1. Consider a European capped call option whose terminal payoff function is given by

\[ c_M(S, 0; X, M) = \min(\max(S - X, 0), M), \]

where \( X \) is the strike price and \( M \) is the cap. Show that the value of the European capped call is given by

\[ c_M(S, \tau; X, M) = c(S, \tau; X) - c(S, \tau; X + M), \]

where \( c(S, \tau; X + M) \) is the value of a European vanilla call with strike price \( X + M \).

2. Consider the value of a European call option written by an issuer whose only asset is \( \alpha \) \( (\alpha < 1) \) units of the underlying asset. At expiration, the terminal payoff of this call is then given by

\[ S_T - X \text{ if } \alpha S_T \geq S_T - X \geq 0 \]
\[ \alpha S_T \text{ if } S_T - X > \alpha S_T \]

and zero otherwise. Show that the value of this European call option is given by

\[ c_L(S, \tau; X, \alpha) = c(S, \tau; X) - (1 - \alpha)c \left( S, \tau; \frac{X}{1 - \alpha} \right), \quad \alpha < 1, \]

where \( c \left( S, \tau; \frac{X}{1 - \alpha} \right) \) is the value of a European vanilla call with strike price \( \frac{X}{1 - \alpha} \).

3. Suppose the dividends and interest incomes are taxed at the rate \( R \) but capital gains taxes are zero. Find the price formulas for the European put and call on an asset which pays a continuous dividend yield at the constant rate \( q \), assuming that the riskless interest rate \( r \) is also constant.

**Hint:** Explain why the riskless interest rate \( r \) and dividend yield \( q \) should be replaced by \( r(1 - R) \) and \( q(1 - R) \), respectively, in the Black-Scholes formulas.

4. Consider a futures on an underlying asset which pays \( N \) discrete dividends between \( t \) and \( T \) and let \( D_i \) denote the amount of the \( i \)th dividend paid on the ex-dividend date \( t_i \). Show that the futures price is given by

\[ F(S, t) = Se^{r(T-t)} - \sum_{i=1}^{N} D_i e^{r(T-t_i)}, \]

where \( S \) is the current asset price and \( r \) is the riskless interest rate. Consider a European call option on the above futures. Show that the governing differential equation for the price of the call, \( c_F(F, t) \), is given by (Bremer et al., 1985)

\[ \frac{\partial c_F}{\partial t} + \frac{\sigma^2}{2} \left[ F + \sum_{i=1}^{N} D_i e^{r(T-t_i)} \right]^2 \frac{\partial^2 c_F}{\partial F^2} - rc_F = 0. \]
5. A forward start option is an option which comes into existence at some future time $T_1$ and expires at $T_2$ ($T_2 > T_1$). The strike price is set equal the asset price at $T_1$ such that the option is at-the-money at the future option’s initiation time $T_1$. Consider a forward start call option whose underlying asset has value $S$ at current time $t$ and constant dividend yield $q$, show that the value of the forward start call is given by

$$e^{-qt_1}c(S, T_2 - T_1; S)$$

where $c(S, T_1 - T_1; S)$ is the value of an at-the-money call (strike price same as asset price) with time to expiry $T_2 - T_1$.

*Hint:* The value of an at-the-money call option is proportional to the asset price.

6. Consider a contingent claim whose value at maturity $T$ is given by

$$\min(S_{T_0}, S_T),$$

where $T_0$ is some intermediate time before maturity, $T_0 < T$, and $S_T$ and $S_{T_0}$ are the asset price at $T$ and $T_0$, respectively. Show that the value of the contingent claim at time $t$ is given by

$$V_t = S_t[1 - N(d_1) + e^{-r(T-T_0)}N(d_2)],$$

where $S_t$ is the asset price at time $t$ and

$$d_1 = \frac{r(T - T_0) + \sigma^2(T - T_0)}{\sigma\sqrt{T - T_0}}, \quad d_2 = d_1 - \sigma\sqrt{T - T_0}.$$

7. Consider the exchange option which entitles the holder the right but not the obligation to exchange risky asset $S_2$ for another risky asset $S_1$. Let the price dynamics of $S_1$ and $S_2$ under the risk neutral measure be governed by

$$dS_i = (r - q_i)dt + \sigma_i dZ_i, \quad i = 1, 2,$$

where $dZ_1 dZ_2 = \rho dt$. Let $V(S_1, S_2, \tau)$ denote the price function of the exchange option, whose terminal payoff takes the form

$$V(S_1, S_2, 0) = \max(S_1 - S_2, 0).$$

Show that the governing equation for $V(S_1, S_2, \tau)$ is given by

$$\frac{\partial V}{\partial \tau} = \frac{\sigma_2^2}{2} \frac{\partial^2 V}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{\sigma_1^2}{2} \frac{\partial^2 V}{\partial S_2^2} + (r - q_1)\frac{\partial V}{\partial S_1} + (r - q_2)\frac{\partial V}{\partial S_2} - rV.$$

By taking $S_2$ as the numeraire and defining the similarity variables

$$x = \frac{S_1}{S_2} \quad \text{and} \quad W(x, \tau) = \frac{V(S_1, S_2, \tau)}{S_2},$$

show that the governing equation for $W(x, \tau)$ becomes

$$\frac{\partial W}{\partial \tau} = \frac{\sigma_2^2}{2} x^2 \frac{\partial^2 W}{\partial x^2} + (q_1 - q_2)x \frac{\partial W}{\partial x}.$$
Verify that the solution to $W(x, \tau)$ is given by

$$W(x, \tau) = e^{-q_1 \tau} x N(d_1) - e^{-q_2 \tau} N(d_2)$$

where

$$d_1 = \frac{\ln \frac{S}{S_0} + (q_2 - q_1 + \frac{\sigma^2}{2}) \tau}{\sigma \sqrt{\tau}}, \quad d_2 = d_1 - \sigma \sqrt{\tau},$$

$$\sigma^2 = \sigma_1^2 - 2 \rho \sigma_1 \sigma_2 + \sigma_2^2.$$

8. Suppose the terminal payoff of an exchange rate option is $F_T \mathbf{1}_{\{F_T > X\}}$. Let $V_d(F, t)$ denote the value of the option in the domestic currency world, show that

$$V_d(F, t) = F e^{-r_f (T - t)} E_{Q_f} \left[ \mathbf{1}_{\{F_T > X\}} | F_t = F \right] = F e^{-r_f \tau} N(d)$$

where

$$d = \frac{\ln \frac{F}{K} + \left( r_d - r_f + \frac{\sigma^2}{2} \right) \tau}{\sigma_F \sqrt{\tau}}, \quad \tau = T - t.$$

9. Let $F_{S\backslash U}$ denote the Singaporean currency price of one unit of US currency and $F_{H\backslash S}$ denote the Hong Kong currency price of one unit of Singaporean currency. We may interpret $F_{S\backslash U}$ as the price process of a tradeable asset in Singaporean currency. Assume $F_{S\backslash U}$ to be governed by the following dynamics under the risk neutral measure $Q_S$ in Singaporean currency world:

$$\frac{dF_{S\backslash U}}{F_{S\backslash U}} = (r_{SGD} - r_{USD}) dt + \sigma_{F_{S\backslash U}} dZ_{F_{S\backslash U}},$$

where $r_{SGD}$ and $r_{USD}$ are the Singaporean and US riskless interest rates, respectively. The digital quanto option pays one Hong Kong dollar if $F_{S\backslash U}$ is above the strike level $X$. Find the value of the digital quanto option in terms of the riskless interest rates of different currency worlds and volatility values $\sigma_{F_{S\backslash U}}$ and $\sigma_{F_{H\backslash S}}$.

10. In the Merton model of risky debt, suppose we define

$$\sigma_V(\tau; d) = \frac{\sigma A \partial V}{V \partial A},$$

which gives the volatility of the value of the risky debt. Also, we denote the credit spread by $s(\tau; d)$, where $s(\tau; d) = Y(\tau) - r$. Show that

(a) $\frac{\partial s}{\partial d} = \frac{1}{\tau d} \sigma_V(\tau; d) > 0$;

(b) $\frac{\partial s}{\partial \sigma^2} = \frac{1}{2\sqrt{\tau} N(d_1)} N'(d_1) \sigma_V(\tau; d) > 0$, where $d_1 = \frac{\ln d}{\sigma \sqrt{\tau}} - \frac{\sigma \sqrt{\tau}}{2}$;

(c) $\frac{\partial s}{\partial r} = -\sigma_V(\tau; d) < 0$.

Give financial interpretation to each of the above results.
11. A firm is an entity consisting of its assets and let $A_t$ denote the market value of firm’s assets. Assume that the total asset value follows a stochastic process modeled by

$$\frac{dA_t}{A_t} = \mu \, dt + \sigma \, dZ_t,$$

where $\mu$ and $\sigma^2$ (assumed to be constant) are the instantaneous mean and variance, respectively, of the rate of return on $A_t$. Let $C$ and $D$ denote the market value of the current liabilities and market value of debt, respectively. Let $T$ be the maturity date of the debt with face value $D_T$. Suppose the current liabilities of amount $C_T$ are also payable at time $T$, and it constitutes a claim senior to the debt. Also, let $F$ denote the present value of total amount of interest and dividends paid over the term $T$. For simplicity, $F$ is assumed to be prepaid at time $t = 0$.

The debt is in default if $A_T$ is less than the total amount payable at maturity date $T$, that is,

$$A_T < D_T + C_T.$$

(a) Show that the probability of default is given by

$$p = N \left( \frac{\ln \frac{D_T + C_T}{A_T} - \mu T + \sigma^2 T}{\sigma \sqrt{T}} \right).$$

(b) Explain why the expected loan loss $L$ on the debt is given by

$$EL = \int_{C_T}^{D_T + C_T} (D_T + C_T - a) f(a) \, da + \int_0^{C_T} D_T f(a) \, da,$$

where $f$ is the density function of $A_T$. Give the financial interpretation to each integral.