MATH 571 —– Mathematical Models of Financial Derivatives

Topic 5 – Extended option models

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5.1 Continuous dividend yield models

Let q denote the constant continuous dividend yield, that is, the holder receives dividend of amount equal to $qS\ dt$ within the interval dt. The asset price dynamics is assumed to follow the Geometric Brownian Motion

$$\frac{dS}{S} = \rho\, dt + \sigma\ dZ.$$

We form a riskless hedging portfolio by short selling one unit of the European call and long holding $\triangle$ units of the underlying asset. The differential of the portfolio value Π is given by

$$\begin{aligned} d\Pi &= -dc + \triangle\ dS + q\triangle S\ dt \\ &= \left(-\frac{\partial c}{\partial t} - \frac{\sigma^2}{2}S^2\frac{\partial^2 c}{\partial S^2} + q\triangle S\right) dt + \left(\triangle - \frac{\partial c}{\partial S}\right) dS. \end{aligned}$$

The last term $q\triangle S\ dt$ is the wealth added to the portfolio due to the dividend payment received. By choosing $\triangle = \frac{\partial c}{\partial S}$, we obtain a riskless hedge for the portfolio. The hedged portfolio should earn the riskless interest rate.

We then have

$$d\Pi = \left(-\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + qS\frac{\partial c}{\partial S} \right) dt = r\left(-c + S\frac{\partial c}{\partial S} \right) dt,$$

which leads to

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + (r-q)S\frac{\partial c}{\partial S} - rc, \quad \tau = T - t, \quad 0 < S < \infty, \quad \tau > 0.$$

Martingale pricing approach

Suppose all the dividend yields received are used to purchase additional units of asset, then the wealth process of holding one unit of asset initially is given by

$$\widehat{S}_t = e^{qt} S_t,$$

where e^{qt} represents the growth factor in the number of units. The wealth process $\widehat{S}_t$ follows

$$\frac{d\widehat{S}_t}{\widehat{S}_t} = (\rho + q)\, dt + \sigma\, dZ.$$

We would like to find the equivalent risk neutral measure Q under which the discounted wealth process $\widehat{S}_t^*$ is Q-martingale. We choose $\gamma(t)$ in the Radon-Nikodym derivative to be

$$\gamma(t) = \frac{\rho + q - r}{\sigma}.$$

Now $\widetilde{Z}$ is a Brownian process under Q and

$$d\widetilde{Z} = dZ + \frac{\rho + q - r}{\sigma}\,dt.$$

Also, $\widehat{S}_t^*$ becomes Q-martingale since

$$\frac{d\widehat{S}_t^*}{\widehat{S}_t^*} = \sigma\,d\widetilde{Z}.$$

The asset price S_t under the equivalent risk neutral measure Q becomes

$$\frac{dS_t}{S_t} = (r-q)\,dt + \sigma\,d\widetilde{Z}.$$

Hence, the risk neutral drift rate of S_t is $r - q$.

Analogy with foreign currency options

The continuous yield model is also applicable to *options on foreign currencies* where the continuous dividend yield can be considered as the yield due to the interest earned by the foreign currency at the foreign interest rate r_f.

Call and put price formulas

The price of a European call option on a continuous dividend paying asset can be obtained by changing S to $Se^{-q\tau}$ in the price formula.

This rule of transformation is justified since the drift rate of the dividend yield paying asset under the risk neutral measure is $r - q$. Now, the European call price formula with continuous dividend yield q is found to be

$$c = Se^{-q\tau}N(\widehat{d_1}) - Xe^{-r\tau}N(\widehat{d_2}),$$

where

$$\widehat{d_1} = \frac{\ln \frac{S}{X} + (r - q + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \quad \widehat{d_2} = \widehat{d_1} - \sigma\sqrt{\tau}.$$

Similarly, the European put formula with continuous dividend yield q can be deduced from the Black-Scholes put price formula to be

$$p = Xe^{-r\tau}N(-\widehat{d_2}) - Se^{-q\tau}N(-\widehat{d_1}).$$

The new put and call prices satisfy the *put-call parity relation*

$$p = c - Se^{-q\tau} + Xe^{-r\tau}.$$

Furthermore, the following *put-call symmetry relation* can also be deduced from the above call and put price formulas

$$c(S, \tau; X, r, q) = p(X, \tau; S, q, r),$$

- The put price formula can be obtained from the corresponding call price formula by interchanging S with X and r with q in the formula. Recall that a call option entitles its holder the right to exchange the riskless asset for the risky asset, and vice versa for a put option. The dividend yield earned from the risky asset is q while that from the riskless asset is r.
- If we interchange the roles of the riskless asset and risky asset in a call option, the call becomes a put option, thus giving the justification for the put-call symmetry relation.

5.2 Time dependent parameters

Suppose the model parameters become deterministic functions of time, the Black-Scholes equation has to be modified as follows

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2(\tau)}{2} S^2 \frac{\partial^2 V}{\partial S^2} + [r(\tau) - q(\tau)]\, S\, \frac{\partial V}{\partial S} - r(\tau)V, \quad 0 < S < \infty, \quad \tau > 0,$$

where V is the price of the derivative security.

When we apply the following transformations: $y = \ln S$ and $w = e^{\int_0^\tau r(u)\, du} V$, then

$$\frac{\partial w}{\partial \tau} = \frac{\sigma^2(\tau)}{2} \frac{\partial^2 w}{\partial y^2} + \left[r(\tau) - q(\tau) - \frac{\sigma^2(\tau)}{2} \right] \frac{\partial w}{\partial y}.$$

Consider the following form of the fundamental solution

$$f(y, \tau) = \frac{1}{\sqrt{2\pi s(\tau)}} \exp\left(-\frac{[y + e(\tau)]^2}{2s(\tau)} \right),$$

it can be shown that $f(y,\tau)$ satisfies the parabolic equation

$$\frac{\partial f}{\partial \tau} = \frac{1}{2} s'(\tau) \frac{\partial^2 f}{\partial y^2} + e'(\tau) \frac{\partial f}{\partial y}.$$

Suppose we let

$$\begin{aligned} s(\tau) &= \int_0^\tau \sigma^2(u)\, du \\ e(\tau) &= \int_0^\tau [r(u) - q(u)]\, du - \frac{s(\tau)}{2}, \end{aligned}$$

one can deduce that the fundamental solution is given by

$$\phi(y,\tau) = \frac{1}{\sqrt{2\pi \int_0^\tau \sigma^2(u)\, du}} \;\; \exp\left(-\frac{\{y + \int_0^\tau [r(u) - q(u) - \frac{\sigma^2(u)}{2}]\, du\}^2}{2 \int_0^\tau \sigma^2(u)\, du} \right).$$

Given the initial condition $w(y,0)$, the solution can be expressed as

$$w(y,\tau) = \int_{-\infty}^{\infty} w(\xi,0)\, \phi(y-\xi,\tau)\, d\xi.$$

Note that the time dependency of the coefficients $r(\tau), q(\tau)$ and $\sigma^2(\tau)$ will not affect the spatial integration with respect to ξ. We make the following substitutions in the option price formulas

$$
\begin{aligned}
&r \text{ is replaced by } \frac{1}{\tau}\int_0^\tau r(u)\,du \\
&q \text{ is replaced by } \frac{1}{\tau}\int_0^\tau q(u)\,du \\
&\sigma^2 \text{ is replaced by } \frac{1}{\tau}\int_0^\tau \sigma^2(u)\,du.
\end{aligned}
$$

For example, the European call price formula is modified as follows:

$$
c = Se^{-\int_0^\tau q(u)\,du}\, N(\widetilde{d_1}) - Xe^{-\int_0^\tau r(u)\,du} N(\widetilde{d_2})
$$

where

$$
\widetilde{d_1} = \frac{\ln \frac{S}{X} + \int_0^\tau [r(u) - q(u) + \frac{\sigma^2(u)}{2}]\,du}{\sqrt{\int_0^\tau \sigma^2(u)\,du}}, \quad \widetilde{d_2} = \widetilde{d_1} - \sqrt{\int_0^\tau \sigma^2(u)\,du}.
$$

Time dependent volatility

The Black-Scholes formulas remain valid for time dependent volatility except that $\sqrt{\frac{1}{T-t}\int_t^T \sigma(\tau)^2\,d\tau}$ is used to replace σ.

How to obtain $\sigma(t)$ given the implied volatility measured at time t^* of a European option expiring at time t? Now

$$\sigma_{imp}(t^*,t) = \sqrt{\frac{1}{t-t^*}\int_{t^*}^t \sigma(u)^2\,du}$$

so that

$$\int_{t^*}^t \sigma(u)^2\,du = \sigma_{imp}^2(t^*,t)(t-t^*).$$

Differentiate with respect to t, we obtain

$$\sigma(t) = \sqrt{\sigma_{imp}(t^*,t)^2 + 2(t-t^*)\sigma_{imp}(t^*,t)\frac{\partial \sigma_{imp}(t^*,t)}{\partial t}}.$$

Practically, we do not have a continuous differentiable implied volatility function $\sigma_{imp}(t^*, t)$, but rather implied volatilities are available at discrete instants t_i. Suppose we assume $\sigma(t)$ to be piecewise constant over (t_{i-1}, t_i), then

$$\begin{aligned}
&\int_{t^*}^{t_i} \sigma^2(\tau)\, d\tau - \int_{t^*}^{t_{i-1}} \sigma^2(\tau)\, d\tau \\
= \; & (t_i - t^*)\sigma_{imp}^2(t^*, t_i) - (t_{i-1} - t^*)\sigma_{imp}^2(t^*, t_{i-1}) \\
= \; & \int_{t_{i-1}}^{t_i} \sigma^2(\tau)\, d\tau = \sigma^2(t)(t_i - t_{i-1}), \quad t_{i-1} < t < t_i,
\end{aligned}$$

$$\sigma(t) = \sqrt{\frac{(t_i - t^*)\sigma_{imp}^2(t^*, t_i) - (t_{i-1} - t^*)\sigma_{imp}^2(t^*, t_{i-1})}{t_i - t_{i-1}}}, \quad t_{i-1} < t < t_i.$$

5.3 Exchange options

- An exchange option is an option which gives the holder the right but not the obligation to exchange one risky asset for another.
- Let X_t and Y_t be the price processes of the two assets.
- The terminal payoff of a European exchange option at maturity T of exchanging Y_T for X_T is given by $\max(X_T - Y_T, 0)$.

Under the risk neutral measure Q, let X_t and Y_t be governed by

$$\frac{dX_t}{X_t} = r\,dt + \sigma_X\,dZ_t^X \quad \text{and} \quad \frac{dY_t}{Y_t} = r\,dt + \sigma_Y dZ_t^Y,$$

where r is the constant riskless interest rate, σ_X and σ_Y are the constant volatility of X_t and Y_t, respectively. Also, $dZ_t^X\,dZ_t^Y = \rho\,dt$, where ρ is correlation coefficient.

Suppose X_t is used as the numeraire, we define the equivalent probability measure Q_X on $\mathcal{F}_T$ by

$$\frac{dQ_X}{dQ} = e^{-rT}\frac{X_T}{X_0}.$$

The price function $V(X, Y, \tau)$ of the exchange option conditional on $X_t = X$ and $Y_t = Y$ is given by

$$\begin{aligned} V(X, Y, \tau) &= e^{-r\tau} E_Q \left[\max(X_T - Y_T, 0) | \mathcal{F}_t\right] \\ &= e^{-r\tau} E_Q \left[X_T \left(1 - \frac{Y_T}{X_T} \right) \mathbf{1}_{\{Y_T/X_T < 1\}} \middle| \mathcal{F}_t \right], \quad \tau = T - t. \end{aligned}$$

Writing $W_t = Y_t/X_t$ and taking X_t as the numeraire, we obtain

$$V(X, Y, \tau) = X E_{Q_X} \left[(1 - W_T) \mathbf{1}_{\{W_T < 1\}} \middle| \mathcal{F}_T \right].$$

The above expectation representation resembles that of a put option on the underlying asset $W_t = Y_t/X_t$ and with unit strike. Note that

$$\frac{dW_t}{W_t} = \frac{d(Y_t/X_t)}{Y_t/X_t} = (-\rho\sigma_X\sigma_Y + \sigma_X^2)\,dt + \sigma_Y\,dZ_t^Y - \sigma_X\,dZ_t^X.$$

By the Girsanov Theorem, $\widetilde{Z}_t^X$ and $\widetilde{Z}_t^Y$ as defined by

$$d\widetilde{Z}_t^X = dZ_t^X - \sigma_X\,dt \quad \text{and} \quad d\widetilde{Z}_t^Y = dZ_t^Y - \rho\sigma_X\,dt$$

are Brownian process under Q_X. Combining the above relations, we obtain

$$\frac{dW_t}{W_t} = \sigma_Y\,d\widetilde{Z}_t^Y - \sigma_X\,d\widetilde{Z}_t^X.$$

Under Q_X, W_t is seen to be a Geometric Brownian motion with zero drift rate and whose volatility $\sigma_{Y/X}$ is given by

$$\sigma_{Y/X}^2 = \sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2.$$

Using the put price formula, the price of the exchange option is then given by

$$V(X, Y, \tau) = XN(d_X) - YN(d_Y),$$

where

$$d_X = \frac{\ln \frac{X}{Y} + \frac{\sigma_{Y/X}^2 \tau}{2}}{\sigma_{Y/X}\sqrt{\tau}}, \quad d_Y = d_X - \sigma_{Y/X}\sqrt{\tau}, \quad \tau = T - t.$$

5.4 Quanto option – equity options with exchange rate risk exposure

- A quanto option is an option on a foreign currency donominated asset but the payoff is in domestic currency.
- The holder of a quanto option is exposed to both exchange rate risk and equity risk.

Some examples of quanto call options are listed below:

1. Foreign equity call struck in foreign currency

$$c_1(S_T, F_T, T) = F_T \max(S_T - X_f, 0).$$

   Here, F_T is the terminal exchange rate, S_T is the terminal price of the underlying foreign currency denominated asset and X_f is the strike price in foreign currency.

2. Foreign equity call struck in domestic currency

$$c_2(S_T, T) = \max(F_T S_T - X_d, 0)$$

Here, X_d is the strike price in domestic currency.

3. Fixed exchange rate foreign equity call

$$c_3(S_T, T) = F_0 \max(S_T - X_f, 0)$$

Here, F_0 is some predetermined fixed exchange rate.

4. Equity-linked foreign exchange call

$$c_4(S_T, T) = S_T \max(F_T - X_F, 0).$$

Here, X_F is the strike price on the exchange rate. The holder plans to purchase the foreign asset any way but wishes to place a floor value X_F on the exchange rate.

Quanto prewashing techniques

- Let S_t and F_t denote the price process of the foreign asset and the exchange rate, respectively.
- Define $S_t^* = F_t S_t$, which is the foreign asset price in domestic currency.
- Let r_d and r_f denote the constant domestic and foreign interest rate, respectively, and let q denote the dividend yield of the foreign asset.
- We assume that both S_t and F_t are Geometric Brownian processes.

- Under the domestic risk neutral measure Q_d, the drift rate of S^* and F are

$$\delta^d_{S^*} = r_d - q \quad \text{and} \quad \delta^d_F = r_d - r_f.$$

- The reciprocal of F can be considered as the foreign currency price of one unit of domestic currency.

- The drift rate of S and $1/F$ under the foreign risk neutral measure Q_f are given by

$$\delta^f_S = r_f - q \quad \text{and} \quad \delta^f_{1/F} = r_f - r_d,$$

respectively. Note that the dividend yield is the same for the foreign asset in the two-currency world. Why?

- "Quanto prewashing" means finding δ^d_S, that is, the drift rate of the price of the foreign currency denominated asset S under the domestic risk neutral measure Q_d.

Let the dynamics of S_t and F_t under Q_d be governed by

$$\begin{aligned} \frac{dS_t}{S_t} &= \delta_S^d \, dt + \sigma_S \, dZ_S^d \\ \frac{dF_t}{F_t} &= \delta_F^d \, dt + \sigma_F \, dZ_F^d, \end{aligned}$$

where $dZ_S^d \, dZ_F^d = \rho \, dt$, σ_S and σ_F are the volatility of S_t and F_t, respectively. Since $S_t^* = F_t S_t$, we then have

$$\delta_{S^*}^d = \delta_{FS}^d = \delta_F^d + \delta_S^d + \rho \sigma_F \sigma_S.$$

We then obtain

$$\delta_S^d = \delta_{S^*}^d - \delta_F^d - \rho \sigma_F \sigma_S = r_f - q - \rho \sigma_F \sigma_S.$$

Comparing with $\delta_S^f = r_f - q$, we need to add the quanto prewashing term $-\rho \sigma_F \sigma_S$ when we move from valuation in Q_f to Q_d.

Siegel's paradox – $\delta^d_{1/F} = r_f - r_d + \sigma_F^2$

Given that the price dynamics of F_t under Q_d is

$$\frac{dF_t}{F_t} = (r_d - r_f)\,dt + \sigma_F\,dZ_d,$$

then the process for $1/F_t$ is

$$\frac{d(1/F_t)}{1/F_t} = (r_f - r_d + \sigma_F^2)\,dt - \sigma_F\,dZ_d.$$

This is seen as a puzzle to many people since the risk neutral drift rate for $1/F$ is expected to be $r_f - r_d$ instead of $r_f - r_d + \sigma_F^2$.

We observe directly from the above SDE's that

$$\sigma_F = \sigma_{1/F} \quad \text{and} \quad \rho_{F,1/F} = -1.$$

This is also consistent with the quanto prewashing technique when it is applied to $1/F$, where the additional term $-\rho\sigma_F\sigma_{1/F}$ becomes $-(-1)\sigma_F^2 = \sigma_F^2$.

An interesting application of Siegel's paradox

Suppose the terminal payoff of an exchange rate option is $F_T \mathbf{1}_{\{F_T > K\}}$. Let $V^d(F,t)$ denote the value of the option in the domestic currency world. Define

$$V^f(F_t,t) = V^d(F_t,t)/F_t,$$

so that the terminal payoff of the exchange rate option in foreign currency world is $\mathbf{1}_{\{F_T > K\}}$. Now

$$V^f(F,t) = e^{-r_f(T-t)} E_t^{Q_f}[\mathbf{1}_{\{F_T > K\}} | F_t = F].$$

From $\delta^d_{1/F} = \delta^f_{1/F} + \sigma^2_F$ and observing $\sigma_F = \sigma_{1/F}$, we deduce that

$$\delta^f_F = \delta^d_F + \sigma^2_F.$$

This is easily seen if we interchange the foreign and domestic currency worlds. We obtain

$$V^d(F,t) = FV^f(F,t) = e^{-r_f(T-t)}FN(d) = e^{-r_d\tau}e^{\delta^d_F\tau}FN(d)$$

where

$$\begin{aligned} d &= \frac{\ln\frac{F}{K} + \left(\delta^f_F - \frac{\sigma^2_F}{2}\right)\tau}{\sigma\sqrt{\tau}} \\ &= \frac{\ln\frac{F}{K} + \left(r_d - r_f + \frac{\sigma^2_F}{2}\right)\tau}{\sigma\sqrt{\tau}}. \end{aligned}$$

Price formulas of various quanto options

1. Foreign equity call struck in foreign currency

   Let $c_1^f(S, \tau)$ denote the usual vanilla call option on the foreign currency asset in the foreign currency world. The terminal payoff is

$$c_1^f(S, 0) = \max(S - X_f, 0).$$

We treat this call as if it is structured in the foreign currency world. Its value can always be converted into domestic currency using the prevailing exchange rate.

$$c_1(S,F,\tau) = Fc_1^f(S,\tau) = F\left[Se^{-q\tau}N(d_1^{(1)}) - X_f e^{-r_f\tau}N(d_2^{(1)})\right],$$

where

$$d_1^{(1)} = \frac{\ln\frac{S}{X_f} + \left(\delta_S^f + \frac{\sigma_S^2}{2}\right)\tau}{\sigma_S\sqrt{\tau}}, \quad d_2^{(1)} = d_1^{(1)} - \sigma_S\sqrt{\tau}.$$

Correlation risk ρ and exchange rate risk σ_F do not appear in the price formula!

2. Foreign equity call struck in domestic currency

The terminal payoff at $\tau = 0$ in domestic currency is

$$c_2(S, F, 0) = \max(S^* - X_d, 0),$$

where $S^* = FS$ is a domestic currency denominated asset. Note that

$$\delta^d_{S^*} = r_d - q \quad \text{and} \quad \sigma^2_{S^*} = \sigma^2_S + 2\rho\sigma_S\sigma_F + \sigma^2_F.$$

The price formula of the foreign equity call is then given by

$$c_2(S, F, \tau) = S^* e^{-q\tau} N(d_1^{(2)}) - X_d e^{-r_d\tau} N(d_2^{(2)}),$$

where

$$d_1^{(2)} = \frac{\ln \frac{S^*}{X_d} + \left(\delta^d_{S^*} + \frac{\sigma^2_{S^*}}{2}\right)\tau}{\sigma_{S^*}\sqrt{\tau}}, \quad d_2^{(2)} = d_1^{(2)} - \sigma_{S^*}\sqrt{\tau}.$$

3. Fixed exchange rate foreign equity call

The terminal payoff is denominated in the domestic currency world, so the drift rate δ_S^d of the foreign asset in Q_d should be used. The price function of the fixed exchange rate foreign equity call is given by

$$c_3(S,\tau) = F_0 e^{-r_d\tau}\left[S e^{\delta_S^d \tau} N(d_1^{(3)}) - X_f N(d_2^{(3)})\right],$$

where

$$d_1^{(3)} = \frac{\ln \frac{S}{X_f} + \left(\delta_S^d + \frac{\sigma_S^2}{2}\right)\tau}{\sigma_S\sqrt{\tau}}, \quad d_2^{(3)} = d_1^{(3)} - \sigma_S\sqrt{\tau}.$$

- The price formula does not depend on the exchange rate F since the exchange rate has been chosen to be the fixed value F_0.
- The currency exposure of the call is embedded in the quanto-prewashing term $-\rho\sigma_S\sigma_F$ in δ_S^d. This call has exposure to both correlation risk and exchange rate risk.

4. Equity-linked foreign exchange call

Write the terminal payoff in the form of an exchange option

$$c_4(S, F, 0) = \max(S^* - XS, 0).$$

Taking the two assets to be exchanged as S^* and XS, the ratio of the two assets is $\frac{S^*}{XS} = \frac{F}{X}$ and the difference of the drift rates under Q_d is $\delta^d_{S^*} - \delta^d_S = r_d - r_f + \rho\sigma_F\sigma_S$.

$$\begin{aligned} c_4(S, \tau) &= e^{-r_d\tau}\left[S^* e^{\delta^d_{S^*}\tau} N(d_1^{(4)}) - XSe^{\delta^d_S\tau} N(d_2^{(4)})\right] \\ &= Se^{-q\tau}\left[FN(d_1^{(4)}) - Xe^{(r_f - r_d - \rho\sigma_F\sigma_S)\tau} N(d_2^{(4)})\right], \end{aligned}$$

where

$$d_1^{(4)} = \frac{\ln\frac{F}{X} + \left(r_d - r_f + \rho\sigma_F\sigma_S + \frac{\sigma_F^2}{2}\right)\tau}{\sigma_F\sqrt{\tau}}, \quad d_2^{(4)} = d_1^{(4)} - \sigma_F\sqrt{\tau}.$$

Digital quanto option relating 3 currency worlds

$F_{S\backslash U}$ = SGD currency price of one unit of USD currency

$F_{H\backslash S}$ = HKD currency price of one unit of SGD currency

- Digital quanto option payoff: pay one HKD if $F_{S\backslash U}$ is above some strike level K.
- We may interpret $F_{S\backslash U}$ as the price process of a tradeable asset in SGD. The dynamics is governed by

$$\frac{dF_{S\backslash U}}{F_{S\backslash U}} = (r_{SGD} - r_{USD})\,dt + \sigma_{F_{S\backslash U}}\,dZ^{S}_{F_{S\backslash U}}.$$

- Given $\delta^S_{F_{S\backslash U}} = r_{SGD} - r_{USD}$, how to find $\delta^H_{F_{S\backslash U}}$, which is the risk neutral drift rate of the SGD asset denominated in Hong Kong dollar?

- By the quanto-prewashing technique

$$\delta^H_{F_{S\backslash U}} = \delta^S_{F_{S\backslash U}} - \rho\sigma_{F_{S\backslash U}}\sigma_{F_{H\backslash S}}.$$

  Note that $F_{S\backslash U}$ can be interpreted as a foreign asset (Singaporean dollar denominated).

- Digital option value $= e^{-r_{HKD}\tau} E^t_{Q^H}\left[\mathbf{1}_{\{F_{S\backslash U} > K\}}\right] = e^{-r_{HKD}\tau} N(d)$

  where

$$d = \frac{\ln \frac{F_{S\backslash U}}{K} + \left(\delta^H_{F_{S\backslash U}} - \frac{\sigma^2_{F_{S\backslash U}}}{2}\right)\tau}{\sigma_{F_{S\backslash U}}\sqrt{\tau}}.$$

Example 1

The quanto option pays $F_{H\backslash S}$ Hong Kong dollars when $F_{S\backslash U} > K$. This is equivalent to pay one Singaporean dollar. Value of the quanto option in Singaporean dollar is

$$e^{-r_{SGD}\tau} E^t_{Q_S}\left[\mathbf{1}_{\{F_{S\backslash U} > K\}}\right] = e^{-r_{SGD}\tau} N(\widehat{d})$$

where

$$\widehat{d} = \frac{\ln \frac{F_{S\backslash U}}{K} + \left(\delta^S_{F_{S\backslash U}} - \frac{\sigma^2_{F_{S\backslash U}}}{2}\right)\tau}{\sigma_{F_{S\backslash U}}\sqrt{\tau}}, \quad \delta^S_{F_{S\backslash U}} = r_{SGD} - r_{USD}.$$

This option model is similar to $c_1(S, F, \tau)$, where the option payoff in foreign currency is converted into domestic currency using the prevailing exchange rate at maturity. The most efficient approach is to perform valuation of the option under the foreign currency world. The value of the quanto option in Hong Kong dollar is $F_{H\backslash S} e^{-r_{SGD}\tau} N(\widehat{d})$.

Example 2

The quanto option pays $F_{H\backslash U}$ Hong Kong dollars when $F_{S\backslash U} > K$. This is equivalent to pay one US dollars.

Method One

Observe that $F_{H\backslash U} = F_{H\backslash S} F_{S\backslash U}$ so that it is like paying $F_{S\backslash U}$ Singaporean dollars when $F_{S\backslash U} > K$.

Value of the quanto option in Hong Kong dollars is

$$\begin{aligned} F_{H\backslash S} e^{-r_{SGD}\tau} E^t_{Q_S}\left[F_{S\backslash U} \mathbf{1}_{\{F_{S\backslash U} > K\}} \right] &= F_{H\backslash S} e^{-r_{SGD}\tau} e^{(r_{SGD}\tau - r_{USD})\tau} F_{S\backslash U} N(d_1) \\ &= F_{H\backslash U} e^{-r_{USD}\tau} N(d_1) \end{aligned}$$

where

$$d_1 = \frac{\ln \frac{F_{S\backslash U}}{K} + \left(r_{SGD} - r_{USD} + \frac{\sigma^2_{F_{S\backslash U}}}{2} \right)\tau}{\sigma_{F_{S\backslash U}} \sqrt{\tau}}.$$

Method Two

The quanto option pays one US dollars when $F_{S\backslash U} > K \Leftrightarrow \dfrac{1}{K} > \dfrac{1}{F_{S\backslash U}} = F_{U\backslash S}$.
Later, we multiply the option value in US currency by the exchange rate $F_{H\backslash U}$ to convert into Hong Kong dollars.

Value of the quanto option in US dollars is

$$e^{-r_{USD}\tau} E^t_{Q_U}\left[\mathbf{1}_{\left\{F_{U\backslash S} < \frac{1}{K}\right\}}\right] = e^{-r_{UDS}\tau} N(-d_2),$$

where

$$d_2 = \frac{\ln \frac{F_{U\backslash S}}{1/K} + \left((r_{USD} - r_{SGD}) - \frac{\sigma^2_{F_{U\backslash S}}}{2}\right)\tau}{\sigma_{F_{U\backslash S}}\sqrt{\tau}} = -d_1.$$

Remark The quanto option value in Hong Kong dollars using the two approaches agree with each other.

5.5 Implied volatilities and volatility smiles

- The difficulties of setting volatility value in the price formulas lie in the fact that the input value should be the forecast volatility value over the remaining life of the option rather than an estimated volatility value *(historical volatility)* from the past market data of the asset price.

- Suppose we treat the option price function $V(\sigma)$ as a function of the volatility σ and let V_{market} denote the option price observed in the market. The implied volatility σ_{imp} is defined by

$$V(\sigma_{imp}) = V_{market}.$$

- The volatility value implied by an observed market option price (*implied volatility*) indicates a consensual view about the volatility level determined by the market.

- In particular, several implied volatility values obtained simultaneously from different options with varying maturities and strike prices on the same underlying asset provide an extensive market view about the volatility at varying strikes and maturities.

- In financial markets, it becomes a common practice for traders to quote an option's market price in terms of implied volatility σ_{imp}.

- Since σ cannot be solved explicitly in terms of S, X, r, τ and option price V from the pricing formulas, the determination of the implied volatility must be accomplished by an iterative algorithm as commonly performed for the root-finding procedure for a non-linear equation.

Numerical calculations of implied volatilities

- When applied to the implied volatility calculations, the Newton-Raphson iterative scheme is given by

$$\sigma_{n+1} = \sigma_n - \frac{V(\sigma_n) - V_{market}}{V'(\sigma_n)},$$

where σ_n denotes the n^{th} iterate of σ_{imp}. Provided that the first iterate σ_1 is properly chosen, the limit of the sequence $\{\sigma_n\}$ converges to the unique solution σ_{imp}.

- The above iterative scheme may be rewritten in the following form

$$\frac{\sigma_{n+1} - \sigma_{imp}}{\sigma_n - \sigma_{imp}} = 1 - \frac{V(\sigma_n) - V(\sigma_{imp})}{\sigma_n - \sigma_{imp}} \frac{1}{V'(\sigma_n)} = 1 - \frac{V'(\sigma_n^*)}{V'(\sigma_n)}.$$

One can show that σ_n^* lies between σ_n and σ_{imp}, by virtue of the Mean Value Theorem in calculus.

- The first iterate σ_1 is chosen such that $V'(\sigma)$ is maximized by $\sigma = \sigma_1$.

- Recall that

$$V'(\sigma) = \frac{S\sqrt{\tau}\, e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} > 0 \quad \text{for all } \sigma,$$

and so

$$V''(\sigma) = \frac{S\sqrt{\tau} d_1 d_2 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}\sigma} = \frac{V'(\sigma) d_1 d_2}{\sigma}.$$

- The critical points of the function $V'(\sigma)$ are given by $d_1 = 0$ and $d_2 = 0$, which lead respectively to

$$\sigma^2 = -2\frac{\ln \frac{S}{X} + r\tau}{\tau} \quad \text{and} \quad \sigma^2 = 2\frac{\ln \frac{S}{X} + r\tau}{\tau}.$$

- The above two values of σ^2 both give $V'''(\sigma) < 0$. We can choose the first iterate σ_1 to be

$$\sigma_1 = \sqrt{\left|\frac{2}{\tau}\left(\ln\frac{S}{X} + r\tau\right)\right|}.$$

- With this choice of $\sigma_1, V'(\sigma)$ is maximized at $\sigma = \sigma_1$. Setting $n = 1$ and observing $V'(\sigma_1^*) < V'(\sigma_1)$ [note that $V'(\sigma)$ is maximized at $\sigma = \sigma_1$], we obtain

$$0 < \frac{\sigma_2 - \sigma_{imp}}{\sigma_1 - \sigma_{imp}} < 1.$$

In general, suppose we can establish

$$0 < \frac{\sigma_{n+1} - \sigma_{imp}}{\sigma_n - \sigma_{imp}} < 1, \qquad n \geq 1,$$

then the sequence $\{\sigma_n\}$ is monotonic and bounded, so $\{\sigma_n\}$ converges to the unique solution σ_{imp}.

Volatility smiles

- The Black-Scholes model assumes a lognormal probability distribution of the asset price at all future times. Since volatility is the only unobservable parameter in the Black-Scholes model, the model gives the option price as a function of volatility.

- If we plot the implied volatility of the exchange-traded options, like index options, against their strike price for a fixed maturity, the curve is typically convex in shape, rather than a straight horizontal line as suggested by the simple Black-Scholes model. This phenomenon is commonly called the *volatility smile* by market practitioners.

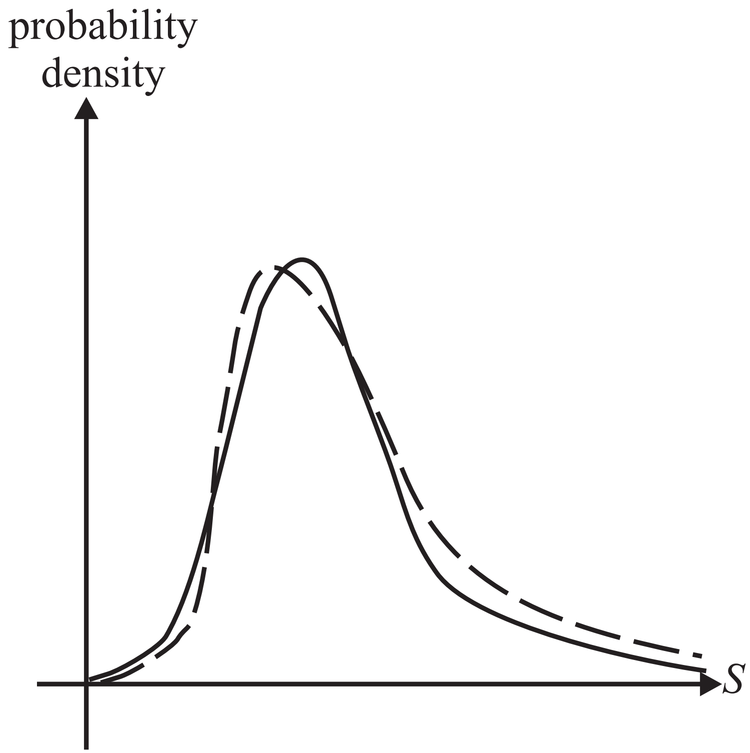
- These smiles exhibit widely differing properties, depending on whether the market data were taken before or after the October, 1987 market crash.

- The figures show the shapes of typical pre-crash smile and post-crash smile of the exchange-traded European index options. The implied volatility values are obtained by averaging options of different maturities.

- In real market situation, it is a common occurrence that when the asset price is high, volatility tends to decrease, making it less probable for a higher asset price to be realized.

- When the asset price is low, volatility tends to increase, that is, it is more probable that the asset price plummets further down.

- Suppose we plot the true probability distribution of the asset price and compare with the lognormal distribution, one observes that the left-hand tail of the true distribution is thicker than that of the lognormal one, while the reverse situation occurs at the right-hand tail.

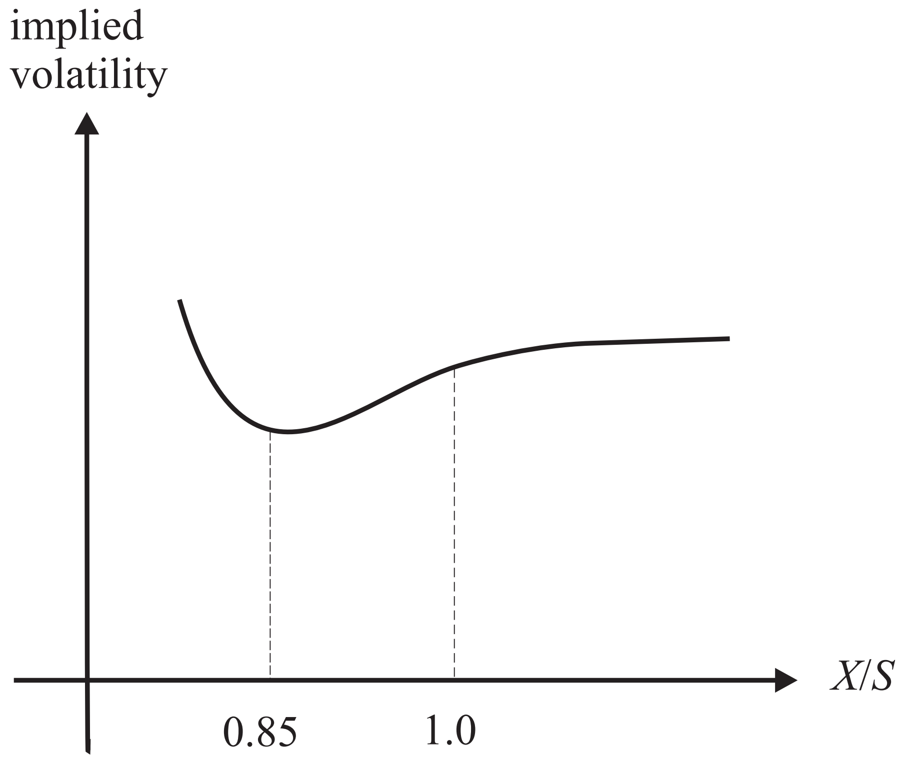


Comparison of the true probability density of asset price (solid curve) implied from market data and the lognormal distribution (dotted curve). The true probability density is thicker at the left tail and thinner at the right tail.

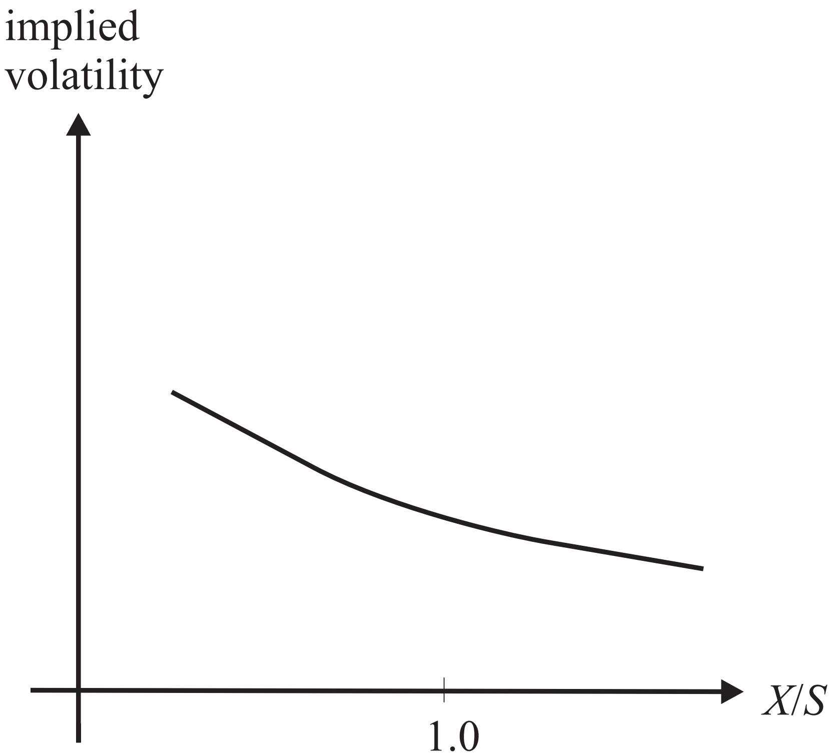
- As reflected from the implied probabilities calculated from the market data of option prices, this market behavior of higher probability of large decline in stock index is better known to market practitioners after the October, 1987 market crash.

- The market price of the out-of-the-money calls (puts) became cheaper (more expensive) than the Black-Scholes price after the 1987 crash because of the thickening (thinning) of the left- (right-) hand tail of the true probability distribution.

- In common market situation, the out-of-the-money stock index puts are traded at higher implied volatilities than the out-of-the-money stock index calls.



A typical pattern of pre-crash smile. The implied volatility curve is convex with a dip.



A typical pattern of post-crash smile. The implied volatility drops against X/S, indicating that out-of-the-money puts ($X/S < 1$) are traded at higher implied volatility than out-of-the-money calls ($X/S > 1$).

5.6 Local volatility and Dupire's equation

- Suppose European option prices at all strikes and maturities are available so that $\sigma_{imp}(t, T; X)$ can be computed, can we find a state-time dependent volatility function $\sigma(S_t, t)$ that gives the theoretical Black-Scholes option prices which are consistent with the market option prices. In the literature, $\sigma(S_t, t)$ is called the *local volatility function*.

- Given that market European option prices are all available, the risk neutral probability distribution of the asset price can be recovered.

Useful calculus formula

$$
\begin{aligned}
& \frac{d}{dx}\int_A^B f(x,t)\,dt \\
= \; & \int_A^B \frac{\partial f}{\partial x}(x,t)\,dt + f(x,B)\frac{dB}{dx} - f(x,A)\frac{dA}{dx}.
\end{aligned}
$$

- Let $\psi(S_T, T; S_t, t)$ denote the transition density function of the asset price. The price at time t of a European call with maturity date T and strike price X is given by

$$c(S_t, t; X, T) = e^{-r(T-t)} \int_X^\infty (S_T - X)\psi(S_T, T; S_t, t)\ dS_T.$$

- If we differentiate c with respect to X, we obtain

$$\frac{\partial c}{\partial X} = -e^{-r(T-t)} \int_X^\infty \psi(S_T, T; S_t, t)\ dS_T;$$

and differentiate once more, we have

$$\psi(X, T; S_t, t) = e^{r(T-t)} \frac{\partial^2 c}{\partial X^2}.$$

- The transition density function can be inferred completely from the market prices of options with the same maturity and different strikes, without knowing the volatility function.

Dupire equation

Assuming that the asset price dynamics under the risk neutral measure is governed by

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma(S_t, t)\ dZ_t,$$

where the volatility has both state and time dependence. Write $c = c(X, T)$, the Dupire equation takes the form

$$\frac{\partial c}{\partial T} = -qc - (r - q)X\frac{\partial c}{\partial X} + \frac{\sigma^2(X, T)}{2}X^2\frac{\partial^2 c}{\partial X^2}.$$

Consider

$$\frac{\partial \psi}{\partial T} = e^{r(T-t)}\left(r\frac{\partial^2 c}{\partial X^2} + \frac{\partial^2}{\partial X^2}\frac{\partial c}{\partial T}\right),$$

that $\psi(X, T; S, t)$ satisfies the forward Fokker-Planck equation, where

$$\frac{\partial \psi}{\partial T} = \frac{\partial^2}{\partial X^2}\left[\frac{\sigma^2(X,T)}{2}X^2\psi\right] - \frac{\partial}{\partial X}[(r-q)X\psi].$$

Combining the above equations and eliminating the common factor $e^{r(T-t)}$, we have

$$\begin{aligned} & r\frac{\partial^2 c}{\partial X^2} + \frac{\partial^2}{\partial X^2}\frac{\partial c}{\partial T} \\ = \; & \frac{\partial^2}{\partial X^2}\left[\frac{\sigma^2(X,T)}{2}X^2\frac{\partial^2 c}{\partial X^2}\right] - \frac{\partial}{\partial X}\left[(r-q)X\frac{\partial^2 c}{\partial X^2}\right]. \end{aligned}$$

Integrating the above equation with respect to X twice, we obtain

$$\begin{aligned} & \frac{\partial c}{\partial T} + rc + (r-q)\left(X\frac{\partial c}{\partial X} - c\right) \\ = \quad & \frac{\sigma^2(X,T)}{2}X^2\frac{\partial^2 c}{\partial X^2} + \alpha(T)X + \beta(T), \end{aligned}$$

where $\alpha(T)$ and $\beta(T)$ are arbitrary functions of T.

Since all functions involving c in the above equation vanish as X tends to infinity, hence $\alpha(T)$ and $\beta(T)$ must be zero.

Grouping the remaining terms in the equation, we obtain the Dupire equation.

We may express the local volatility $\sigma(X,T)$ explicitly in terms of the call price function and its derivatives, where

$$\sigma^2(X,T) = \frac{2\left[\frac{\partial c}{\partial T} + qc + (r-q)X\frac{\partial c}{\partial X}\right]}{X^2\frac{\partial^2 c}{\partial X^2}}.$$

- Suppose a sufficiently large number of market option prices are available at many maturities and strikes, we can estimate the local volatility from the above equation by approximating the derivatives of c with respect to X and T using the market data.

- In real market conditions, market prices of options are available only at limited of number of maturities and strikes.

5.7 Stochastic volatility models

- The daily fluctuations of the return of stock prices typically exhibit volatility clustering where large moves follow large moves and small moves follow small moves. Also, the distribution of stock price returns is highly peaked and fat-failed, indicating mixtures of distribution with different variances.

- It is natural to model volatility as a random variable. The volatility clustering feature reflects the mean reversion characteristic of volatility.

- The modeling of the stochastic behavior of volatility is more difficult since volatility is a *hidden* process. Though volatility is driving stock prices, it is not directly observable.

Differential equation formulation

The asset price S_t and the variance of asset price v_t follow the stochastic processes

$$\begin{aligned} dS_t &= \mu S_t\, dt + \sqrt{v_t} S_t\, dZ_S \\ dv_t &= k(\overline{v} - v_t)\, dt + \eta\sqrt{v_t}\, dZ_v \end{aligned}$$

where the Brownian processes are correlated with $dZ_S\, dZ_v = \rho\, dt$.

- The variance process is seen to have a mean reversion level $\overline{v}$ and reversion speed k, and η is the volatility of variance.
- The asset price has the drift rate μ under the physical measure.
- All model parameters are assumed to be constant.

Let $V(S, v, t; T)$ denote the price of an option with maturity date T. Applying the Ito lemma, the differential dV is given by

$$
\begin{aligned}
dV = & \frac{\partial V}{dt} + \frac{v}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \rho \eta v S \frac{\partial^2 V}{\partial S \partial v} + \frac{\eta^2 v}{2} \frac{\partial^2 V}{\partial v^2} \\
& + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial v} dv.
\end{aligned}
$$

- Since variance v is not a traded security, we need to include options of different maturity dates T_1 and T_2 and the underlying asset in order to construct a riskless hedging portfolio.

- Let the portfolio contain Δ_1 units of the option with maturity date T_1, Δ_2 units of the option with maturity date T_2 and Δ_S units of the underlying asset. The value of the portfolio is given by

$$
\Pi = \Delta_1 V(S, v, t; T_1) + \Delta_2 V(S, v, t; T_2) + \Delta_S S.
$$

- Suppose we write

$$\frac{dV(T_i)}{V(T_i)} = \alpha_i\,dt + \sigma_i^S\,dZ_S + \sigma_i^v\,dZ_v, \qquad i = 1, 2,$$

then

$$\begin{aligned}
\alpha_i \;&=\; \frac{1}{V(T_i)}\left[\frac{\partial V(T_i)}{\partial t} + \frac{v}{2}S^2\frac{\partial^2 V(T_i)}{\partial S^2} + \rho\eta v S\frac{\partial^2 V(T_i)}{\partial S\partial v} + \frac{\eta^2 v}{2}\frac{\partial V(T_i)}{\partial v^2}\right.\\
&\left.\quad + \mu S\frac{\partial V(T_i)}{\partial S} + k(\overline{v} - v)\frac{\partial V(T_i)}{\partial v}\right],\\
\sigma_i^S \;&=\; \frac{1}{V(T_i)}\sqrt{v}S\frac{\partial V(T_i)}{\partial S}, \qquad \sigma_i^v = \frac{1}{V(T_i)}\eta\sqrt{v}\frac{\partial V(T_i)}{\partial v}, \qquad i = 1, 2.
\end{aligned}$$

- Since there are only two risk factors (as modeled by the two Brownian processes) and three traded securities are available, it is always possible to form an instantaneously riskless portfolio.

Assuming the trading strategy to be self-financing so that the change in portfolio value arises from changes in the prices of the securities.

The differential change in portfolio value is given by

$$\begin{aligned} d\Pi &= \Delta_1\, dV(T_1) + \Delta_2\, dV(T_2) + \Delta\, dS \\ &= [\Delta_1 \alpha_1 V(T_1) + \Delta_2 \alpha_2 V(T_2) + \Delta_S \mu S]\, dt \\ &\quad + [\Delta_1 \sigma_1^S V(T_1) + \Delta_2 \sigma_2^S V(T_2) + \Delta_S \sqrt{v} S]\, dZ_S \\ &\quad + [\Delta_1 \sigma_1^v V(T_1) + \Delta_2 \sigma_2^v V(T_2)]\, dZ_v. \end{aligned}$$

In order to cancel the stochastic terms in $d\Pi$, we must choose Δ_1, Δ_2 and Δ_S such that they satisfy the following system of equations

$$\begin{aligned} &\Delta_1 \sigma_1^S V(T_1) + \Delta_2 \sigma_2^S V(T_2) + \Delta_S \sqrt{v} S = 0 \\ &\Delta_1 \sigma_1^v V(T_1) + \Delta_2 \sigma_2^v V(T_2) = 0. \end{aligned}$$

The instantaneously riskless portfolio must earn the riskless interest rate r, that is,

$$\begin{aligned} d\Pi &= [\Delta_1 \alpha_1 V(T_1) + \Delta_2 \alpha_2 V(T_2) + \Delta_S \mu S]\, dt \\ &= r[\Delta_1 V(T_1) + \Delta_2 V(T_2) + \Delta_S S]\, dt \end{aligned}$$

giving the third equation for Δ_1, Δ_2 and Δ_S:

$$\Delta_1(\alpha_1 - r)V(T_1) + \Delta_2(\alpha_2 - r)V(T_2) + \Delta_S(\alpha - r)S = 0.$$

In matrix form:

$$\begin{pmatrix} (\alpha_1 - r)V(T_1) & (\alpha_2 - r)V(T_2) & (\mu - r)S \\ \sigma_1^S V(T_1) & \sigma_2^S V(T_2) & \sqrt{v}S \\ \sigma_1^v V(T_1) & \sigma_2^v V(T_2) & 0 \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_S \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

- Non-trivial solutions for Δ_1, Δ_2 and Δ_S exist in the above homogeneous system of equations when the first row in the above coefficient matrix can be expressed as a linear combination of the second and third rows.

- This is equivalent to the existence of multipliers $\lambda_S(S, v, t)$ and $\lambda_v(S, v, t)$ such that

$$\alpha_i - r = \lambda_S \sigma_i^S + \lambda_v \sigma_i^v, \quad i = 1, 2, \quad \text{and} \quad \mu - r = \lambda_S \sqrt{v}.$$

- The multipliers λ_S and λ_v are seen to be the market price of risk of the asset price and variance, respectively. In general, they are functions of S, v and t.

We obtain

$$\frac{\partial V}{\partial t} + \frac{v}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \rho \eta v S \frac{\partial^2 V}{\partial S \partial v} + \frac{\eta^2 v}{2} \frac{\partial V}{\partial v^2} + rS \frac{\partial V}{\partial S}$$
$$+ \ [k(\overline{v} - v) - \lambda_v \eta \sqrt{v}] \frac{\partial V}{\partial v} - rV = 0.$$

- Heston makes the assumption that $\lambda_v(S, v, t)$ is a constant multiple of $\sqrt{v}$ so that the coefficient of $\dfrac{\partial V}{\partial v}$ becomes a linear function of v.

- Without loss of generality, we may express the drift term as $k'(\overline{v}' - v)$ for some constants k' and $\overline{v}'$, where k' and $\overline{v}'$ can be treated as risk adjusted parameters for the drift of v.

Price function of a European call option

- It may be more convenient to work with the futures call option. Let f_t denote the time-t price of the futures on the underlying asset with expiration date T and define $x_t = \ln \frac{f_t}{X}$.

- Let $c(x, v, \tau; X)$ denote the futures call price function, $\tau = T - t$, whose governing equation is given by

$$\frac{\partial c}{\partial \tau} = \frac{v}{2}\frac{\partial^2 c}{\partial x^2} - \frac{v}{2}\frac{\partial c}{\partial x} + \frac{\eta^2 v}{2}\frac{\partial^2 c}{\partial v^2} + \rho\eta v\frac{\partial^2 c}{\partial x \partial v} + k'(\overline{v}' - v)\frac{\partial c}{\partial v}$$

with initial condition:

$$c(x, v, 0) = \max(e^x - 1, 0).$$

- The futures call price function takes the form:

$$c(x,v,\tau) = e^x G_1(x,v,\tau) - G_0(x,v,\tau),$$

where $G_0(x,v,\tau)$ is the risk neutral probability that the futures call option is in-the-money at expiration and $G_1(x,v,\tau)$ is related to the risk neutral expectation of the terminal futures price given that the option expires in-the-money.

- The two functions $G_j(x,v,\tau)$, $j = 0,1$, satisfy the following differential equations:

$$\begin{aligned}\frac{\partial G_j}{\partial \tau} &= \frac{v}{2}\frac{\partial^2 G_j}{\partial x^2} - \left(\frac{1}{2} - j\right) v \frac{\partial G_j}{\partial x} + \frac{\eta^2 v}{2}\frac{\partial^2 G_j}{\partial v^2} \\ &\quad + \rho\eta v \frac{\partial^2 G_j}{\partial x \partial v} + k'(\overline{v}' - v)\frac{\partial G_j}{\partial v}, \quad j = 0,1,\end{aligned}$$

with initial condition:

$$G_j(x,v,0) = \mathbf{1}_{\{x \geq 0\}}.$$

- The Fourier transform method is used to solve the above differential equation.

- Let $\widehat{G}_j(m, v, \tau)$ denote the Fourier transform of $G_j(x, v, \tau)$, where

$$\widehat{G}_j(m, v, \tau) = \int_{-\infty}^{\infty} e^{-imx} G_j(x, v, \tau)\, dx, \quad j = 0, 1.$$

- The Fourier transform of the initial condition is

$$\begin{aligned} \widehat{G}_j(m, v, 0) &= \int_{-\infty}^{\infty} e^{-imx} G_j(x, v, 0)\, dx \\ &= \int_{0}^{\infty} e^{-imx}\, dx = \frac{1}{im}, \quad j = 0, 1. \end{aligned}$$

Taking the Fourier transform of the differential equation, we obtain

$$\begin{aligned}
\frac{\partial \widehat{G}_j}{\partial \tau} &= -\frac{m^2}{2} v \widehat{G}_j - imv\left(\frac{1}{2} - j\right) \widehat{G}_j \\
&\quad + \frac{\eta^2}{2} v \frac{\partial^2 \widehat{G}_j}{\partial v^2} + im\rho\eta v \frac{\partial \widehat{G}_j}{\partial v} + k'(\overline{v}' - v)\frac{\partial \widehat{G}_j}{\partial v} \\
&= v\left(\alpha \widehat{G}_j + \beta \frac{\partial \widehat{G}_j}{\partial v} + \gamma \frac{\partial^2 \widehat{G}_j}{\partial v^2}\right) + \delta \frac{\partial \widehat{G}_j}{\partial v}, \quad j = 0, 1,
\end{aligned}$$

where

$$\begin{aligned}
\alpha &= -\frac{m^2}{2} - im\left(\frac{1}{2} - j\right), \quad \beta = im\rho\eta - k', \\
\gamma &= \frac{\eta^2}{2}, \quad \delta = k'\overline{v}'.
\end{aligned}$$

We seek solution of the affine form for $\widehat{G}_j$ such that

$$\widehat{G}_j(m, v, \tau) = \exp(A(m, \tau) + B(m, \tau)v) G_j(m, v, 0).$$

- By substituting the above assumed form into the governing equation, we obtain

$$\begin{aligned}\frac{\partial B}{\partial \tau} &= \alpha + \beta B + \gamma B^2 = \gamma(B - \rho_+)(B - \rho_-) \\ \frac{\partial A}{\partial \tau} &= \delta B\end{aligned}$$

with $B(m, 0) = 0$ and $A(m, 0) = 0$. Here, $\rho_\pm = \dfrac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\gamma}$.

- Writing

$$\rho = \rho_-/\rho_+ \quad \text{and} \quad \xi = \sqrt{\beta^2 - 4\alpha\gamma},$$

the solution to $B(m, \tau)$ and $A(m, \tau)$ are found to be

$$
\begin{aligned}
B(m,\tau) &= \rho_- \frac{1-e^{-\xi\tau}}{1-\rho e^{-\xi\tau}} \\
A(m,\tau) &= \delta\left(\rho_-\tau - \frac{2}{\eta^2}\ln\frac{1-\rho e^{-\xi\tau}}{1-\rho}\right).
\end{aligned}
$$

Finally, the solution to $G_j(x,v,\tau)$ is obtained by taking the Fourier inversion of $\widehat{G}_j(m,v,\tau)$, giving

$$
\begin{aligned}
& G_j(x,v,\tau) \\
= & \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{\exp(imx + A(m,\tau) + B(m,\tau)v)}{im}\right) dm, \quad j = 0,1.
\end{aligned}
$$

5.8 Merton's model for risky debts

- Default is assumed to occur when the market value of the issuer's assets has fallen to a low level such that the issuer cannot meet the par payment at maturity.

- The issuer is essentially granted an option to default on its debt. When the value of firm's assets is less than the total debt, the debt holders can only receive the value of the firm. In the literature, the approach that uses the firm value as the fundamental state variable determining default is termed the *structural approach* or *firm value approach*.

- To analyze the credit risk structure of a risky debt using the structural approach, it is necessary to characterize the issuer's firm value process together with the information on the capital structure of the firm.

Firm value

- The value of a firm is the value of its business as a going concern. The firm's business constitutes its assets, and the present assessment of the future returns from the firm's business constitutes the current value of the firm's assets.

- The value of the firm's assets is different from the bottom line on the firm's balance sheet. When the firm is bought or sold, the value traded is the ongoing business. The difference between the amount paid for that value and the amount of book assets is usually accounted for as the "good will".

- The value of the firm's assets can be measured by *the price at which the total of the firm's liabilities can be bought or sold*. The various liabilities of the firm are *claims on its assets*. The claimants may include the debt holders, equity holder, etc.

- market value of firm asset

  = market value of equity + market value of bonds

  = share price $times$ no. of shares + sum of market bond prices

The debt issuer's firm A_t evolves according to the stochastic process of the form

$$\frac{dA_t}{A_t} = \mu_A \; dt + \sigma \; dZ_t,$$

where μ_A is the instantaneous expected rate of return, σ is the volatility of the firm asset value process. The liabilities of the firm consist only of a single debt with face value F. The debt has zero coupon and no embedded option features.

- At debt's maturity, the payment to the debt holders is the minimum of the face value F and the firm value at maturity A_T.
- Default can be triggered only at maturity and this occurs when $A_T < F$, that is, the firm asset value cannot meet its debt claim.
- Upon default, the firm is liquidated at zero cost and all the proceeds from the liquidation are transferred to the debt holder.

- The terminal payoff to the debt holders can be expressed as

$$\min(A_T, F) = F - \max(F - A_T, 0),$$

where the last term can be visualized as a put payoff. The debt holders have essentially sold a put option to the issuer since the issuer has the right to put the firm assets at the price of the par F.

- Let A denote the firm asset value at current time, $\tau = T - t$ is the time to expiry and we view the value of the risky debt $V(A, \tau)$ as a contingent claim on the firm asset value.

- By invoking the standard assumption of continuous time no-arbitrage pricing framework (continuous trading and short selling of the firm assets, perfectly divisible assets, no borrowing-lending spread, etc.), we obtain the usual Black-Scholes pricing equation:

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2} A^2 \frac{\partial^2 V}{\partial A^2} + rA \frac{\partial V}{\partial A} - rA,$$

The terminal payoff becomes the "initial" condition at $\tau = 0$:

$$V(A, 0) = F - \max(F - A, 0).$$

By linearity of the Black-Scholes equation, $V(A, \tau)$ can be decomposed into

$$V(A, \tau) = Fe^{-r\tau} - p(A, \tau), \quad \tau = T - t,$$

where $p(A, \tau)$ is the price function of a European put option.

$$\begin{aligned} p(A, \tau) &= Fe^{-r\tau} N(-d_2) - AN(-d_1), \\ d_1 &= \frac{\ln \frac{A}{F} + \left(r + \frac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}}, \quad d_2 = d_1 - \sigma \sqrt{\tau}. \end{aligned}$$

The value of the risky debt $V(A, \tau)$ is seen to be the value of the default free debt $Fe^{-r\tau}$ less the present value of the expected loss to the debt holders. The expected loss is simply the value of the put option granted to the issuer.

The equity value $E(A, \tau)$ (or shareholders' stake) is the firm value less the debt liability.

$$\begin{aligned} E(A, \tau) &= A - V(A, \tau) \\ &= A - [Fe^{-r\tau} - p(A, \tau)] = c(A, \tau), \end{aligned}$$

where $c(A, \tau)$ is the price function of the European call. This is not surprising since the shareholders have the call payoff at maturity equals $\max(A_T - F, 0)$.

Interpretation of the put option value sold to the issuer

Write the expected loss (put option value) as

$$N(-d_2)\left[Fe^{-r(T-t)} - \frac{N(-d_1)}{N(-d_2)}A\right],$$

where $\dfrac{N(-d_1)}{N(-d_2)}$ is considered as the *expected discounted recovery rate*.

$$\begin{aligned} & \text{Risky bond value} \\ = \; & \text{present value of par} - \text{default probability} \times \text{expected discounted} \\ & \text{loss given default} \end{aligned}$$

and

$$\text{default probability } = N(-d_2) = P_r[A_T \leq F].$$

Numerical example

Data

$A_t = 100, \sigma_V = 40\%, d_t =$ quasi-debt-leverage ratio $= 60\%$,

$T - t = 1$ year and $r = \ln(1 + 5\%)$.

Calculations

1. Given $d_t = \dfrac{Fe^{-r(T-t)}}{V} = 0.6$,

   then $F = 100 \times 0.6 \times (1 + 5\%) = 63$.

2. Discounted expected recovery value

$$= \frac{N(-d_1)}{N(-d_2)} A = \frac{0.069829}{0.140726} \times 100 = 49.62.$$

3. Expected discounted shortfall amounts $= 63 - 49.62 = 10.38$.

4. Cost of default = put value

$$= N(-d_2) \times \text{ expected discounted shortfall}$$

$$= 14.07\% \times 10.38 = 1.46;$$

value of credit risky bond is given by

$$60 - 1.46 = 58.54.$$

Term structure of credit spreads

The yield to maturity $Y(\tau)$ of the risky debt is defined as the rate of return of the debt, where

$$V(A,\tau) = Fe^{-Y(\tau)\tau}.$$

Rearranging the terms, we have

$$Y(\tau) = -\frac{1}{\tau} \ln \frac{V(A,\tau)}{F}.$$

The credit spread is the difference between the yields of risky and default free zero-coupon debts. This represents the risk premium demanded by the debt holders to compensate for the potential risk of default.

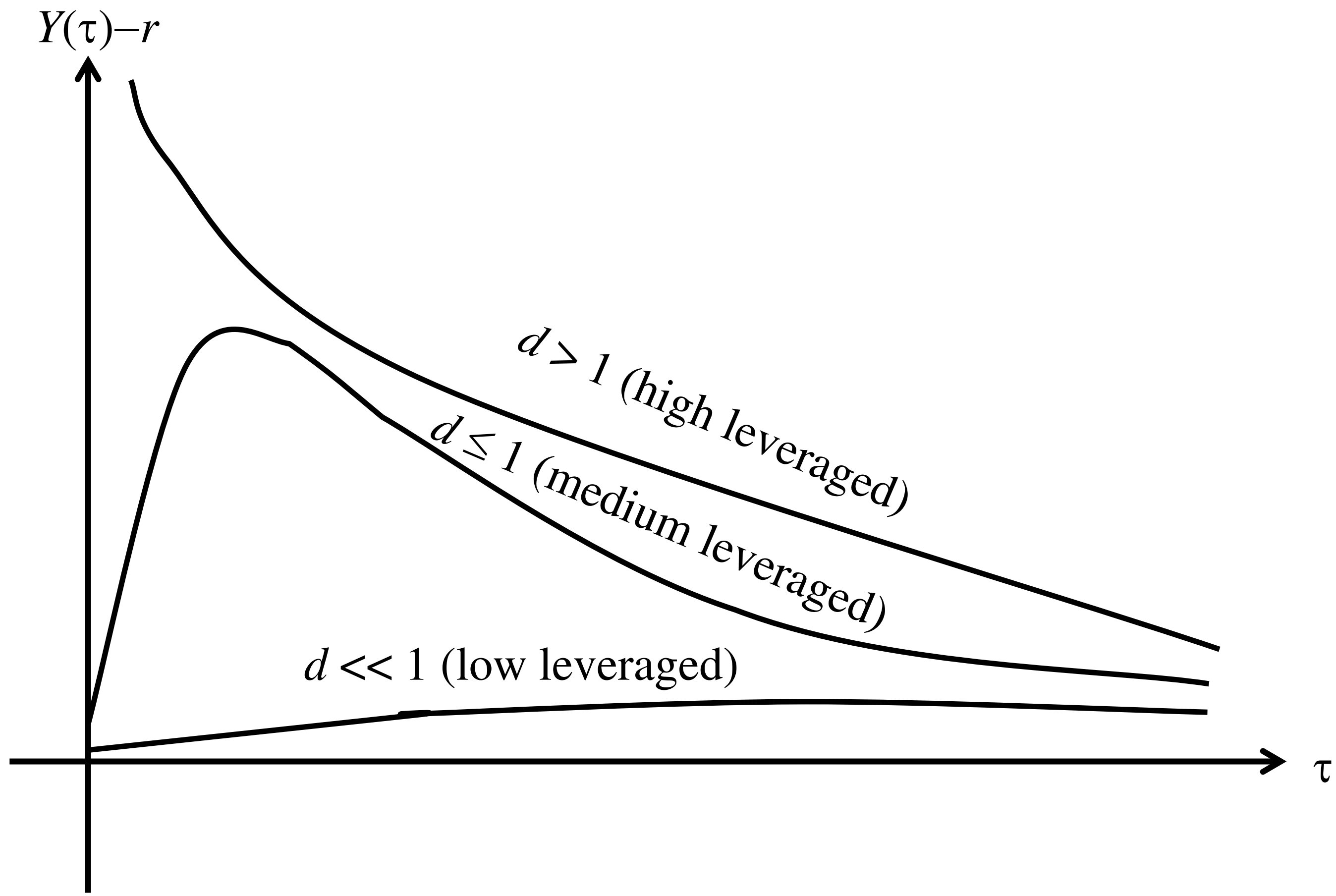
Under the assumption of constant riskfree interest rate, the credit spread is found to be

$$Y(\tau) - r = -\frac{1}{\tau}\ln\left(N(d_2) + \frac{1}{d}N(d_1)\right),$$

where

$$d = \frac{Fe^{-r\tau}}{A}, \quad d_1 = \frac{\ln d}{\sigma\sqrt{\tau}} - \frac{\sigma\sqrt{\tau}}{2} \quad \text{and} \quad d_2 = -\frac{\ln d}{\sigma\sqrt{\tau}} - \frac{\sigma\sqrt{\tau}}{2}.$$

- The quantity d is the ratio of the default free debt $Fe^{-r\tau}$ to the firm value A, thus it is coined the term "quasi" debt-to-firm ratio. The adjective "quasi" is added since all valuations are performed under the risk neutral measure instead of the "physical" measure.



- As time approaches maturity, the credit spread always tends to zero when $d \leq 1$ but tends toward infinity when $d > 1$.

- At times far from maturity, the credit spread has low value for all values of d since sufficient time has been allowed for the firm value to have a higher potential to grow beyond F.

Time dependent behaviors of credit spreads

- Downward-sloping for highly leveraged firms.
- Hump shaped for medium leveraged firms.
- Upward-sloping for low leveraged firms.

Possible explanation

- For high-quality bonds, credit spreads widen as maturity increases since the upside potential is limited and the downside risk is substantial.

Remark

Most banking regulations do not recognize the term structure of credit spreads. When allocating capital to cover potential defaults and credit downgrades, a one-year risky bond is treated the same as a ten-year counterpart.

Shortcomings

1. Default can never occur by surprise since the firm value is assumed to follow a diffusion process – may be partially remedied by introducing jump effect into the firm value process.

2. Actual spreads are larger than those predicted by Merton's model.

3. Default premiums are shown to be inversely related to firm size as revealed from empirical studies. In Merton's model, $Y(\tau) - r$ is a function of d and $\sigma^2\tau$ only, with no explicit dependence on A.

Reference

H.Y. Wong and Y.K. Kwok, "Jump diffusion model for risky debts: quality spread differentials," *International Journal of Theoretical and Applied Finance*, vol. 6(6) (2003) p.655-662.

Example – Risky commodity-linked bond

- A silver mining company offered bond issues backed by silver. Each $\$1,000$ bond is linked to 50 ounces of silver, pays a coupon rate of 8.5% and has a maturity of 15 years.
- At maturity, the company guarantees to pay the holders either $\$1,000$ or the market value of 50 ounces of silver.

Rationale The issuer is willing to share the potential price appreciation in exchange for a lower coupon rate or other favorable bond indentures.

Terminal payoff of bond value

$$\overline{B}(V,S,T) = \min(V, F + \max(S - F, 0)),$$

where V is the firm value, r is the interest rate, S is the value of 50 ounces of silver, F is the face value.

Potential extensions in risky debt models

1. *Interest rate uncertainty*

   Debts are relatively long-term interest rate sensitive instruments. The assumption of constant rates is embarrassing.

2. Jump-diffusion process of the firm value.

   - Allows for a jump process to shock the firm value process.
   - Remedy the realistic small short-maturity spreads in pure diffusion model. Default may occur by surprise.

3. *Bankruptcy–triggering mechanism*

   Black-Cox (1976) assume a cut-off level whereby intertemporal default can occur. The cut-off may be considered as a safety covenant which protects bondholder or liability level for the firm below which the firm bankrupts.

4. *Deviation from the strict priority rule*

   Empirical studies show that the absolute priority rule is enforced in only 25% of corporate bankruptcy cases. The write-down of creditor claims is usually the outcome of a bargaining process which results in shifts of gains and losses among corporate claimants relative to their contractual rights.

Quality spread differentials between fixed rate debt and floating rate debt

- In *fixed rate debts*, the par paid at maturity is fixed.

- A *floating rate debt* is similar to a *money market account*, where the par at maturity is the sum of principal and accured interests. The amount of accrued interests depends on the realization of the stochastic interest rate over the life of bond.

  Whether the default premiums demanded by investors are equal for both types of debts?

Related question: Does the swap rate in an interest rate swap depend on which party is serving as the fixed rate payer?

Empirical studies reveal that the yield premiums for fixed rate debts are in general higher than those for floating rate debts. Why? On the other hand, when the yield curve is upward sloping, floating rate debt holders should demand a higher floating spread.

Credit valuation model

1. Credit risk should be measured in terms of *probabilities and mathematical expectations*, rather than assessed by qualitative ratings.

2. Credit risk model should be based on current, rather than historical measurements. The relevant variables are the *actual market values rather than accounting values*. It should reflect the development in the borrower's credit standing through time.

3. An assessment of the future earning power of the firm, company's operations, projection of cash flows, etc., has already been made by the aggregate of the market participants in the stock market. The stock price will be the first to reflect the changing prospects. The challenge is how to *interpret the changing share prices* properly.

4. The various liabilities of a firm are *claims on the firm's value*, which often take the form of options, so the credit model should be consistent with the theory of option pricing.

Industrial implementation: KMV model

Expected default frequency

- *Expected default frequency* (EDF) is a forward-looking measure of actual probability of default. EDF is firm specific.

- KMV model is based on the structural approach to calculate EDF (credit risk is driven by the firm value process).
  - It is best when applied to publicly traded companies, where the value of equity is determined by the stock market.
  - The market information contained in the firm′s stock price and balance sheet are translated into an implied risk of default.

- Accurate and timely information from the equity market provides a continuous credit monitoring process that is difficult and expensive to duplicate using traditional credit analysis.

- Annual reviews and other traditional credit processes cannot maintain the same degree of "on guard" that EDFs calculated on a monthly or a daily basis can provide.

Key features in KMV model

1. Distance to default ratio determines the level of default risk.

   - This key ratio compares the firm's net worth $E(V_T) - d^*$ to its volatility.
   - The net worth is based on values from the equity market, so it is both timely and superior estimate of the firm value.

2. Ability to adjust to the credit cycle and ability to quickly reflect any deterioration in credit quality.

3. Work best in highly efficient liquid market conditions.

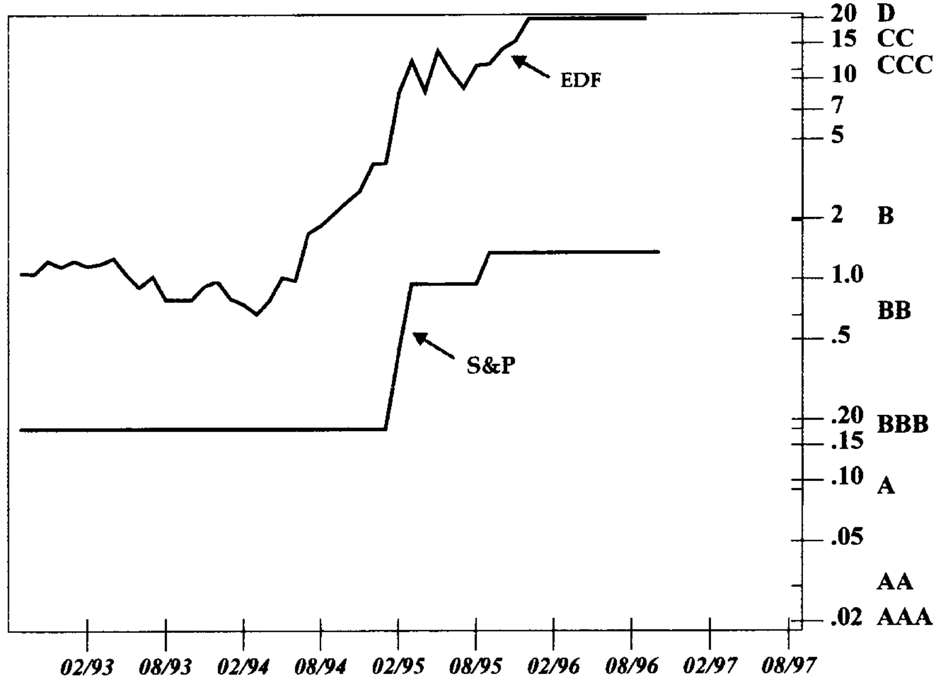
Three steps to derive the actual probabilities of default:

1. Estimation of the market value and volatility of the firm asset value.

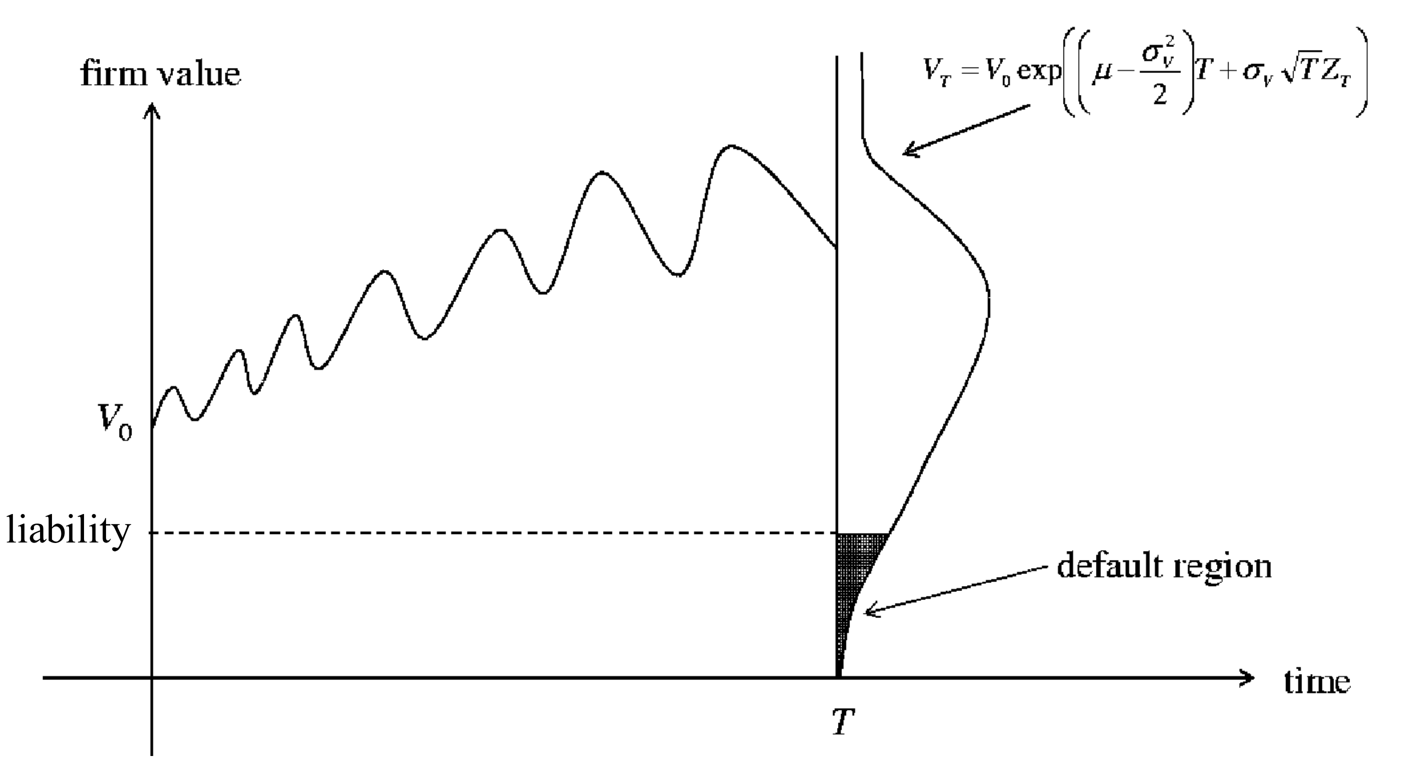
2. Calculation of the distance to default, an index measure of default risk.

3. Scaling of the distance to default to actual probabilities of default using a default database.

- Changes in EDF tend to anticipate at least one year earlier than the downgrading of the issuer by rating agencies like Moodys and S & Ps



Distribution of terminal firm value at maturity of debt



- According to KMV′s empirical studies, log-asset returns confirm quite well to a normal distribution, and σ_V stays relatively constant.
- From the sample of several hundred companies, firms default when the asset value reaches a level somewhere between the value of total liabilities and the value of the short-term debt.

Distance to default

Default point, $d^* =$ short-term debt $+ \frac{1}{2} \times$ long-term debt. Why $\frac{1}{2}$? Why not!

From $V_T = V_0 \exp\left(\left(\mu - \frac{\sigma_V^2}{2}\right)T + \sigma_V Z_T\right)$, the probability of finishing below d^* at date T is

$$N\left(-\frac{\ln\frac{V_0}{d^*} + \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma_V\sqrt{T}}\right).$$

Distance to default is defined by

$$d_f = \frac{E(V_T) - d^*}{\widehat{\sigma}_V\sqrt{T}} = \frac{\ln\frac{V_0}{d^*} + \left(\mu - \frac{\widehat{\sigma}_V^2}{2}\right)T}{\widehat{\sigma}_V\sqrt{T}},$$

where V_0 is the current market value of firm, μ is the expected rate of return on firm value and $\widehat{\sigma}_V$ is the annualized firm value volatility. The probability of default is a function of the firm's capital structure, the volatility of the asset returns and the current asset value.

Estimation of firm value V and volatility of firm value σ_V

- Usually, only the price of equity for most public firms is directly observable. In some cases, part of the debt is directly traded.

- Using option pricing approach:

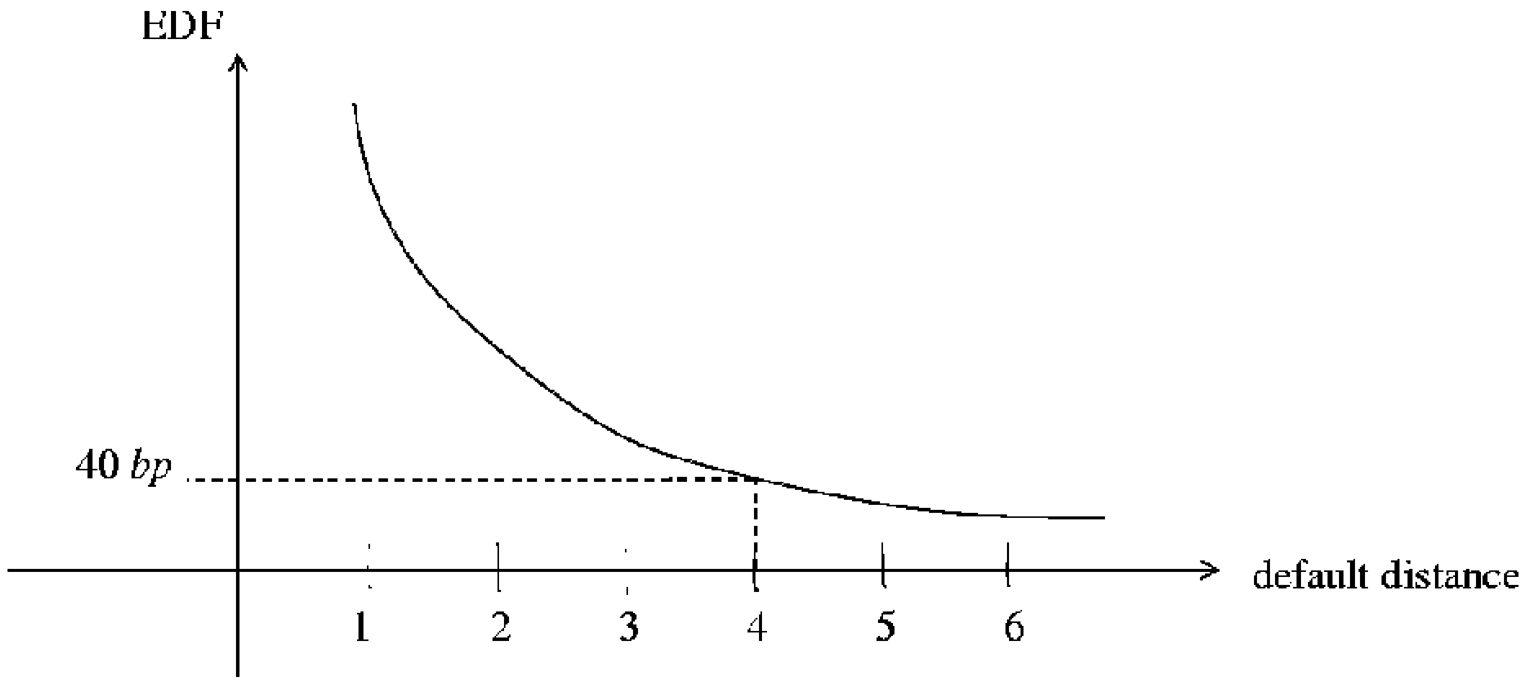
  equity value, $E = f(V, \sigma_V, K, c, r)$

  volatility of equity, $\sigma_E = g(V, \sigma_V, K, c, r)$

  where K denotes the leverage ratio in the capital structure, c is the average coupon paid on the long-term debt, r is the riskfree rate. Actually, the relation between σ_E and σ_V is obtained via the Ito lemma: $E\sigma_E = \frac{\partial f}{\partial V} V \sigma_V$.

- Solve for V and σ_V from the above 2 equations.

Probabilities of default from the default distance



Based on historical information on a large sample of firms, for each time horizon, one can estimate the proportion of firms of a given default distance (say, $d_f = 4.0$) which actually defaulted after one year.

Example *Federal Express* (dollars in billion of US$)

	November 1997	*February 1998*
Market capitalization (price $\times$ shares outstanding)	\$7.9	\$7.3
Book liabilities	\$4.7	\$4.9
Market value of assets	\$12.6	\$12.2
Asset volatility	15%	17%
Default point	\$3.4	\$3.5
Default distance	$\dfrac{12.6 - 3.4}{0.15}$	$\dfrac{12.2 - 3.5}{0.17}$
EDF	$0.06\%(6bp) = AA^-$	$0.11\%(11bp) = A^-$

The causes of change for the EDF are due to variations in the *stock price, debt level* (leverage ratio) and *asset volatility*.

Weaknesses of the KMV approach

- It requires some *subjective estimation* of the input parameters.
- It is difficult to construct theoretical EDFs without the *assumption of normality* of asset returns.
- *Private firms EDFs* can be calculated only by using some compabability analysis based on accounting data.
- It does not *distinguish* among different types of long-term bonds according to their seniority, collateral, covenants or convertibility.