MATH 571
Mathematical Models of Financial Derivatives

Homework Five

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1. Consider a European capped call option whose terminal payoff function is given by

$$c_M(S, 0; X, M) = \min(\max(S - X, 0), M),$$

where X is the strike price and M is the cap. Show that the value of the European capped call is given by

$$c_M(S, \tau; X, M) = c(S, \tau; X) - c(S, \tau; X + M),$$

where $c(S, \tau; X + M)$ is the value of a European vanilla call with strike price $X + M$.

2. Consider the value of a European call option written by an issuer whose only asset is α (< 1) units of the underlying asset. At expiration, the terminal payoff of this call is then given by

$$\begin{array}{rl} S_T - X & \text{if} \quad \alpha S_T \geq S_T - X \geq 0 \\ \alpha S_T & \text{if} \quad S_T - X > \alpha S_T \end{array}$$

and zero otherwise. Show that the value of this European call option is given by

$$c_L(S, \tau; X, \alpha) = c(S, \tau; X) - (1 - \alpha)c\left(S, \tau; \frac{X}{1 - \alpha}\right), \quad \alpha < 1,$$

where $c\left(S, \tau; \frac{X}{1 - \alpha}\right)$ is the value of a European vanilla call with strike price $\frac{X}{1 - \alpha}$.

3. Suppose the dividends and interest incomes are taxed at the rate R but capital gains taxes are zero. Find the price formulas for the European put and call on an asset which pays a continuous dividend yield at the constant rate q, assuming that the riskless interest rate r is also constant.

Hint: Explain why the riskless interest rate r and dividend yield q should be replaced by $r(1 - R)$ and $q(1 - R)$, respectively, in the Black-Scholes formulas.

4. Consider a futures on an underlying asset which pays N discrete dividends between t and T and let D_i denote the amount of the ith dividend paid on the ex-dividend date t_i. Show that the futures price is given by

$$F(S, t) = Se^{r(T-t)} - \sum_{i=1}^{N} D_i e^{r(T-t_i)},$$

where S is the current asset price and r is the riskless interest rate. Consider a European call option on the above futures. Show that the governing differential equation for the price of the call, $c_F(F, t)$, is given by (Brenner *et al.*, 1985)

$$\frac{\partial c_F}{\partial t} + \frac{\sigma^2}{2}\left[F + \sum_{i=1}^{N} D_i e^{r(T-t_i)}\right]^2 \frac{\partial^2 c_F}{\partial F^2} - rc_F = 0.$$

5. A *forward start* option is an option which comes into existence at some future time T_1 and expires at T_2 ($T_2 > T_1$). The strike price is set equal the asset price at T_1 such that the option is at-the-money at the future option's initiation time T_1. Consider a forward start call option whose underlying asset has value S at current time t and constant dividend yield q, show that the value of the forward start call is given by

$$e^{-qT_1}c(S, T_2 - T_1; S)$$

where $c(S, T_1 - T_1; S)$ is the value of an at-the-money call (strike price same as asset price) with time to expiry $T_2 - T_1$.

Hint: The value of an at-the-money call option is proportional to the asset price.

6. Consider a contingent claim whose value at maturity T is given by

$$\min(S_{T_0}, S_T),$$

where T_0 is some intermediate time before maturity, $T_0 < T$, and S_T and S_{T_0} are the asset price at T and T_0, respectively. Show that the value of the contingent claim at time t is given by

$$V_t = S_t[1 - N(d_1) + e^{-r(T-T_0)}N(d_2)],$$

where S_t is the asset price at time t and

$$d_1 = \frac{r(T - T_0) + \frac{\sigma^2}{2}(T - T_0)}{\sigma\sqrt{T - T_0}}, \quad d_2 = d_1 - \sigma\sqrt{T - T_0}.$$

7. Consider the exchange option which entities the holder the right but not the obligation to exchange risky asset S_2 for another risky asset S_1. Let the price dynamics of S_1 and S_2 under the risk neutral measure be governed by

$$\frac{dS_i}{S_i} = (r - q_i)\, dt + \sigma_i\, dZ_i, \quad i = 1, 2,$$

where $dZ_1\, dZ_2 = \rho\, dt$. Let $V(S_1, S_2, \tau)$ denote the price function of the exchange option, whose terminal payoff takes the form

$$V(S_1, S_2, 0) = \max(S_1 - S_2, 0).$$

Show that the governing equation for $V(S_1, S_2, \tau)$ is given by

$$\begin{aligned} \frac{\partial V}{\partial \tau} &= \frac{\sigma_1^2}{2}S_1^2\frac{\partial^2 V}{\partial S_1^2} + \rho\sigma_1\sigma_2 S_1 S_2\frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{\sigma_2^2}{2}S_2^2\frac{\partial^2 V}{\partial S_2^2} \\ &+ (r - q_2)S_1\frac{\partial V}{\partial S_2} + (r - q_2)S_2\frac{\partial V}{\partial S_2} - rV. \end{aligned}$$

By taking S_2 as the numeraire and defining the similarity variables

$$x = \frac{S_1}{S_2} \quad \text{and} \quad W(x, \tau) = \frac{V(S_1, S_2, \tau)}{S_2},$$

show that the governing equation for $W(x, \tau)$ becomes

$$\frac{\partial W}{\partial \tau} = \frac{\sigma^2}{2}x^2\frac{\partial^2 W}{\partial x^2} + (q_1 - q_2)x\frac{\partial W}{\partial x}.$$

Verify that the solution to $W(x,\tau)$ is given by

$$W(x,\tau) = e^{-q_1\tau} x N(d_1) - e^{-q_2\tau} N(d_2)$$

where

$$d_1 = \frac{\ln\frac{S_1}{S_2} + \left(q_2 - q_1 + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau},$$
$$\sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2.$$

8. Suppose the terminal payoff of an exchange rate option is $F_T \mathbf{1}_{\{F_T > X\}}$. Let $V_d(F,t)$ denote the value of the option in the domestic currency world, show that

$$\begin{aligned} V_d(F,t) &= Fe^{-r_f(T-t)} E_{Q_f}\left[\mathbf{1}_{\{F_T>X\}} | F_t = F\right] \\ &= Fe^{-r_f\tau} N(d) \end{aligned}$$

where

$$d = \frac{\ln\frac{F}{K} + \left(r_d - r_f + \frac{\sigma_F^2}{2}\right)\tau}{\sigma_F\sqrt{\tau}}, \quad \tau = T - t.$$

9. Let $F_{S\backslash U}$ denote the Singaporean currency price of one unit of US currency and $F_{H\backslash S}$ denote the Hong Kong currency price of one unit of Singaporean currency. We may interpret $F_{S\backslash U}$ as the price process of a tradeable asset in Singaporean currency. Assume $F_{S\backslash U}$ to be governed by the following dynamics under the risk neutral measure Q_S in Singaporean currency world:

$$\frac{dF_{S\backslash U}}{F_{S\backslash U}} = (r_{SGD} - r_{USD})\,dt + \sigma_{F_{S\backslash U}}\,dZ^S_{F_{S\backslash U}},$$

where r_{SGD} and r_{USD} are the Singaporean and US riskless interest rates, respectively. The digital quanto option pays one Hong Kong dollar if $F_{S\backslash U}$ is above the strike level X. Find the value of the digital quanto option in terms of the riskless interest rates of different currency worlds and volatility values $\sigma_{F_{S\backslash U}}$ and $\sigma_{F_{H\backslash S}}$.

10. In the Merton model of risky debt, suppose we define

$$\sigma_V(\tau; d) = \frac{\sigma A}{V}\frac{\partial V}{\partial A},$$

which gives the volatility of the value of the risky debt. Also, we denote the credit spread by $s(\tau; d)$, where $s(\tau; d) = Y(\tau) - r$. Show that

(a) $\dfrac{\partial s}{\partial d} = \dfrac{1}{\tau d}\sigma_V(\tau; d) > 0$;

(b) $\dfrac{\partial s}{\partial \sigma^2} = \dfrac{1}{2\sqrt{\tau}}\dfrac{N'(d_1)}{N(d_1)}\sigma_V(\tau; d) > 0$, where $d_1 = \dfrac{\ln d}{\sigma\sqrt{\tau}} - \dfrac{\sigma\sqrt{\tau}}{2}$;

(c) $\dfrac{\partial s}{\partial r} = -\sigma_V(\tau; d) < 0$.

Give financial interpretation to each of the above results.

11. A firm is an entity consisting of its assets and let A_t denote the market value of firm's assets. Assume that the total asset value follows a stochastic process modeled by

$$\frac{dA_t}{A_t} = \mu\ dt + \sigma\ dZ_t,$$

where μ and σ^2 (assumed to be constant) are the instantaneous mean and variance, respectively, of the rate of return on A_t. Let C and D denote the market value of the current liabilities and market value of debt, respectively. Let T be the maturity date of the debt with face value D_T. Suppose the current liabilities of amount C_T are also payable at time T, and it constitutes a claim senior to the debt. Also, let F denote the present value of total amount of interest and dividends paid over the term T. For simplicity, F is assumed to be prepaid at time $t = 0$.

The debt is in default if A_T is less than the total amount payable at maturity date T, that is,

$$A_T < D_T + C_T.$$

(a) Show that the probability of default is given by

$$p = N\left(\frac{\ln \frac{D_T + C_T}{A_t - F} - \mu T + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}\right).$$

(b) Explain why the expected loan loss L on the debt is given by

$$EL = \int_{C_T}^{D_T + C_T} (D_T + C_T - a) f(a)\ da + \int_0^{C_T} D_T f(a)\ da,$$

where f is the density function of A_T. Give the financial interpretation to each integral.