^2 c}{\partial S_t^2}=r\left(-c+S_t\frac{\partial c}{\partial S_t}\right) \\
\Leftrightarrow\ & \frac{\partial c}{\partial t}+rS\frac{\partial c}{\partial S}+\frac{\sigma^2}{2}S^2\frac{\partial^2 c}{\partial S^2}-rV=0, \quad \text{where } c=c(S,t).
\end{aligned}
$$

Integral formulation

The financial gain on the portfolio from zero time to time t is given by the following

$$
\begin{aligned}
G(\Pi(S_t, t)) &= \int_0^t -dc + \int_0^t \Delta_u \, dS_u \\
= & \int_0^t \left[-\frac{\partial c}{\partial u} - \frac{\sigma^2}{2} S_u^2 \frac{\partial^2 c}{\partial S_u^2} + \left(\Delta_u - \frac{\partial c}{\partial S_u} \right) \rho S_u \right] du \\
& + \int_0^t \left(\Delta_u - \frac{\partial c}{\partial S_u} \right) \sigma S_u \, dZ_u.
\end{aligned}
$$

- Recall that Δ_u is non-anticipative in the stochastic integral.

This fits well with the financial scenario where Δ_u is held fixed over $(u, u + du)$ and the differential change on the asset position is attributed to the change in the asset price dS_u.

- The stochastic component of the portfolio gain stems from the last term: $\int_0^t \left(\Delta_u - \frac{\partial c}{\partial S_u}\right)\sigma S_u \; dZ_u$. Suppose we adopt the dynamic hedging strategy by choosing $\Delta_u = \dfrac{\partial c}{\partial S_u}$, for all times $u < t$, then the financial gain becomes deterministic at all times.

- Interestingly, by setting $\Delta_u = \frac{\partial c}{\partial Su}$, both the stochastic term and drift term disappear. The dependence of gain on ρ disappears together with disappearance of randomness.

- By virtue of no arbitrage, the financial gain should be the same as the gain from investing on the riskfree asset with dynamic position whose value equals $-c + S_u \dfrac{\partial c}{\partial S_u}$.

The deterministic gain from this dynamic position of riskless asset is given by

$$M_t = \int_0^t r\left(-c + S_u\frac{\partial c}{\partial S_u}\right)\,du.$$

By equating these two deterministic gains $G(\Pi(S_t, t))$ and M_t, we have

$$\begin{aligned} & G(\Pi(S_t,t)) = M_t \\ \Leftrightarrow\quad & 0 = \int_0^t \left[\frac{\partial c}{\partial u} + \frac{\sigma^2}{2}S_u^2\frac{\partial^2 c}{\partial S_u^2} + r\left(-c + S_u\frac{\partial c}{\partial S_u}\right)\right]du, \qquad 0 < u < t, \end{aligned}$$

which is satisfied for any asset price S_u provided that $c(S,t)$ satisfies the equation

$$\frac{\partial c}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 c}{\partial S^2} + rS\frac{\partial c}{\partial S} - rc = 0.$$

- The above parabolic partial differential equation is called the *Black-Scholes equation*. Note that the parameter ρ, which is the expected rate of return of the asset, does not appear in the equation.

- The terminal payoff at time T of the European call with strike price X is translated into the following terminal condition:

$$c(S, T) = \max(S - X, 0).$$

- The option pricing model involves five parameters: S, T, X, r and σ, all except the volatility σ are directly observable parameters.

- The independence of the pricing model on ρ is related to the concept of *risk neutrality*.

Deficiencies in the model

1. Geometric Brownian process assumption of the asset price process? Actual asset price dynamics is much more complicated. Later models allow the asset price process to follow the jump-diffusion process and exhibit stochastic volatility.

2. Continuous hedging at all times

   — trading usually involves transaction costs.

3. Interest rate should be stochastic instead of deterministic.

Black and Scholes use the differential formulation of $d\Pi$ and follow the "pragmatic" approach of keeping the hedge ratio Δ_t to be instantaneously "frozen". Mathematicians may be puzzled since the simple product rule in calculus is not observed, where $d(\Delta t S_t) = \Delta t\, dS_t + S_t\, d\Delta t$.

Merton's formulation – Dynamic replication strategy

$Q_S(t) =$ number of units of asset

$Q_V(t) =$ number of units of option

$M_S(t) =$ dollar value of $Q_S(t)$ units of asset

$M_V(t) =$ dollar value of $Q_V(t)$ units of option

$M(t) =$ value of riskless asset invested in money market account

- Construction of a self-financing and dynamically hedged portfolio containing risky asset, option and riskless asset (in the form of money market account).

- Dynamic replication: Composition is allowed to change at all times in the replication process.
- The self-financing portfolio is set up with zero initial net investment cost and no additional funds added or withdrawn afterwards.

The zero net investment condition at time t is

$$\begin{aligned}\Pi(t) &= M_S(t) + M_V(t) + M(t) \\ &= Q_S(t)S + Q_V(t)V + M(t) = 0.\end{aligned}$$

Differential of option value V:

$$\begin{aligned}dV &= \frac{\partial V}{\partial t}\,dt + \frac{\partial V}{\partial S}\,dS + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}\,dt \\ &= \left(\frac{\partial V}{\partial t} + \rho S\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S\frac{\partial V}{\partial S}\,dZ.\end{aligned}$$

Formally, we write the stochastic dynamics of V as

$$\frac{dV}{V} = \rho_V\, dt + \sigma_V\, dZ$$

where

$$\rho_V = \frac{\frac{\partial V}{\partial t} + \rho S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2}}{V} \quad \text{and} \quad \sigma_V = \frac{\sigma S \frac{\partial V}{\partial S}}{V}.$$

$$\begin{aligned} d\Pi(t) &= [Q_S(t)\, dS + Q_V(t)\, dV + rM(t)\, dt] \\ &\quad + \underbrace{[S\, dQ_S(t) + V\, dQ_V(t) + dM(t)]}_{\text{zero due to self-financing trading strategy}} \end{aligned}$$

- The additional term $rM(t)\, dt$ gives the interest amount earned from the money market account over dt.

- $dM(t)$ represents the change in the money market account due to the net dollar gained/lost from the sale of the underlying asset and option in the portfolio.

The instantaneous portfolio return $d\Pi(t)$ can be expressed in terms of $M_S(t)$ and $M_V(t)$ as follows:

$$\begin{aligned}
d\Pi(t) &= Q_S(t)\,dS + Q_V(t)\,dV + rM(t)\,dt \\
&= M_S(t)\frac{dS}{S} + M_V(t)\frac{dV}{V} + rM(t)\,dt \\
&= [(\rho - r)M_S(t) + (\rho_V - r)M_V(t)]\,dt \\
&\quad + \ [\sigma M_S(t) + \sigma_V M_V(t)]\,dZ.
\end{aligned}$$

We make the self-financing portfolio to be instantaneously riskless by choosing $M_S(t)$ and $M_V(t)$ such that the stochastic term becomes zero.

From the relation:

$$\sigma M_S(t) + \sigma_V M_V(t) = \sigma S Q_S(t) + \frac{\sigma S \frac{\partial V}{\partial S}}{V} V Q_V(t) = 0,$$

we obtain the following ratio of the units of asset and derivative to be held

$$\frac{Q_S(t)}{Q_V(t)} = -\frac{\partial V}{\partial S}.$$

Taking $Q_V(t) = -1$, and knowing

$$0 = \Pi(t) = -V + \Delta S + M(t)$$

we obtain

$$V = \Delta S + M(t), \text{ where } \Delta = \frac{\partial V}{\partial S}.$$

- This corresponds to the case of shorting one unit of the option. The above equation implies that the position of one unit of option can be replicated by a self-financing trading strategy using S and $M(t)$, where $\Delta = \dfrac{\partial V}{\partial S}$.

Numerical example Suppose the call option value increases by \$0.3 when the underlying asset increases \$1 in value, then $\partial V/\partial S \approx 0.3$. To hedge the sale of one unit of the call, the hedger holds 0.3 units of the underlying asset so that

$$\$1 \times 0.3 + \$0.3 \times (-1) = 0.$$

The dynamic replicating portfolio is riskless and requires no net investment, so $d\Pi(t) = 0$.

$$0 = [(\rho - r)M_S(t) + (\rho_V - r)M_V(t)]\,dt.$$

Putting $\dfrac{Q_S(t)}{Q_V(T)} = -\dfrac{\partial V}{\partial S}$, we obtain

$$(\rho - r)S\frac{\partial V}{\partial S} = (\rho_V - r)V.$$

Substituting ρ_V by $\left[\dfrac{\partial V}{\partial t} + \rho S\dfrac{\partial V}{\partial S} + \dfrac{\sigma^2}{2}S^2\dfrac{\partial^2 V}{\partial S^2}\right]\Big/ V$, we obtain the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Alternative perspective on risk neutral valuation

From $\rho_V = \frac{\frac{\partial V}{\partial t} + \rho S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2}}{V}$, we obtain

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \rho S \frac{\partial V}{\partial S} - \rho_V V = 0.$$

We need to calibrate the parameters ρ and ρ_V, or find some other means to avoid such nuisance.

Combining $\sigma_V = \frac{\sigma S \frac{\partial V}{\partial S}}{V}$ and $(\rho - r) S \frac{\partial V}{\partial S} = (\rho_V - r) V$, we obtain

$$\underbrace{\frac{\rho_V - r}{\sigma_V}}_{\lambda_V} = \underbrace{\frac{\rho - r}{\sigma}}_{\lambda_S} \quad \Rightarrow \text{ Black-Scholes equation.}$$

λ_V and λ_S are the market price of risk of V and S, respectively. For risk aversion (risk neutral) investors, they demand positive (zero) market price of risk.

- The market price of risk is the rate of extra return above r per unit risk. Two hedgeable securities should have the same market price of risk.

- The two risky instruments, option and asset, are hedgeable. The riskless hedged portfolio should earn the riskless interest rate. Apparently, the market prices of risk of the option and asset become irrelevant. For convenience, we set $\rho = \rho_V = r$ (implying zero market price of risk). This gives the Black-Scholes equation.

- Option valuation can be performed in the risk neutral world by *artificially* taking the expected rate of returns of the asset and option to be r. Apparently, we choose a pricing measure (called risk neutral measure or martingale measure) such that the expected rate of return of any risky instrument is r or the discounted value has zero rate of return.

Arguments of risk neutrality

- We find the price of a derivative *relative* to that of the underlying asset $\Rightarrow$ mathematical relationship between the prices is invariant to the risk preference.
- Be careful that the actual rate of return of the underlying asset would affect the asset price and thus indirectly affects the *absolute* derivative price.
- We simply use the convenience of risk neutrality to arrive at the mathematical relationship.

Remark It would be mistaken to interpret risk neutrality as "independence of risk in the underlying asset" in the option pricing model. The volatility parameter σ remains in the pricing model.

“How we came up with the option formula?” — Black (1989)

- It started with tinkering (笨拙的修補) and ended with delayed recognition.
- The expected return on a warrant should depend on the risk of the warrant in the same way that a common stock’s expected return depends on its risk.
- I spent many, many days trying to find the solution to that (differential) equation. I have a PhD in applied mathematics, but had never spent much time on differential equations, so I didn’t know the standard methods used to solve problems like that. I have an A.B. in physics, but I didn’t recognize the equation as a version of the heat equation, which has well-known solutions.

Expectation representation of derivative price

Under the actual probability measure P, the governing pde is

$$\frac{\partial V}{\partial t} + \rho S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - \rho_V V = 0, \quad V(S,T) = h(S).$$

By the Feynman-Kac representation, $V(S,t)$ admits the expectation representation

$$V(S,t) = e^{-\rho_V (T-t)} E_P^t[h(S_T)],$$

when E_P^t denotes the expectation under P conditional on filtration $\mathcal{F}_t$.

Suppose the governing pde is the Black-Scholes equation, where

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} - rV = 0,$$

then the derivative price function admits the expectation representation

$$V(S,t) = e^{-r(T-t)}E_Q^t[h(S_T)].$$

Under the pricing (risk neutral) measure Q, the dynamics of S_t is governed by

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dZ_t^Q, \quad Z_t^Q \text{ is } Q\text{-Brownian.}$$

What happens when the underlying is not a tradeable security?

Suppose the derivative price $V(Q,t;T)$ is dependent on some price index Q whose dynamics is

$$dQ_t = \mu(Q_t,t)\,dt + \sigma_Q(Q_t,t)\,dZ_t.$$

Now, Q is not a traded security. We can only hedge two derivatives with respective maturity T_1 and T_2, whose values are dependent on Q.

The portfolio value Π is given by

$$\Pi = V_1(Q,t;T_1) - V_2(Q,t,T_2),$$

where

$$\frac{dV_i}{V_i} = \mu_V(Q,t;T_i)\,dt + \sigma_V(Q,t;T_i)\,dZ_t, \quad i = 1,2.$$

By Ito's lemma:

$$\begin{aligned}\mu_V(Q,t;T_i) &= \frac{1}{V_i}\left(\frac{\partial V_i}{\partial t} + \mu\frac{\partial V}{\partial Q} + \frac{\sigma_Q^2}{2}\frac{\partial^2 V}{\partial Q^2}\right)\\ \sigma_V(Q,t;T_i) &= \frac{\sigma_Q}{V_i}\frac{\partial V_i}{\partial Q}, \quad i = 1, 2.\end{aligned}$$

The change in portfolio value is

$$\begin{aligned}d\Pi &= [V_1\mu_V(Q,t;T_1) - V_2\mu_V(Q,t;T_2)]\ dt\\ &\quad + [V_1\sigma_V(Q,t;T_1) - V_2\sigma_V(Q,t;T_2)]\ dZ_t.\end{aligned}$$

Suppose V_1 and V_2 are chosen such that

$$V_1 = \frac{\sigma_V(T_2)}{\sigma_V(T_2) - \sigma_V(T_1)}\Pi \quad \text{and} \quad V_2 = \frac{\sigma_V(T_1)}{\sigma_V(T_2) - \sigma_V(T_1)}\Pi,$$

then the stochastic term vanishes and $\Pi = V_1 - V_2$ is satisfied.

Now

$$\frac{d\Pi}{\Pi} = \frac{\mu_V(T_1)\sigma_V(T_2) - \mu_V(T_2)\sigma_V(T_1)}{\sigma_V(T_2) - \sigma_V(T_1)}\, dt \equiv r\, dt.$$

Rearranging, we obtain

$$\frac{\mu_V(T_1) - r}{\sigma_V(T_1)} = \frac{\mu_V(T_2) - r}{\sigma_V(T_2)}.$$

The relation is valid for arbitrary maturity dates T_1 and T_2. Hence,

$$\frac{\mu_V(Q,t) - r}{\sigma_V(Q,t)} = \lambda(r,t) = \text{ market price of risk of } Q.$$

We obtain

$$\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial Q} + \frac{\sigma_Q^2}{2}\frac{\partial^2 V}{\partial Q^2} - rV = \lambda\sigma_Q\frac{\partial V}{\partial Q}.$$

The governing equation for derivative value becomes

$$\frac{\partial V}{\partial t} + (\mu - \lambda \sigma_Q)\frac{\partial V}{\partial Q} + \frac{\sigma_Q^2}{2}\frac{\partial^2 V}{\partial Q^2} - rV = 0,$$

where the market price of risk is involved.

When Q is a traded security, then $V = Q$ also satisfies the above equation. (注意:不許交易，何來價格) This gives

$$\mu - \lambda \sigma_Q = rQ.$$

Furthermore, we set $\sigma_Q = \sigma Q$, where σ is a constant. We recover the Black-Scholes equation

$$\frac{\partial V}{\partial t} + rQ\frac{\partial V}{\partial Q} + \frac{\sigma^2}{2}Q^2\frac{\partial^2 V}{\partial Q^2} - rV = 0.$$

4.4 Martingale pricing theory

Continuous time securities model

- Uncertainty in the financial market is modeled by the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$, where Ω is a sample space, $\mathcal{F}$ is a σ-algebra on Ω, P is a probability measure on $(\Omega, \mathcal{F}), \mathcal{F}_t$ is the filtration and $\mathcal{F}_T = \mathcal{F}$.
- There are $M + 1$ securities whose price processes are modeled by adapted stochastic processes $S_m(t), m = 0, 1, \cdots, M$.
- We define $h_m(t)$ to be the number of units of the m^{th} security held in the portfolio at time t.
- The trading strategy $H(t)$ is the vector stochastic process $(h_0(t)$ $h_1(t) \cdots h_M(t))^T$, where $H(t)$ is a $(M+1)$-dimensional predictable process since the portfolio composition is determined by the investor based on the information available before time t.

- The value process associated with a trading strategy $H(t)$ is defined by

$$V(t) = \sum_{m=0}^{M} h_m(t) S_m(t), \quad 0 \le t \le T,$$

and the gain process $G(t)$ is given by

$$G(t) = \sum_{m=0}^{M} \int_0^t h_m(u)\ dS_m(u), \quad 0 \le t \le T.$$

- Similar to that in discrete models, $H(t)$ is self-financing if and only if

$$V(t) = V(0) + G(t).$$

- We use $S_0(t)$ to denote the money market account process that grows at the riskless interest rate $r(t)$, that is,

$$dS_0(t) = r(t)S_0(t)\ dt.$$

- The discounted security price process $S_m^*(t)$ is defined as

$$S_m^*(t) = S_m(t)/S_0(t), \quad m = 1, 2, \cdots, M.$$

- The discounted value process $V^*(t)$ is defined by dividing $V(t)$ by $S_0(t)$. The discounted gain process $G^*(t)$ is defined by

$$G^*(t) = V^*(t) - V^*(0).$$

Arbitrage and equivalent martingale measure

- A self-financing trading strategy H represents an arbitrage opportunity if and only if (i) $G^*(T) \geq 0$ and (ii) $E_P G^*(T) > 0$ where P is the actual probability measure of the states of occurrence associated with the securities model.
- A probability measure Q on the space $(\Omega, \mathcal{F})$ is said to be an equivalent martingale measure if it satisfies

  (i) Q is equivalent to P, that is, both P and Q have the same null set;

  (ii) the discounted security price processes $S_m^*(t), m = 1, 2, \cdots, M$ are martingales under Q, that is,

$$E_Q[S_m^*(u)|\mathcal{F}_t] = S_m^*(t), \quad \text{for all } 0 \leq t \leq u \leq T.$$

Remark The restriction on trading strategies based on "no arbitrage" is not sufficient for the existence of an equivalent martingale measure.

existence of an equivalent martingale measure $\Rightarrow$ absence of arbitrage

- Assume that an equivalent martingale measure exists and $\boldsymbol{H}$ is a self-financing strategy under P so it is also self-financing under Q.

- The time-t discounted value $V^*(t)$ of the portfolio generated by $\boldsymbol{H}$ is a Q-martingale so that $V^*(0) = E_Q[V^*(T)]$.

- We start with $V(0) = V^*(0) = 0$, and suppose we claim that $V^*(T) \geq 0$ with strict inequality for some states of the world. Since $Q(\omega) > 0$ and $E_Q[V^*(T)] = V^*(0) = 0$ should be observed, we can only have $V^*(T) = 0$.

In conclusion, starting with $V^*(0) = 0$, it is impossible to have "$V^*(T) \geq 0$ and $V^*(T)$ is strictly positive for some states". Hence, there cannot exist any arbitrage opportunities.

A self-financing trading strategy is said to be Q-admissible if the discounted gain process $G^*(t)$ is a Q-martingale.

Contingent claims are modeled as $\mathcal{F}_T$-measurable random variables. A contingent claim is said to be attainable if there exists at least an admissible trading strategy H such that $V(T) = Y$.

Theorem

Assume that an equivalent martingale measure Q exists. Let Y be an attainable contingent claim generated by a Q-admissible self-financing trading strategy H. Then for each time $t, 0 \leq t \leq T$, the arbitrage price of Y is given by

$$V(t; H) = S_0(t) E_Q \left[\frac{Y}{S_0(T)} \middle| \mathcal{F}_t \right].$$

The validity of the Theorem is readily seen if we consider the discounted value process $V^*(t;H)$ to be a martingale under Q. This leads to

$$V(t;H) = S_0(t)V^*(t;H) = S_0(t)E_Q[V^*(T;H)|\mathcal{F}_t].$$

Furthermore, by observing that $V^*(T;H) = Y/S_0(T)$, so the risk neutral valuation formula follows.

Black-Scholes model revisited

The price processes of $S(t)$ and $M(t)$ are governed by

$$\begin{aligned}\frac{dS(t)}{S(t)} &= \rho\ dt + \sigma\ dZ \\ dM(t) &= rM(t)\ dt.\end{aligned}$$

The price process of $S^*(t) = S(t)/M(t)$ becomes

$$\frac{dS^*(t)}{S^*(t)} = (\rho - r)dt + \sigma\ dZ.$$

We would like to find the equivalent martingale measure Q such that the discounted asset price S^* is Q-martingale. By the Girsanov Theorem, suppose we choose $\gamma(t)$ in the Radon-Nikodym derivative such that

$$\gamma(t) = \frac{\rho - r}{\sigma},$$

then $\widetilde{Z}$ is a Brownian motion under the probability measure Q and

$$d\widetilde{Z} = dZ + \frac{\rho - r}{\sigma}dt.$$

Under the Q-measure, the process of $S^*(t)$ now becomes

$$\frac{dS^*(t)}{S^*(t)} = \sigma \ d\widetilde{Z},$$

hence $S^*(t)$ is Q-martingale. The asset price $S(t)$ under the Q-measure is governed by

$$\frac{dS(t)}{S(t)} = r \ dt + \sigma \ d\widetilde{Z}.$$

When the money market account is used as the numeraire, the corresponding equivalent martingale measure is called the *risk neutral measure* and the drift rate of S under the Q-measure is called the *risk neutral drift rate*.

The arbitrage price of a derivative is given by

$$V(S,t) = e^{-r(T-t)} E_Q^{t,S}[h(S_T)]$$

where $E_Q^{t,S}$ is the expectation under the risk neutral measure Q conditional on the filtration $\mathcal{F}_t$ with $S_t = S$. By the Feynman-Kac representation formula, when $V(S,t)$ satisfies the pde

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

$V(S,t)$ admits the above expectation representation.

Consider the European call option whose terminal payoff is $\max(S_T - X, 0)$. The call price $c(S,t)$ is given by

$$\begin{aligned} c(S,t) &= e^{-r(T-t)} E_Q[\max(S_T - X, 0)] \\ &= e^{-r(T-t)} \{E_Q[S_T \mathbf{1}_{\{S_T \geq X\}}] - XE_Q[\mathbf{1}_{\{S_T \geq X\}}]\}. \end{aligned}$$

Exchange rate process under domestic risk neutral measure

- Consider a foreign currency option whose payoff function depends on the exchange rate F, which is defined to be the domestic currency price of one unit of foreign currency.
- Let M_d and M_f denote the money market account process in the domestic market and foreign market, respectively. The processes of $M_d(t), M_f(t)$ and $F(t)$ are governed by

$$dM_d(t) = rM_d(t)\,dt, \quad dM_f(t) = r_f M_f(t)\,dt, \quad \frac{dF(t)}{F(t)} = \mu\,dt + \sigma\,dZ_F,$$

  where r and r_f denote the riskless domestic and foreign interest rates, respectively.

- We may treat the domestic money market account and the foreign money market account in domestic dollars (whose value is given by FM_f) as traded securities in the domestic currency world.

- With reference to the domestic equivalent martingale measure, M_d is used as the numeraire.

- By Ito's lemma, the relative price process $X(t) = F(t)M_f(t)/M_d(t)$ is governed by

$$\frac{dX(t)}{X(t)} = (r_f - r + \mu)\,dt + \sigma\,dZ_F.$$

Summary

Option pricing equation before Black-Scholes-Merton

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \rho S \frac{\partial V}{\partial S} - \rho_V V = 0$$

where

$$dS_t = \rho\, dt + \sigma\, dZ_t \quad \text{and} \quad \frac{dV_t}{V_t} = \rho_V\, dt + \sigma_V\, dZ_t.$$

By the Feynman-Kac formula, we obtain

$$V_t = e^{-\rho_V (T-t)} E_t^P [V_T], P \text{ is is the physical measure.}$$

One has to estimate ρ and ρ_V.

Application of hedging

1. The underlying is tradeable so that

$$\frac{\rho_V - r}{\sigma_V} = \frac{\rho - r}{\sigma}, \text{ same market price of risk.}$$

We obtain

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 v}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$
$$V_t = e^{-r(T-t)} E_t^Q[V_T], \text{where } Q \text{ is the martingale measure.}$$

Under Q, the dynamics of S_t is governed by

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dZ_t \quad \text{or} \quad \frac{dS_t^*}{S_t^*} = \sigma\,dZ_t.$$

Apparently, we can set $\rho = \rho_V = r$, equivalent to say the investor is risk neutral since she demands zero excess rate of return above the risk free rate on risky instruments.

2. The underlying is non-tradeable so that

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (\mu - \lambda\sigma)\frac{\partial V}{\partial S} - rV = 0,$$

where

$$\lambda = \frac{\rho_V - r}{\sigma_V}.$$

Under hedgeability of two derivatives on S, the rate of return on V can be set to be r. However, the drift rate is modified to $\mu - \lambda\sigma$. When S becomes tradeable, we have $\lambda = \frac{\rho_V - r}{\rho_V} = \frac{\rho - r}{\sigma}$ so $\rho - \lambda\sigma$ becomes r. This recovers the standard Black-Scholes-Merton equation.

- With the choice of $\gamma = (r_f - r + \mu)/\sigma$ in the Girsanov Theorem, we define

$$dZ_d = dZ_F + \gamma\, dt,$$

where Z_d is a Brownian process under Q_d.

- Under the domestic equivalent martingale measure Q_d, the process of X now becomes

$$\frac{dX(t)}{X(t)} = \sigma\, dZ_d$$

so that X is Q_d-martingale.

- The exchange rate process F under the Q_d-measure is given by

$$\frac{dF(t)}{F(t)} = (r - r_f)\, dt + \sigma\, dZ_d.$$

- The risk neutral drift rate of F under Q_d is found to be $r - r_f$.

4.5 European option pricing formulas and their greeks

Recall that the Black-Scholes equation for a European vanilla call option takes the form

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + rS\frac{\partial c}{\partial S} - rc, \qquad 0 < S < \infty,\ \tau > 0, \quad \tau = T - t.$$

Initial condition (payoff at expiry)

$$c(S, 0) = \max(S - X, 0), \quad X \text{ is the strike price.}$$

Using the transformation: $y = \ln S$ and $c(y,\tau) = e^{-r\tau} w(y,\tau)$, the Black-Scholes equation is transformed into

$$\frac{\partial w}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial y^2} + \left(r - \frac{\sigma^2}{2}\right) \frac{\partial w}{\partial y}, \quad -\infty < y < \infty,\ \tau > 0.$$

The initial condition for the model now becomes

$$w(y, 0) = \max(e^y - X, 0).$$

Green function approach

The infinite domain Green function is known to be

$$\phi(y,\tau) = \frac{1}{\sigma\,\sqrt{2\pi\tau}}\ \exp\left(-\frac{[y+(r-\frac{\sigma^2}{2})\tau]^2}{2\sigma^2\tau}\right).$$

Here, $\phi(y,\tau)$ satisfies the initial condition:

$$\lim_{\tau\to 0^+} \phi(y,\tau) = \delta(y),$$

where $\delta(y)$ is the Dirac function representing a unit impulse at the origin.

The initial condition can be expressed as

$$w(y,0) = \int_{-\infty}^{\infty} w(\xi,0)\delta(y-\xi)\,d\xi,$$

so that $w(y,0)$ can be considered as the superposition of impulses with varying magnitude $w(\xi,0)$ ranging from $\xi \to -\infty$ to $\xi \to \infty$.

- Since the Black-Scholes equation is linear, the response in position y and at time to expiry τ due to an impulse of magnitude $w(\xi, 0)$ in position ξ at $\tau = 0$ is given by $w(\xi, 0)\phi(y - \xi, \tau)$.
- From the principle of superposition for a linear differential equation, the solution is obtained by summing up the responses due to these impulses.

$$
\begin{aligned}
c(y, \tau) &= e^{-r\tau} w(y, \tau) \\
&= e^{-r\tau} \int_{-\infty}^{\infty} w(\xi, 0)\, \phi(y - \xi, \tau)\, d\xi \\
&= e^{-r\tau} \int_{\ln X}^{\infty} (e^{\xi} - X)\, \frac{1}{\sigma\sqrt{2\pi\tau}} \\
&\qquad \exp\left(-\frac{[y + (r - \frac{\sigma^2}{2})\tau - \xi]^2}{2\sigma^2\tau} \right)\, d\xi.
\end{aligned}
$$

Note that

$$\begin{aligned}
& \int_{\ln X}^{\infty} e^{\xi}\, \frac{1}{\sigma\, \sqrt{2\pi\tau}} \;\; \exp\left(-\frac{[y + (r - \frac{\sigma^2}{2})\tau - \xi]^2}{2\sigma^2\tau} \right)\, d\xi \\
= \;\; & \exp(y + r\tau) \int_{\ln X}^{\infty} \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{\left[y + \left(r + \frac{\sigma^2}{2} \right) \tau - \xi \right]^2}{2\sigma^2\tau} \right)\, d\xi \\
= \;\; & e^{r\tau} S N \left(\frac{\ln \frac{S}{X} + (r + \frac{\sigma^2}{2})\tau}{\sigma\, \sqrt{\tau}} \right), \quad y = \ln S;
\end{aligned}$$

$$\begin{aligned}
& \int_{\ln X}^{\infty} \frac{1}{\sigma\, \sqrt{2\pi\tau}} \;\; \exp\left(-\frac{[y + (r - \frac{\sigma^2}{2})\tau - \xi]^2}{2\sigma^2\tau} \right)\, d\xi \\
= \;\; & N \left(\frac{y + (r - \frac{\sigma^2}{2})\tau - \ln X}{\sigma\, \sqrt{\tau}} \right) = N \left(\frac{\ln \frac{S}{X} + (r - \frac{\sigma^2}{2})\tau}{\sigma\, \sqrt{\tau}} \right), \quad y = \ln S.
\end{aligned}$$

Hence, the price formula of the European call option is found to be

$$c(S,\tau) = SN(d_1) - Xe^{-r\tau}N(d_2),$$

where

$$d_1 = \frac{\ln \frac{S}{X} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}.$$

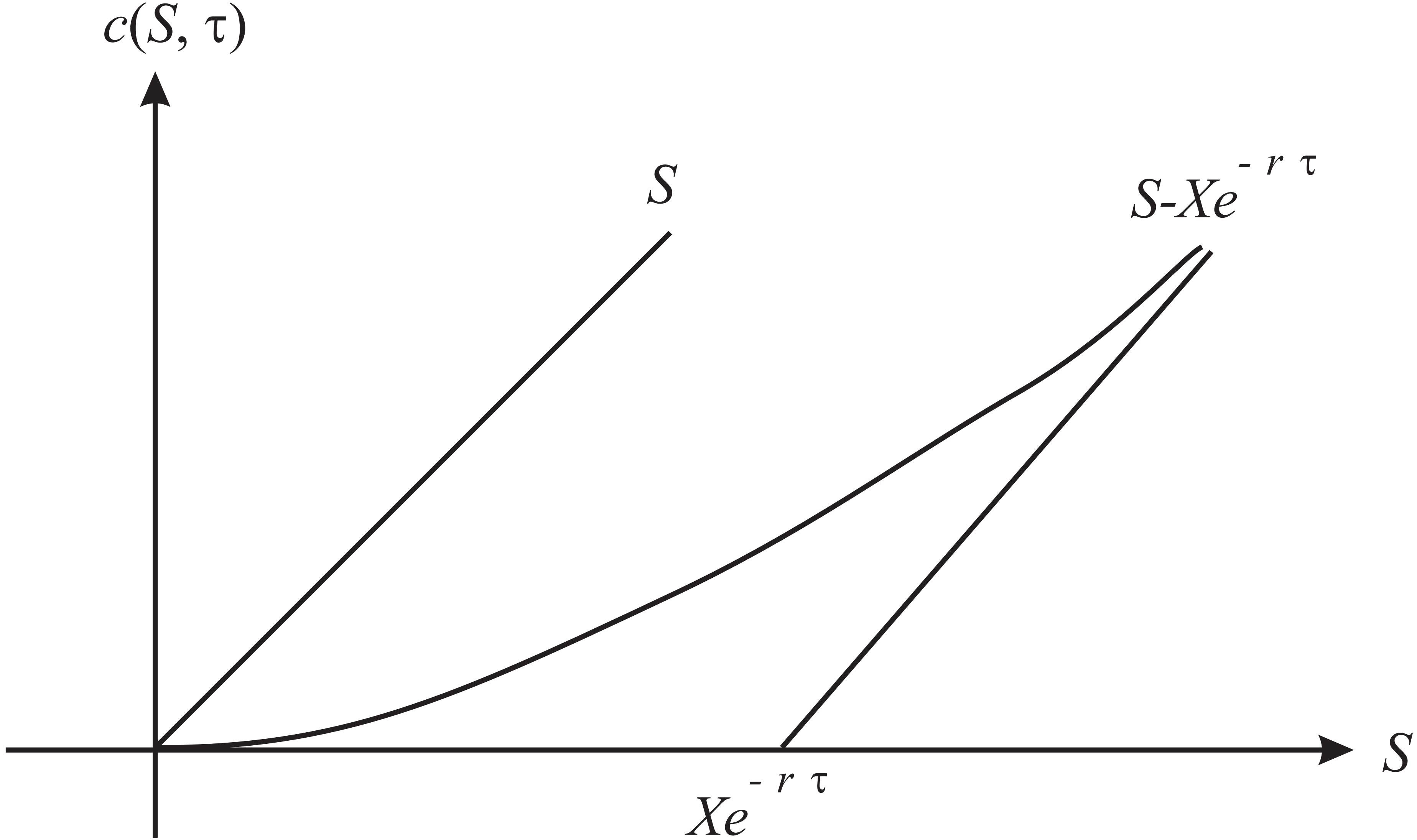
- The initial condition is seen to be satisfied by observing that the limits of d_1 and d_2 tend to 1 or 0, depending on $S > X$ or $S < X$.

- The boundary conditions are satisfied by observing

  $$\lim_{S\to\infty} N(d_1) = \lim_{S\to\infty} N(d_2) = 1$$

  and

  $$\lim_{S\to 0^+} N(d_1) = \lim_{S\to 0^+} N(d_2) = 0.$$



The call value lies within the bounds

$$\max(S - Xe^{-r\tau}, 0) \le c(S, \tau) \le S, \quad S \ge 0, \tau \ge 0.$$

$$
\begin{aligned}
c(S,\tau) &= e^{-r\tau} E_Q[(S_T - X)\mathbf{1}_{\{S_T \geq X\}}] \\
&= e^{-r\tau} \int_0^\infty \max(S_T - X, 0)\psi(S_T, T; S, t)\ dS_T.
\end{aligned}
$$

- Under the risk neutral measure Q,

$$
\ln \frac{S_T}{S} = \left(r - \frac{\sigma^2}{2}\right)\tau + \sigma \widetilde{Z}(\tau)
$$

so that $\ln \dfrac{S_T}{S}$ is normally distributed with mean $\left(r - \dfrac{\sigma^2}{2}\right)\tau$ and variance $\sigma^2\tau, \tau = T - t$.

- From the density function of a normal random variable, the transition density function is given by

$$
\psi(S_T, T; S, t) = \frac{1}{S_T \sigma \sqrt{2\pi\tau}} \exp\left(-\frac{\left[\ln \frac{S_T}{S} - \left(r - \frac{\sigma^2}{2}\right)\tau\right]^2}{2\sigma^2\tau}\right).
$$

If we compare the price formula with the expectation representation we deduce that

$$\begin{aligned} N(d_2) &= E_Q[\mathbf{1}_{\{S_T \geq X\}}] = Q[S_T \geq X] \\ SN(d_1) &= e^{-r\tau} E_Q[S_T \mathbf{1}_{\{S_T \geq X\}}]. \end{aligned}$$

- $N(d_2)$ is recognized as the probability under the risk neutral measure Q that the call expires in-the-money, so $Xe^{-r\tau}N(d_2)$ represents the present value of the risk neutral expectation of payment paid by the option holder at expiry.
- $SN(d_1)$ is the discounted risk neutral expectation of the terminal asset price conditional on the call being in-the-money at expiry.

Delta - derivative with respect to asset price

$$
\begin{aligned}
\triangle_c = \frac{\partial c}{\partial S} &= N(d_1) + S\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{d_1^2}{2}}\frac{\partial d_1}{\partial S} - Xe^{-r\tau}\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{d_2^2}{2}}\frac{\partial d_2}{\partial S} \\
&= N(d_1) + \frac{1}{\sigma\sqrt{2\pi\tau}}[e^{-\frac{d_1^2}{2}} - e^{-(r\tau+\ln\frac{S}{X})}e^{-\frac{d_2^2}{2}}] \\
&= N(d_1) > 0.
\end{aligned}
$$

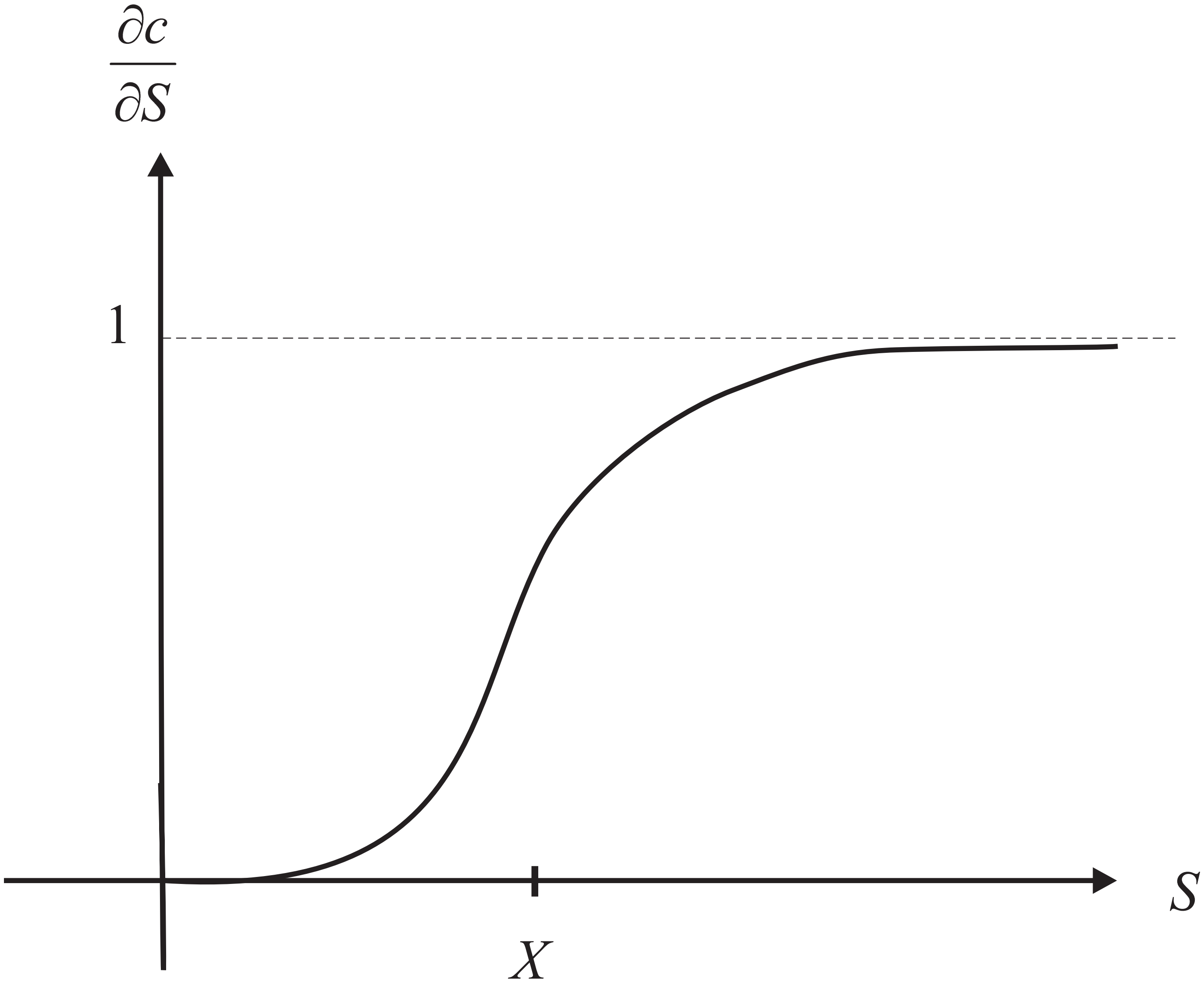
Knowing that a European call can be replicated by Δ units of asset and riskless asset in the form of money market account, the factor $N(d_1)$ in front of S in the call price formula thus gives the hedge ratio Δ.

- $\triangle_c$ is an increasing function of S since $\dfrac{\partial}{\partial S}N(d_1)$ is always positive. Also, the value of $\triangle_c$ is bounded between 0 and 1.
- The curve of $\triangle_c$ against S changes concavity at
$$S_c = X \exp\left(-\left(r + \frac{3\sigma^2}{2}\right)\tau\right)$$
so that the curve is concave upward for $0 \leq S < S_c$ and concave downward for $S_c < S < \infty$.

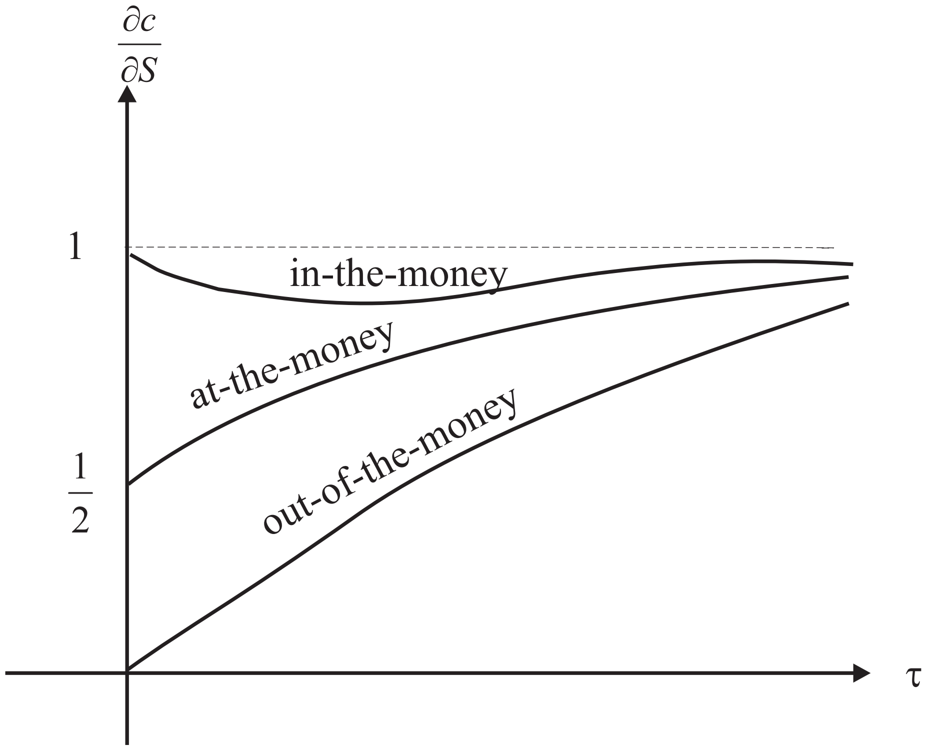
$$\lim_{\tau\to\infty} \frac{\partial c}{\partial S} = 1 \quad \text{for all values of } S,$$

while

$$\lim_{\tau\to 0^+} \frac{\partial c}{\partial S} = \begin{cases} 1 & \text{if } S > X \\ \frac{1}{2} & \text{if } S = X \\ 0 & \text{if } S < X \end{cases}.$$



Variation of the delta of the European call value with respect to the asset price S. The curve changes concavity at $S = Xe^{-\left(r+\frac{3\sigma^2}{2}\right)\tau}$.



Variation of the delta of the European call value with respect to time to expiry τ.

- The delta value always tends to one from below when the time to expiry tends to infinity.
- The delta value tends to different asymptotic limits as time comes close to expiry, depending on the moneyness of the option.

4.6 Implied volatilities and volatility smiles

- The difficulties of setting volatility value in the price formulas lie in the fact that the input value should be the forecast volatility value over the remaining life of the option rather than an estimated volatility value *(historical volatility)* from the past market data of the asset price.

- Suppose we treat the option price function $V(\sigma)$ as a function of the volatility σ and let V_{market} denote the option price observed in the market. The implied volatility σ_{imp} is defined by

$$V(\sigma_{imp}) = V_{market}.$$

- The volatility value implied by an observed market option price (*implied volatility*) indicates a consensual view about the volatility level determined by the market.

- In particular, several implied volatility values obtained simultaneously from different options with varying maturities and strike prices on the same underlying asset provide an extensive market view about the volatility at varying strikes and maturities.

- In financial markets, it becomes a common practice for traders to quote an option's market price in terms of implied volatility σ_{imp}.

- Since σ cannot be solved explicitly in terms of S, X, r, τ and option price V from the pricing formulas, the determination of the implied volatility must be accomplished by an iterative algorithm as commonly performed for the root-finding procedure for a non-linear equation.

Numerical calculations of implied volatilities

- When applied to the implied volatility calculations, the Newton-Raphson iterative scheme is given by

$$\sigma_{n+1} = \sigma_n - \frac{V(\sigma_n) - V_{market}}{V'(\sigma_n)},$$

where σ_n denotes the n^{th} iterate of σ_{imp}. Provided that the first iterate σ_1 is properly chosen, the limit of the sequence $\{\sigma_n\}$ converges to the unique solution σ_{imp}.

- The above iterative scheme may be rewritten in the following form

$$\frac{\sigma_{n+1} - \sigma_{imp}}{\sigma_n - \sigma_{imp}} = 1 - \frac{V(\sigma_n) - V(\sigma_{imp})}{\sigma_n - \sigma_{imp}} \frac{1}{V'(\sigma_n)} = 1 - \frac{V'(\sigma_n^*)}{V'(\sigma_n)}.$$

One can show that σ_n^* lies between σ_n and σ_{imp}, by virtue of the Mean Value Theorem in calculus.

- The first iterate σ_1 is chosen such that $V'(\sigma)$ is maximized by $\sigma = \sigma_1$.

- Recall that

$$V'(\sigma) = \frac{S\sqrt{\tau}\; e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} > 0 \quad \text{for all } \sigma,$$

and so

$$V''(\sigma) = \frac{S\sqrt{\tau} d_1 d_2 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}\sigma} = \frac{V'(\sigma) d_1 d_2}{\sigma}.$$

- The critical points of the function $V'(\sigma)$ are given by $d_1 = 0$ and $d_2 = 0$, which lead respectively to

$$\sigma^2 = -2\frac{\ln \frac{S}{X} + r\tau}{\tau} \quad \text{and} \quad \sigma^2 = \; 2\frac{\ln \frac{S}{X} + r\tau}{\tau}.$$

- The above two values of σ^2 both give $V'''(\sigma) < 0$. We can choose the first iterate σ_1 to be

$$\sigma_1 = \sqrt{\left|\frac{2}{\tau}\left(\ln\frac{S}{X} + r\tau\right)\right|}.$$

- With this choice of $\sigma_1, V'(\sigma)$ is maximized at $\sigma = \sigma_1$. Setting $n = 1$ and observing $V'(\sigma_1^*) < V'(\sigma_1)$ [note that $V'(\sigma)$ is maximized at $\sigma = \sigma_1$], we obtain

$$0 < \frac{\sigma_2 - \sigma_{imp}}{\sigma_1 - \sigma_{imp}} < 1.$$

In general, suppose we can establish

$$0 < \frac{\sigma_{n+1} - \sigma_{imp}}{\sigma_n - \sigma_{imp}} < 1, \qquad n \geq 1,$$

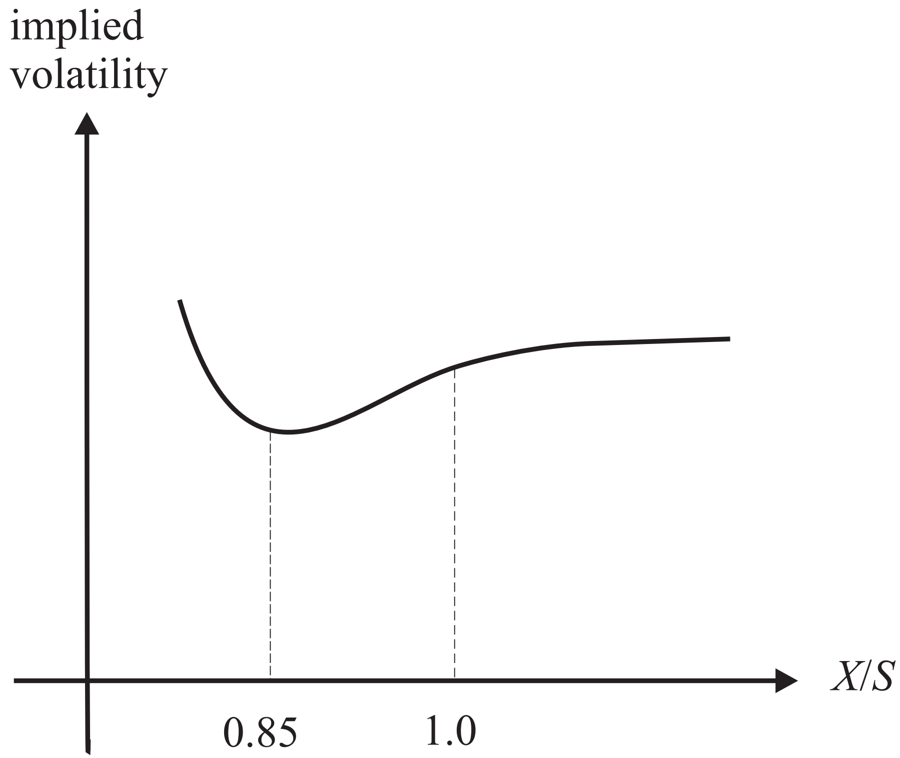
then the sequence $\{\sigma_n\}$ is monotonic and bounded, so $\{\sigma_n\}$ converges to the unique solution σ_{imp}.

Volatility smiles

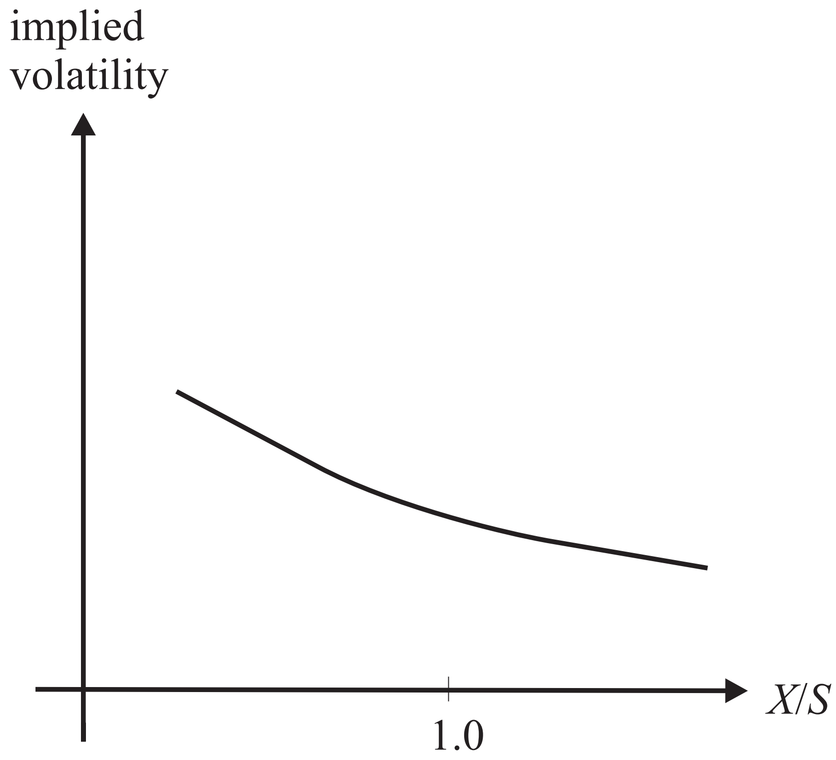
- The Black-Scholes model assumes a lognormal probability distribution of the asset price at all future times. Since volatility is the only unobservable parameter in the Black-Scholes model, the model gives the option price as a function of volatility.

- If we plot the implied volatility of the exchange-traded options, like index options, against their strike price for a fixed maturity, the curve is typically convex in shape, rather than a straight horizontal line as suggested by the simple Black-Scholes model. This phenomenon is commonly called the *volatility smile* by market practitioners.

- These smiles exhibit widely differing properties, depending on whether the market data were taken before or after the October, 1987 market crash.



A typical pattern of pre-crash smile. The implied volatility curve is convex with a dip.



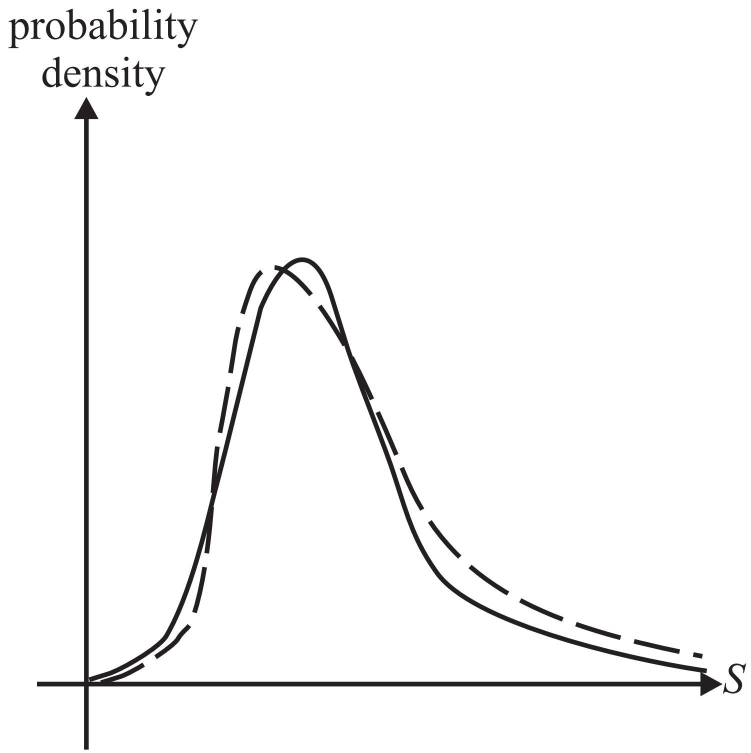
A typical pattern of post-crash smile. The implied volatility drops against X/S, indicating that out-of-the-money puts ($X/S < 1$) are traded at higher implied volatility than out-of-the-money calls ($X/S > 1$).

- The figures show the shapes of typical pre-crash smile and post-crash smile of the exchange-traded European index options. The implied volatility values are obtained by averaging options of different maturities.

- In real market situation, it is a common occurrence that when the asset price is high, volatility tends to decrease, making it less probable for a higher asset price to be realized.

- When the asset price is low, volatility tends to increase, that is, it is more probable that the asset price plummets further down.

- Suppose we plot the true probability distribution of the asset price and compare with the lognormal distribution, one observes that the left-hand tail of the true distribution is thicker than that of the lognormal one, while the reverse situation occurs at the right-hand tail.



Comparison of the true probability density of asset price (solid curve) implied from market data and the theoretical lognormal distribution (dotted curve). The true probability density is thicker at the left tail and thinner at the right tail.

- As reflected from the implied probabilities calculated from the market data of option prices, this market behavior of higher probability of large decline in stock index is better known to market practitioners after the October, 1987 market crash.

- The market price of the out-of-the-money calls (puts) became cheaper (more expensive) than the Black-Scholes price after the 1987 crash because of the thickening (thinning) of the left- (right-) hand tail of the true probability distribution.

- In common market situation, the out-of-the-money stock index puts are traded at higher implied volatilities than the out-of-the-money stock index calls.