MATH 571
Mathematical Models of Financial Derivatives

Homework Four

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1. Let X be a normally distributed random variable with mean μ and variance σ^2. Show that the higher central moments of the normal random variable are given by

$$\mu_n(X;\mu) = E[(X-\mu)^n] = \begin{cases} 0, & n \text{ odd} \\ (n-1)(n-3)\cdots 3\cdot 1\sigma^n, & n \text{ even.} \end{cases}$$

2. Suppose $Z(t)$ is a standard Brownian process, show that the following processes defined by

$$X_1(t) = kZ(t/k^2), \quad k > 0$$
$$X_2(t) = \begin{cases} tZ\left(\frac{1}{t}\right) & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$$

and

$$X_3(t) = Z(t+h) - Z(h), \quad h > 0$$

are also Brownian processes.

Hint: To show that $X_i(t)$ is a Brownian process, $i = 1, 2, 3$, it is necessary to show that

$$X_i(t+s) - X_i(s)$$

is normally distributed with zero mean, and

$$E[[X_i(t+s) - X_i(s)]^2] = t.$$

Also, the increments over disjoint time intervals are independent, and $X_i(t)$ is continuous at $t = 0$.

3. Consider the Brownian motion with drift defined by

$$X(t) = \mu t + \sigma Z(t), \quad X(0) = 0, Z(t) \text{ is the standard Brownian motion,}$$

find $E[X(t)|X(t_0)]$, $\mathrm{var}(X(t)|X(t_0))$ and $\mathrm{cov}(X(t_1), X(t_2))$.

4. Let $Z(t)$ denote the standard Brownian process. Show that
   (a) $\displaystyle\int_{t_0}^{t_1} Z(t)^n \, dZ(t) = \frac{1}{n+1}[Z(t_1)^{n+1} - Z(t_0)^{n+1}] - \frac{n}{2}\int_{t_0}^{t_1} Z(t)^{n-1} \, dt,$
   for any positive integer n,
   (b) $E[Z^4(t)] = 3t^2$.

5. Let $Z(t), t \geq 0$, be the standard Brownian process, $f(t)$ and $g(t)$ are differentiable functions over $[a, b]$. Show that

$$E\left[\int_a^b f'(t)[Z(t) - Z(a)] \, dt \int_a^b g'(t)[Z(t) - Z(a)] \, dt\right]$$
$$= \int_a^b [f(b) - f(t)][g(b) - g(t)] \, dt.$$

   Hint: Interchange the order of expectation and integration, and observe

$$E[[Z(t) - Z(a)][Z(s) - Z(a)]] = \min(t, s) - a.$$

6. Show that

$$\sigma \int_t^T [Z(u) - Z(t)] \, du$$

   has zero mean and variance $\sigma^2(T - t)^3/3$.
   Hint: Consider

$$\begin{aligned} &\operatorname{var}\left(\int_t^T [Z(u) - Z(t)] \, du\right) \\ &= E\left[\int_t^T \int_t^T [Z(u) - Z(t)][Z(v) - Z(t)] \, dudv\right] \\ &= \int_t^T \int_t^T E[\{Z(u) - Z(t)\}\{Z(v) - Z(t)\}] \, dudv \\ &= \int_t^T \int_t^T [\min(u, v) - t] \, dudv. \end{aligned}$$

7. Suppose the stochastic variables S_1 and S_2 follow the Geometric Brownian processes where

$$\frac{dS_i}{S_i} = \mu_i \, dt + \sigma_i \, dZ_i, \qquad i = 1, 2.$$

   Let ρ_{12} denote the correlation coefficient between the Wiener processes dZ_1 and dZ_2. Let $f = S_1S_2$, show that f also follows the Geometric Brownian process of the form

$$\frac{df}{f} = \mu \, dt + \sigma \, dZ_f$$

   where $\mu = \mu_1 + \mu_2 + \rho_{12}\sigma_1\sigma_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2$. Similarly, let $g = \dfrac{S_1}{S_2}$, show that

$$\frac{dg}{g} = \widetilde{\mu} \, dt + \widetilde{\sigma} \, dZ_g$$

where $\widetilde{\mu} = \mu_1 - \mu_2 - \rho_{12}\sigma_1\sigma_2 + \sigma_2^2$ and $\widetilde{\sigma}^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2$.

Hint: Note that

$$\frac{d\left(\frac{1}{S_2}\right)}{\frac{1}{S_2}} = -\mu_2\,dt + \sigma_2^2\,dt - \sigma_2\,dZ_2.$$

Treat S_1/S_2 as the product of S_1 and $1/S_2$ and use the result obtained for the product of Geometric Brownian processes.

8. Define the discrete random variable X by

$$X(\omega) = \begin{cases} 2 & \text{if } \omega = \omega_1 \\ 3 & \text{if } \omega = \omega_2 \\ 4 & \text{if } \omega = \omega_3 \end{cases},$$

where the sample space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $P[\omega_1] = P[\omega_2] = P[\omega_3] = 1/3$. Find a new probability measure $\widetilde{P}$ such that the mean becomes $E_{\widetilde{P}}[X] = 3.5$ while the variance remains unchanged. Is $\widetilde{P}$ unique?

9. Given that S_t is a Geometric Brownian motion which follows

$$\frac{dS_t}{S_t} = \mu\ dt + \sigma\ dZ_t$$

where Z_t is P-Brownian motion. Find another measure $\widetilde{P}$ by specifying the Radon-Nikodym derivative $\frac{d\widetilde{P}}{dP}$ such that S_t is governed by

$$\frac{dS_t}{S_t} = \mu'\ dt + \sigma\ d\widetilde{Z}_t$$

under the measure $\widetilde{P}$, where $\widetilde{Z}_t$ is $\widetilde{P}$-Brownian motion and μ' is the new drift rate.

10. Consider a forward contract on an underlying commodity, find the portfolio consisting of the underlying commodity and bond (bond's maturity coincides with forward's maturity) that replicates the forward contract. Show that the hedge ratio $\triangle$ is always equal to one. Give the financial argument to justify why the hedge ratio is one. Let $B(t,T)$ denote the price at current time t of the unit-par zero-coupon bond maturing at time T and S denote the price of commodity at time t. Show that the forward price $F(S,\tau)$ is given by

$$F(S,\tau) = S/B(t,T), \quad \tau = T - t.$$

11. Consider a portfolio containing Δ units of asset and M dollars of riskless asset in the form of money market account. The portfolio is dynamically adjusted so as to replicate an option. Let S and $V(S,t)$ denote the value of the underlying asset and the option, respectively. Let r denote the riskless interest rate and Π denote the value of the *self-financing* replicating portfolio. When the self-financing trading strategy is adopted, explain why

$$\Pi = \Delta S + M \quad \text{and} \quad d\Pi = \Delta\ dS + rM\ dt,$$

where r is the riskless interest rate. Here, the differential term $S\ d\Delta$ does not enter into $d\Pi$. Assume that the asset price dynamics follows the Geometric Brownian process:

$$\frac{dS}{S} = \rho\ dt + \sigma\ dZ.$$

Using the condition that the option value and the value of the replicating portfolio should match at all times, show that the number of units of asset held must be given by

$$\Delta = \frac{\partial V}{\partial S}.$$

How to proceed further in order to obtain the Black-Scholes equation for V?

12. From the Black-Scholes price function $c(S, \tau)$ for a European vanilla call on a non-dividend paying asset, show that the limiting values of the call price at vanishing volatility and infinite volatility are the lower and upper bounds of the European call price respectively, namely,

$$\lim_{\sigma \to 0^+} c(S, \tau) = \max(S - Xe^{-r\tau}, 0).$$

Give the appropriate financial interpretation of the above result.

13. When a European option is currently out-of-the-money, show that a higher volatility of the asset price or a longer time to expiry makes it more likely for the option to expire in-the-money. What would be the impact on the value of delta? Do we have the same effect or opposite effect when the option is currently in-the-money?

14. Show that when the European call price is a convex function of the asset price, the elasticity of the call price is always greater than or equal to one. Give the financial argument to explain why the elasticity of the price of a European option increases in absolute value when the option becomes more out-of-the-money and closer to expiry. Can you think of a situation where the European put's elasticity has absolute value less than one, that is, the European put option is less riskier than the underlying asset?

15. Suppose the greeks of the value of a derivative security are defined by

$$\Theta = \frac{\partial f}{\partial t}, \quad \triangle = \frac{\partial}{\partial S}, \quad \Gamma = \frac{\partial^2 f}{\partial S^2}.$$

   (a) Find the relation between Θ and Γ for a delta-neutral portfolio where $\triangle = 0$.
   (b) Show that the theta may become positive for an in-the-money European call option on a continuous dividend paying asset when the dividend yield is sufficiently high.
   (c) Explain by financial argument why the theta value tends asymptotically to $-rXe^{-r\tau}$ from below when the asset value is sufficiently high.

16. Let Q^* denote the equivalent martingale measure where the asset price S_t is used as the numeraire. Suppose S_t follows the lognormal distribution with drift rate r and volatility σ under Q^*, where r is the riskless interest rate. Show that

$$\frac{dQ^*}{dQ} = \frac{S_T}{S_0} e^{-rT} = e^{-\frac{\sigma^2}{2} + \sigma Z_T},$$

where Q is the martingale measure with the money market account as the numeraire and Z_T is a Brownian motion under Q. Using the Girsanov Theorem, show that

$$Z_T^* = Z_T - \sigma T$$

is a Brownian motion under Q^*. Explain why

$$E_{Q^*}[\mathbf{1}_{\{S_T \geq X\}}] = N\left(\frac{\ln \frac{S_0}{X} + \left(r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right),$$

then deduce that

$$E_Q[S_T \mathbf{1}_{\{S_T \geq X\}}] = e^{rT} S_0 N\left(\frac{\ln \frac{S_0}{X} + \left(r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right).$$