Convexity meets replication: hedging of swap derivatives and annuity options

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Abstract

Convexity correction arises when one computes the expected value of an interest rate index under a probability measure other than its own natural martingale measure. As a typical example, the natural martingale measure of the swap rate is the swap measure with annuity as the numeraire. However, the evaluation of the discounted expectation of the payoff in a constant maturity swap (CMS) derivative is performed under the forward measure corresponding to the payment date. In this paper, we propose an extension of Carr-Madan's static replication approach by exploring the linkage between replication, convexity correction and numeraire change. We illustrate how the static replication of a CMS caplet by a portfolio of payer swaptions is related to convexity correction associated with the bond-annuity numeraire ratio. We also demonstrate the use of the extended static replication approach for hedging in-arrears clean index principal swaps and annuity options.

Keywords: Convexity adjustment, static replication, constant maturity swap, clean index principal swap, annuity option.

1 Introduction

A constant maturity swap (CMS) is a variant of the vanilla interest rate swap. One of the legs, known as the CMS leg, is indexed to a swap rate of fixed maturity (say, 10-year swap rate). This swap rate is reset at each of the preset fixing dates. The other leg can be either floating (say, LIBOR) or fixed. The floating rate of the CMS leg is called the CMS rate. The tenor and payment frequency of the CMS swap are allowed to be different from those that are associated with the reference swap rate. The class of CMS derivatives, like CMS caps and floors, are derivative products whose payoff structures have dependence on the CMS rates.

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When we consider pricing of the CMS derivatives, it is necessary to compute the expectation of the future CMS rates under the forward measure that is associated with the payment date. However, the natural martingale measure of the CMS rate is the underlying annuity. Convexity correction arises when one computes the expected value of the CMS rate under a forward measure that differs from the natural swap measure with annuity as the numeraire. The CMS convexity correction is then the difference between the expectation of the CMS rate under the forward measure and the forward swap rate. A good review of the mathematics of convexity correction can be found in Pelsser (2003b).

Carr and Madan (1998) propose the static replication formula which demonstrates how to replicate a European contingent claim with a twice differentiable payoff using a static replication strategy involving zero coupon bonds and vanilla call and put options. It has been well known among practitioners in the fixed income markets that a CMS caplet can be replicated by a portfolio of payer swaptions whose terminal payoff matches with that of the CMS caplet. The linkage between convexity correction and static replication of CMS derivatives has been explored in the literature (Hagan, 2003; Reiner and Selami, 2006; Mercurio and Pallavicini, 2006). Since most convexity adjustments can be identified as a result of the change of martingale measure, we would like to consider the extension of the Carr-Madan approach that incorporates measure change. Also, one may use tradeable option type products other than the vanilla options in the replicating portfolio. Our extended static replication approach can be easily applied to hedge and price other exotic fixed income derivatives, like in-arrears clean index principal swaps and annuity options.

This paper is organized as follows. In the next section, we illustrate the static replication of the CMS caplet using a portfolio of payer swaptions with varying discrete strike prices. By taking the continuous limit of the differential interval width between strike prices to be vanishingly small, we relate our continuous formulation of static replication with other known results in the literature. In Section 3, we derive the extension of the Carr-Madan static replication approach that takes measure change into the formulation and adopts the use of wider class of option type products in the replicating portfolio. We then illustrate how to apply the extended Carr-Madan static replication approach to pricing and hedging of the in-arrears clean index principal swaps and annuity options. In Section 4, we present sample calculations of finding the replication portfolios of CMS caplets and annuity options. Concluding remarks are presented in the last section.

2 Static replication of a CMS caplet

By observing the similarity between a CMS caplet and a payer swaption, we would like to demonstrate how to perform the static replication of a CMS caplet using a portfolio of payer swaptions with discrete strikes. Also, we illustrate how to relate the static replication procedure with convexity correction.

First, we fix the mathematical notations for the tenor structure, bond price function and swap rate. The spot time is taken to be time 0 and the tenor structure of the reference swap rate is assumed to be $\{T_0, T_1, \cdots, T_n\}$. Here, T_0

is the start date of the swap, $\{T_i,\ i = 0, 1, \cdots, n-1\}$ are the reset dates and $\{T_i,\ i = 1, \cdots, n\}$ are the payment dates. We let $\delta_i = T_i - T_{i-1}$ be the year fraction of the time interval $[T_{i-1}, T_i]$, and write δ for all δ_i if a constant year fraction is assumed. The price of the T_i-maturity discount bond is denoted by $B(t, T_i),\ i = 0, 1, \cdots, n$, where $t \in [0, T_0]$. The annuity stream with k payments is defined by $A_k(t) = \sum_{i=1}^{k} \delta_i B(t, T_i)$, for $k = 1, 2, \cdots, n$, and $A(t)$ is used to denote $A_n(t)$ for short. We let $S(t; T_0, T_i)$ denote the forward swap rate at time t, $t \in [0, T_0]$, with start date T_0 and payment dates $\{T_1, T_2, \cdots, T_i\},\ i = 1, 2, \cdots, n$. Provided that no confusion arises, we may use S_t to denote $S(t; T_0, T_n)$ for short. As usual, we use $E_{T_i}[\cdot]$ to denote the expectation under the forward measure $\mathbb{Q}_{T_i}$ where the T_i-maturity discount bond $B(t, T_i)$ is used as the numeraire, $i = 1, \cdots, n$. Also, $E_{\mathbb{Q}_N}[\cdot]$ denotes the expectation under the martingale measure $\mathbb{Q}_N$ associated with the numeraire $N(t)$.

The most basic CMS derivative is the CMS caplet whose payoff at the payment date T_p, where $T_p \geq T_0$, is given by

$$F(S_{T_0}, T_p; K) = (S_{T_0} - K)^+ \tag{2.1}$$

where notation $(\ \cdot\ )^+$ denotes $\max\{\cdot\ , 0\}$ and K is the strike of the caplet. Recall that a European swaption allows its holder to enter a fixed-floating swap of a preset tenor and pre-specified fixed rate X on maturity. A payer (receiver) swaption allows the holder to pay (receive) the fixed rate. Swaptions can be either physically settled or cash-settled. We would like to illustrate how to replicate the caplet payment on the payment date T_p using a portfolio of physically settled payer swaptions with discrete strikes $K + m\Delta x,\ m = 0, 1, 2, \cdots$, where Δx represents a small increment on the strike rate starting from K, then $K + \Delta x, K + 2\Delta x$, etc. Let N_m denote the notional amount of the payer swaption with strike $K + m\Delta x,\ m = 0, 1, 2, \cdots$. We illustrate how to determine $N_0, N_1, N_2, \cdots$ successively in order that the caplet payoff at T_p agrees with that of the replicating portfolio of payer swaptions under various scenarios of the observed swap rate at T_0, that is, the realized value of S_{T_0}.

When $S_{T_0} \leq K$, the caplet does not have positive payoff and all the payer swaptions in the replicating portfolio are not in-the-money. Thus, matching of the two payoffs is achieved. Next, we determine the notional amount of each of the payer swaptions successively by matching the payoffs of the caplet and replicating portfolio at various possible discrete values assumed by S_{T_0}.

First, suppose $S_{T_0} = K + \Delta x$, the caplet's payoff is Δx at T_p and the corresponding discounted value at T_0 is $\Delta x\, B(T_0, T_p)\big|_{S_{T_0} = K + \Delta x}$. On the other hand, only the payer swaption with strike rate K is in-the-money while all other swaptions with higher strike rate do not have positive payoff. The time-T_0 payoff of the payer swaption with strike rate K when $S_{T_0} = K + \Delta x$ is given in the form of an annuity $\Delta x \sum_{i=1}^{n} \delta_i B(T_0, T_i)\big|_{S_{T_0} = K + \Delta x}$. To achieve matching of the payoffs of the caplet and replicating portfolio when $S_{T_0} = K + \Delta x$, the notional amount N_0 must be set uniquely equal to the following bond-annuity ratio:

$$N_0 = \left. \frac{B(T_0, T_p)}{\sum_{i=1}^{n} \delta_i B(T_0, T_i)} \right|_{S_{T_0} = K + \Delta x}. \tag{2.2}$$

Naturally, the bond-annuity ratio exhibits dependence on the swap rate S_{T_0}. Similar to Hagan (2003), we write formally the functional dependence in the

form
$$G(S_{T_0}) = \frac{B(T_0, T_p)}{\sum_{i=1}^n \delta_i B(T_0, T_i)}. \tag{2.3}$$
Hence, the notional of the payer swaption with strike rate K can be expressed as
$$N_0 = G(K + \Delta x). \tag{2.4}$$
Next, we determine the notional amount N_1 of the payer swaption with strike $K + \Delta x$ by matching the payoffs when S_{T_0} assumes the value $K + 2\Delta x$. In such scenario, only the two swaptions with respective strike K and $K + \Delta x$ are in-the-money so that

$$(2N_0 + N_1)\Delta x \sum_{i=1}^n \delta_i B(T_0, T_i)\big|_{S_{T_0}=K+2\Delta x} = 2\Delta x B(T_0, T_p)\big|_{S_{T_0}=K+2\Delta x}$$

giving
$$N_1 = 2\big[G(K + 2\Delta x) - G(K + \Delta x)\big].$$
In general, by matching the payoffs of the caplet and the replicating portfolio when $S_{T_0} = K + (m + 1)\Delta x$, the notional amount N_m, $m \geq 1$, must be set uniquely equal to

$$\begin{aligned} N_m = {} & (m+1)\big[G(K + (m+1)\Delta x) - G(K + m\Delta x)\big] \\ & - (m-1)\big[G(K + m\Delta x) - G(K + (m-1)\Delta x)\big]. \end{aligned} \tag{2.5}$$

Let $C_0(K)$ denote the time-0 value of the payer swaption with strike rate K and V_0^{caplet} denote the time-0 value of the CMS caplet. Since the payoff of the replicating portfolio agrees with that of the caplet at discrete strikes, by applying the no-arbitrage priciple, the fair value of the CMS caplet is approximately given by the value of this portfolio of payer swaptions with discrete strikes:

$$V_0^{\text{caplet}} = N_0 C_0(K) + \sum_{m=1}^{\infty} N_m C_0(K + m\Delta x). \tag{2.6}$$

There will be a slight mismatch if S_{T_0} does not fall exactly on one of these discrete strike values. The order of approximation can be shown to be $O(\Delta x^2)$. In reality, the replicating portfolio may include payer swaptions with strike up to $K + M\Delta x$ for some sufficiently large positive integer value M. The value of the payer swaption with exceedingly large value of strike would be vanishingly small.

Continuous limit

Apparently, the replication using payer swaptions with discrete strikes corresponds to the discretization in the space of the strike price of the interval $[K, \infty)$ into discrete sub-intervals of uniform width Δx. It would be interesting to consider the continuous limit of the approximation formula (2.6) when $\Delta x \to 0$. Writing $x_m = K + m\Delta x$ for notational convenience, and assuming G to be twice differentiable, we take $\Delta x \to 0$ and obtain

$$N_m = 2G'(x_m) + G''(x_m)(x_m - K)\Delta x + O(\Delta x^2).$$

The price formula of the CMS caplet in the continuous limit $\Delta x \to 0$ then becomes

$$V_0^{\text{caplet}} = G(K)C_0(K) + \int_K^\infty \big[2G'(x) + G''(x)(x-K)\big]C_0(x)\, dx. \tag{2.7}$$

The above caplet price formula is essentially the same as that derived independently by Hagan (2003) and Mercurio and Pallavicini (2006) using alternative approaches. Indeed, Hagan obtains the caplet price formula in the form:

$$\begin{aligned} V_0^{\text{caplet}} &= \frac{B(0,T_p)}{A(0)}C_0(K) + \left[G(K) - \frac{B(0,T_p)}{A(0)}\right]C_0(K) \\ &\quad + \int_K^\infty \big[2G'(x) + G''(x)(x-K)\big]C_0(x)\, dx. \end{aligned} \tag{2.8}$$

The sum of the last two terms is considered by Hagan to be the convexity correction for the CMS caplet. Thus, we observe the analogy between replication and convexity correction. In particular, when the caplet is at-the-money so that the strike observes the following relation:

$$G(K) = G(S_0) = \frac{B(0,T_p)}{A(0)},$$

then the convexity correction is merely given by the integral term in Eq. (2.8).

3 Extended Carr-Madan static replication formula

Let X_t be a stochastic underdying asset price process and $f(X_T)$ denote the time-T payoff of a European contingent claim, where f is a function that is at least twice differentiable. Carr and Madan (1998) show that

$$\begin{aligned} f(X_T) &= f(\kappa) + f'(\kappa)\big[(X_T-\kappa)^+ - (\kappa - X_T)^+\big] \\ &\quad + \int_0^\kappa f''(x)(x-X_T)^+ dx + \int_\kappa^\infty f''(x)(X_T-x)^+ dx, \end{aligned} \tag{3.1}$$

where $\kappa \geq 0$ is an arbitrary real number. The above static replication formula reveals the static replication of a European contingent claim using zero-coupon bonds and a portfolio of vanilla call and put options. We have observed the static replication of the CMS caplet using a portfolio of tradable payer swaptions, where these replicating instruments do not resemble the vanilla call or put options on the swap rate. Therefore, the Carr-Madan formula in the above form cannot be used directly to deduce the corresponding replication procedure of the CMS caplet. Recall that the terminal payoff of the payer swaption becomes that of the vanilla call option when its value is normalized by the annuity numeraire. In this section, we would like to derive an extended version of the Carr-Madan formula that allows the use of tradable instruments with more generalized form of options type payoffs in the static replication procedure.

Consider a European contingent claim Y_T at time T whose value relative to the numeraire $M(T)$ has the form $f(X_T)$, where f is twice differentiable, we would like to replicate the contingent claim using tradable option type instruments. The terminal payoffs of these instruments at maturity T satisfy the following condition: Their relative value normalized by the numeraire $N(T)$ has the form either $g(X_T)(X_T - x)^+$ or $g(X_T)(x - X_T)^+$ for some twice differentiable function g. Let $C_0(x)$ and $P_0(x)$ denote the time-0 values of these replicating instruments. Let $\mathbb{Q}_M$ and $\mathbb{Q}_N$ denote the relevant martingale measures associated with the numeraire $M(t)$ and $N(t)$, respectively. Write $\Lambda_t = \frac{M(t)}{N(t)}$ as the numeraire ratio with respect to the martingale measures $\mathbb{Q}_M$ and $\mathbb{Q}_N$. The extended version of the Carr-Madan static replication formula that incorporates the relevant numeraires is stated in the following proposition.

Proposition 1. *Assume that Λ_T can be expressed as a function of X_T, i.e., $\Lambda_T = \Lambda(X_T)$. The functions Λ and f are assumed to be twice differentiable. Suppose there exists $\kappa \geq 0$ such that $f(\kappa) = 0$ and $g'(\kappa) \neq 0$. Write $h(x) = \frac{f(x)}{g(x)}$ and $w(x) = \frac{d^2}{dx^2}\left[h(x)\Lambda(x)\right]$ and let $V_0(Y_T)$ denote the time-0 value of the contingent claim, then we have the following static replication formula:*

$$\begin{aligned} V_0(Y_T) &= h'(\kappa)\Lambda(\kappa)[C_0(\kappa) - P_0(\kappa)] \\ &\quad + \int_0^\kappa w(x)P_0(x)\,dx + \int_\kappa^\infty w(x)C_0(x)\,dx. \end{aligned} \tag{3.2}$$

The proof of the proposition is presented in Appendix A.

The above extended version of the static replication formula provides more flexibility with respect to the adaption of option type instruments in the replication and the choices of numeraires that are more convenient for effective valuation of a contingent claim. The challenge is to find the functional dependence of the numeraire ratio Λ_T in terms of the underlying asset price process X_T. Next, we would like to illustrate how to apply the extended Carr-Madan replcation formula to perform the static replication of the CMS caplets, in-arrears clean index principal swaps and annuity options.

CMS caplets

Using the extended Carr-Madan formula (3.2), it is relatively straightforward to reproduce the replication formula (2.8) which reveals the static replication of the CMS caplet using a portfolio of payer swaptions. When normalized by the annuity numeraire $A(T_0)$, the payoff of the physically settled payer swaption is $(S_{T_0} - x)^+$, giving $g(S_{T_0}) \equiv 1$. The payoff of the caplet when normalized by $B(T_0, T_p)$ is $f(S_{T_0}) = (S_{T_0} - K)^+$. Now, the numeraire ratio $\Lambda(T_0)$ is simply $\frac{B(T_0,T_p)}{A(T_0)}$, which has been defined earlier to be $G(S_{T_0})$. Here, $h = f$ and the derivatives of h are

$$h'(x) = \mathbf{1}_{\{x \geq K\}} \text{ and } h''(x) = \delta(x - K)$$

Substituting the above results into formula (3.2), the resulting replication of the CMS caplet using physically settled payer swaptions agrees with that given in Eq. (2.8).

Suppose the terminal payoff of the payer swaption is cash-settled at T_0 and the forward swap rate is used as the discount rate to calculate the annuity, then the corresponding annuity term becomes

$$A(T_0) = \sum_{i=1}^{n} \frac{\delta}{(1+\delta S_{T_0})^i},$$

where a constant year fraction δ is assumed. In this case, we take $N(t)$ to be the bond price $B(t, T_0)$ so that the time-T_0 payoff of the payer swaption is $g(S_{T_0})(S_{T_0} - x)^+$, where $g(S_{T_0}) = A(T_0)$. The corresponding numeraire ratio is

$$\Lambda_T = \Lambda(S_{T_0}) = \frac{B(T_0, T_p)}{B(T_0, T_0)}$$

and

$$h(x) = \frac{(x-K)^+}{g(x)} = \frac{(x-K)^+}{\sum_{i=1}^{n} \frac{\delta}{(1+\delta x)^i}}.$$

By taking $\kappa = 0$ in the extended Carr-Madan formula, the replication of the CMS caplet by a portfolio of cash-settled payer swaptions can be expressed as

$$V_0^{\text{caplet}} = h'(K)\Lambda(K)C_0(K) + \int_K^{\infty} w(x)C_0(x)\,dx. \tag{3.3}$$

When we take $T_p = T_0$, then $\Lambda(x) \equiv 1$. In this case, the replication formula agrees with that given by Reiner and Sellami (2006).

In-arrears clean index principal swaps

The in-arrears clean index principal swap (IPS) is a variation of the standard IPS embedded with two additional features: (i) the LIBORs are reset in arrears; (ii) the notional principal P is reset according to the LIBOR prevailing at the payment date. To be specific, the time-T_i net value of the swap transaction payments to the floating rate receiver is given by $\delta_i P\big(L_i(T_i)\big)\big[L_i(T_i) - K\big]$. Here, the notional principal $P\big(L_i(T_i)\big)$ has dependence on the in-arrears LIBOR L_i observed at the prevailing time T_i. The appropriate replicating instruments would be vanilla caplets and floorlets whose terminal payoffs at time T_{i+1} are

$$\delta_{i+1}\big(L_i(T_i) - x\big)^+ \text{ and } \delta_{i+1}\big(x - L_i(T_i)\big)^+,$$

respectively. Now, the numeraire ratio should be

$$\Lambda\big(L_i(T_i)\big) = \frac{B(T_i, T_i)}{B(T_i, T_{i+1})} = 1 + \delta_{i+1}L_i(T_i);$$

and $g\big(L_i(T_i)\big) = \delta_{i+1}$. Also, κ should be conveniently chosen to be K. Substituting all of the above quantities into the extended Carr-Madan formula, we obtain the static replication of this IPS by a portfolio of caplets and floorlets.

As an illustrative example, suppose we choose the notional $P\big(L_i(T_i)\big)$ to be $\mathbf{1}_{\{x \geq K\}} + [b(x-K)+1]^+\mathbf{1}_{\{x<K\}}$, where $b > 0$. The notional, which is bounded below by zero, has its value decrease linearly when LIBOR falls below the strike rate K. This feature provides the floating rate receiver the protection of mitigating the loss by reducing the principal of the swap. By the extended Carr-Madan formula, the time-0 net value of the swap payment transacted at T_i is given by

$$V_0^i = \frac{\delta_i}{\delta_{i+1}} \left\{ (1+\delta_{i+1}K)[C_0(K) - P_0(K)] - [1+\delta_{i+1}\big(K - \frac{1}{b}\big)]P_0\big(K - \frac{1}{b}\big)\mathbf{1}_{\{K \geq \frac{1}{b}\}} \right\} + \int_{\left(K-\frac{1}{b}\right)^+}^{K} 2b\frac{\delta_i}{\delta_{i+1}}(1+\delta_{i+1}x)P_0(x)\,dx + \int_0^K 2\delta_i P_0(x)\,dx + \int_K^\infty 2\delta_i C_0(x)\,dx. \tag{3.4}$$

Here, $C_0(x)$ and $P_0(x)$ denote the time-0 value of the vanilla caplet and floorlet with respective terminal payoff $\delta_{i+1}\big[L_i(T_i) - x\big]^+$ and $\delta_{i+1}\big[x - L_i(T_i)\big]^+$ at T_{i+1}. If we let $b \to 0$, then the notional principal tends to the constant unit value. The in-arrears IPS reduces to an in-arrears swap, and formula (3.4) becomes

$$V_0^i = \frac{\delta_i}{\delta_{i+1}}(1+\delta_{i+1}K)[C_0(K) - P_0(K)] + \int_0^K 2\delta_i P_0(x)\,dx + \int_K^\infty 2\delta_i C_0(x)\,dx; \tag{3.5a}$$

which is the static replication formula for the in-arrears swaplet. On the other hand, if we let $b \to \infty$, then the payoff equals $\delta_i[L_i(T_i) - K]^+$. The in-arrears IPS swaplet becomes a caplet on the in-arrears LIBOR. In this case, the static replication formula (3.4) reduces to

$$V_0^i = \frac{\delta_i}{\delta_{i+1}}(1+\delta_{i+1}K)C_0(K) + \int_K^\infty 2\delta_i C_0(x)\,dx; \tag{3.5b}$$

which is the usual convexity correction formula for the T_i-maturity caplet on in-arrears LIBOR.

Annuity options

A forward start annuity pays the amount c_i at the future time $T_i,\ i = 0, 1, \cdots, n$. An option on this annuity gives the holder at the expiration date T_0 the right but the obligation to enter into this annuity at a fixed strike A. The terminal payoff at time T_0 of the annuity option is given by

$$V_{T_0}^{\text{a.o.}} = \max\left\{ \sum_{i=0}^n c_i B(T_0, T_i) - A,\ 0 \right\}$$

Pelsser (2003a) has proposed a procedure of performing static replication of the annuity option using a portfolio of receiver swaptions with different swap tenors. We would like to illustrate how to derive an alternative static replication portfolio via the extended Carr-Madan formula (3.2).

It is convenient to rewrite the terminal payoff $V_{T_0}^{\text{a.o.}}$ into the following form:

$$A_n(T_0)\left[\sum_{i=1}^n \frac{c_i B(T_0, T_i)}{A_n(T_0)} - \frac{A - c_0}{A_n(T_0)} \right]^+.$$

Similar to the assumption made by Pelsser (2003a), all interest rates are perfectly correlated so that we may assume

$$\frac{B(T_0, T_i)}{A_n(T_0)} = f_i(S_{T_0}),\quad i = 0, 1, \cdots, n.$$

That is, the bond prices $B(T_0, T_i)$, $i = 0, 1, \cdots, n$, move in perfect lock step. With the use of the numeraire $M(t) = A_n(t)$, the normalized terminal payoff of the annuity option can be expressed as

$$f(S_{T_0}) = \frac{V_{T_0}^{\text{a.o.}}}{M(T_0)} = \left[\sum_{i=1}^{n} c_i f_i(S_{T_0}) - (A - c_0) f_0(S_{T_0}) \right]^+ .$$

Note that $A - c_0$ must be chosen to be positive, otherwise the terminal payoff of the annuity option becomes strictly positive. When the physically settled swaptions are adopted as the replicating instruments, the corresponding numeraire $N(t)$ for the swaptions is the annuity $A_n(t)$. The numeraire ratio $\Lambda(t)$ is then given by

$$\Lambda(t) = \frac{M(t)}{N(t)} = 1,$$

and $g(S_{T_0}) = 1$. The extended Carr-Madan formula gives the following static replication procedure:

$$V_0^{\text{a.o.}} = f'(\kappa)[C_0(\kappa) - P_0(\kappa)] + \int_0^{\kappa} f''(x) P_0(x)\, dx + \int_{\kappa}^{\infty} f''(x) C_0(x)\, dx \tag{3.6}$$

where κ has been chosen such that $f(\kappa) = 0$, $C_0(x)$ and $P_0(x)$ denote the time-0 value of unit notional payer swaption and receiver swaption with strike x, respectively.

As an illustrative example, suppose we take the annual annuity rate μ to be constant, that is, $\mu = \frac{c_i}{\delta_i}$, $i = 1, \cdots, n$. In this case, the terminal payoff $V_{T_0}^{\text{a.o.}}$ can be simplified as

$$A_n(T_0) \left[\mu - \frac{A - c_0}{A_n(T_0)} \right]^+ .$$

Take $T_p = T_0$ in the function $G(S_{T_0})$ defined earlier, now we have

$$G(S_{T_0}) = \frac{B(T_0, T_0)}{A_n(T_0)}.$$

We may write $f(x) = [\widetilde{f}(x)]^+$, where

$$\widetilde{f}(x) = \mu - (A - c_0) G(x).$$

The derivatives of f are found to be

$$\begin{aligned} f'(x) &= \widetilde{f}'(x) \mathbf{1}_{\{\widetilde{f}(x) \geq 0\}}, \\ f''(x) &= \delta\big(\widetilde{f}(x)\big) \widetilde{f}'(x)^2 + \mathbf{1}_{\{\widetilde{f}(x) \geq 0\}} \widetilde{f}''(x), \end{aligned}$$

where $\delta(\cdot)$ is the Dirac delta function. It is observed that $G(x) \to \infty$ monotonically as $x \to \infty$, and $G(x)$ grows asymptotically like a linear function. Since $A - c_0$ and μ are positive, thus $\widetilde{f}(x)$ is monotonically decreasing with $\widetilde{f}(0) > 0$ and $\widetilde{f}(\infty) = -\infty$. Therefore, there exists $K > 0$ such that $\widetilde{f}(x) \geq 0$ if and only if $x \leq K$. Consequently, we may choose κ to be infinite value so that the static replication involves receiver swaptions only. Observing that $f'(\kappa) = 0$, we obtain the following static replication of the annuity option using a portfolio of receiver swaptions:

$$\begin{aligned} V_0^{\text{a.o.}} &= \int_0^\infty f''(x)P_0(x)\,dx \\ &= (A-c_0)G'(K)P_0(K) - \int_0^K (A-c_0)G''(x)P_0(x)\,dx. \end{aligned} \tag{3.7}$$

4 Numerical results

We performed the numerical calculations to find the corresponding replicating portfolio for the CMS caplet and the annuity option based on formulas (2.6) and (3.7). First of all, it is necessary to establish the functional depedence of the bond-annuity ratio on S_{T_0} [see Eq. (2.3)] based on certain assumptions on the yield curve. Here, we consider two common types of yield curve models: flat yield curve model and parallel shifts model. These two models are known to work well when the pricing problems are more concerned with the long-maturity segment of the yield curve. In the present context, the CMS rates are usually medium-term to long-term rates, and the underlying annuity is usually long-term; so the adoption of these two yield curve models is justified.

In the *flat yield curve model*, we assume the yield curve at time T_0 to be flat. Let $\frac{1}{q}$ denote the constant year fraction ($q = 1$ for annual, $q = 2$ for semi-annual). Under the assumptions of flat yield curve and constant year fraction, the discount bond prices can be expressed in terms of S_{T_0} as follows:

$$B(T_0, T_i) = \frac{1}{\left(1 + \frac{S_{T_0}}{q}\right)^i}, \; i = 1, 2, \cdots, n.$$

Also, we assume

$$B(T_0, T_p) = \frac{1}{\left(1 + \frac{S_{T_0}}{q}\right)^{\delta_p}}, \; \text{where } \delta_p = \frac{T_p - T_0}{T_1 - T_0};$$

so that

$$G(S_{T_0}) = \frac{S_{T_0}}{(1 + \frac{S_{T_0}}{q})^{\delta_p}\left[1 - \frac{1}{(1+\frac{S_{T_0}}{q})^n}\right]}. \tag{4.1}$$

On the other hand, when we assume a parallel yield curve shift of a fixed amount x in the *parallel shifts model*, the bond prices at time 0 and time t are related by

$$\frac{B(t, T_i)}{B(t, T_0)} = \frac{B(0, T_i)}{B(0, T_0)} e^{-(T_i - T_0)x}, \; i = 1, 2, \cdots, n.$$

The parallel shift amount x is implicitly determined by the following equation:

$$S_{T_0} = \frac{B(0, T_0) - B(0, T_n)e^{-(T_n - T_0)x}}{\sum_{k=1}^n \delta_k B(0, T_k)e^{-(T_k - T_0)x}}.$$

Under the parallel shifts model, we have

$$G(S_{T_0}) = \frac{S_{T_0} e^{-(T_p - T_0)x}}{1 - \frac{B(0,T_n)}{B(0,T_0)} e^{-(T_n - T_0)x}}, \tag{4.2}$$

where x has an implicit dependence on S_{T_0}.

In Figures 1(a) and 1(b), we plot the dependence of $G(S_{T_0})$ on the swap rate S_{T_0} based on the flat yield curve model and parallel shifts model, respectively, for various values of n (the number of payment dates in the tenor structure $\{T_0, T_1, \cdots, T_n\}$). We assume $T_p = T_1$, and $q = 2$ in our calculations. For $T_p = T_0$, $G(S_{T_0})$ possesses essentially the same local shape. The values of the discount factors in the initial discount curve are tabulated in Table 1. The "almost linear" property of G is observed in the figures even when S_{T_0} assumes a conceivably large value, thus verifying the appropriate use of the linear approximation assumption in the Linear Swap Rate Model (Pelsser, 2003b).

Discount Factors from T_0 to T_6

T_0	T_1	T_2	T_3	T_4	T_5	T_6
0.9537	0.9330	0.9139	0.8953	0.8568	0.8383	0.8204

Discount Factors from T_7 to T_{13}

T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}
0.8032	0.7863	0.7695	0.7527	0.7361	0.7196	0.7031

Discount Factors from T_{14} to T_{20}

T_{14}	T_{15}	T_{16}	T_{17}	T_{18}	T_{19}	T_{20}
0.6868	0.6707	0.6546	0.6387	0.6231	0.6080	0.5932

Table 1: Discount factors in the initial discount curve.

CMS caplets

Based on the replication formula (2.6), we find the replicating portfolios of a set of at-the-money CMS caplets using payer swaptions of varying discrete strikes. The respective strike rate K corresponding to different tenor structures (signified by the number of payment dates n) are presented in Table 2. The subinterval of the strike price Δx is set to be 1%. The upper bound on the strikes of the these swaptions is set to be 20%. The notional of each of these swaptions can be computed using Eq. (2.5).

n	8	10	20
K	4.89%	4.80%	4.81%

Table 2: Values of strike rate under various tenor structures.

In Figures 2(a) and 2(b), we show the plot of the notional values of the payer swaptions with varying strikes corresponding to different tenors, where $n = 8, 10, 20$, based on the flat yield curve model and parallel shifts model, respectively. The swaption with strike K (the same strike of the CMS caplet) is seen to be the dominant one (in terms of notional and actual dollar value) in the replicating portfolio. The convexity correction is provided by the other payer swaptions at higher strikes. As revealed in Figures 2(a) and 2(b), the notional amounts of these swaptions are relatively small compared to that of the dominant swaption at strike K. Since $G(S_{T_0})$ is almost linear in S_{T_0}, the notional values of the swaptions relevant for the convexity correction are almost equal in value. When the tenor of the underlying CMS rate is lengthened, corresponding to an increase in value of n, the convexity correction becomes more significant. Thus the difference between the notional of the dominant swaption at strike K and that of any swaption at a higher strike decreases. This phenomenon is confirmed by the plots shown in Figures 2(a) and 2(b).

Annuity options

We apply the replication formula (3.7) to find the notional values of the receiver swaptions in the replication of an annuity option. We assume a uniform annuity rate of $1 in our calculations. The strike price A of the annuity option is set equal to the forward price of the annuity such that the annuity option is initiated at-the-money.

In a similar manner, we plot the notional values of the receiver swaptions in the replicating portfolio corresponding to different tenors in Figures 3(a) and 3(b). Since $G(x)$ is "almost" linear in both the flat yield curve model and parallel shifts model and so $G''(x) \approx 0$, the notional values of all receiver swaptions other than the dominant swaption are very small. If we neglect the "small" convexity of the function $G(x)$, the annuity option is almost like a receiver swaption with notional $(A - c_0)G'(K)$ and strike K, where K is the unique root of $\widetilde{f}(x) = \mu - (A - c_0)G(x)$. The values of K under various tenor structures are summarized in Table 3.

No. of payment dates, n	8	10	20
Flat Yield Model	4.89%	4.80%	4.81%
Parallel Shifts Model	4.89%	4.80%	4.81%

Table 3: Values of K under various tenor structures and yield curve models.

With the presence of a small convexity in $G(x)$, one has to short hold small amounts of receiver swaptions at strikes lower than K in the replicating portfolio [see Figures 3(a) and 3(b)].

5 Conclusion

We have proposed an extension of the Carr-Madan static replication formula by exploring the linkage between replication, convexity correction and numeraire change. The extended version of the replication formula allows the adoption of wider choices of option type products in the replicating portfolio. We demonstrate the use of the extended replication approach for hedging exotic swap products and annuity options using various types of swaptions. Interestingly, the convexity of the functional dependence of bond-annuity ratio on swap rate enters into the replication formulas of these exotic interest rate instruments. The degree of the convexity determines the notional amounts of the swaptions in the replicating portfolio, thus illustrating the interplay between convexity and replication.

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Appendix A Proof of Proposition 1

First, we recall the martingale pricing formulas:

$$\begin{aligned} V_0(Y_T) &= M(0)E_{\mathbb{Q}_M}\left[\frac{Y_T}{M(T)}\right] = N(0)E_{\mathbb{Q}_N}\left[f(X_T)\frac{M(T)}{N(T)}\right] \\ &= N(0)E_{\mathbb{Q}_N}\left[f(X_T)\Lambda(X_T)\right]. \end{aligned}$$

Suppose we choose $\kappa \geq 0$ such that $f(\kappa) = 0$ but $g(\kappa) \neq 0$. Since $h(x) = \frac{f(x)}{g(x)}$, so $h(\kappa) = 0$. We apply the Carr-Madan formula (3.1) to the payoff function $h(X_T)\Lambda(X_T)$ and obtain

$$\begin{aligned} h(X_T)\Lambda(X_T) &= h'(\kappa)\Lambda(\kappa)[(X_T-\kappa)^+ - (\kappa - X_T)^+] \\ &\quad + \int_0^\kappa w(x)(x-X_T)^+ dx + \int_\kappa^\infty w(x)(X_T - x)^+ dx, \end{aligned}$$

where $w(x) = h''(x)\Lambda(x) + 2h'(x)\Lambda'(x) + h(x)\Lambda''(x)$. By multiplying both sides of the above equation by $g(X_T)$ and taking expectation under the martingale measure $\mathbb{Q}_N$, we obtain

$$\begin{aligned} E_{\mathbb{Q}_N}[f(X_T)\Lambda(X_T)] &= h'(\kappa)\Lambda(\kappa)E_{\mathbb{Q}_N}[g(X_T)((X_T-\kappa)^+ - (x - X_T)^+)] \\ &\quad + \int_0^\kappa w(x)E_{\mathbb{Q}_N}[g(X_T)(x-X_T)]^+ dx \\ &\quad + \int_\kappa^\infty w(x)E_{\mathbb{Q}_N}[g(X_T)(X_T-x)]^+ dx. \end{aligned}$$

The values of the call-type and put-type replicating instruments are given by

$$C_0(x) = N(0)E_{\mathbb{Q}_N}[g(X_T)(X_T-x)^+]$$

and

$$P_0(x) = N(0)E_{\mathbb{Q}_N}[g(X_T)(x-X_T)^+];$$

so that we finally obtain

$$\begin{aligned} V_0(Y_T) &= h'(\kappa)\Lambda(\kappa)[C_0(\kappa) - P_0(\kappa)] \\ &\quad + \int_0^\kappa w(x)P_0(x)\,dx + \int_\kappa^\infty w(x)C_0(x)\,dx. \end{aligned}$$

Appendix B Relevant figures

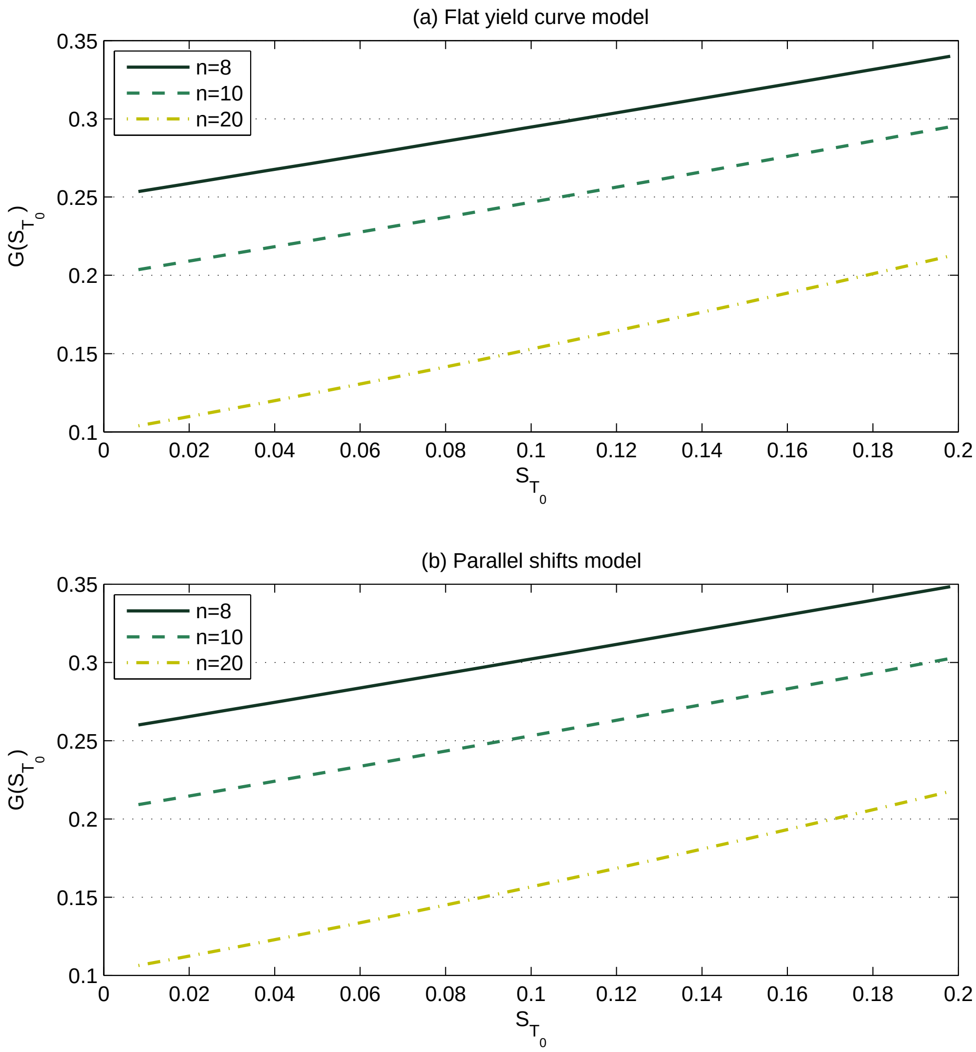

Figure 1: Plot of $G(S_{T_0})$ against S_{T_0} under the assumption of (a) flat yield curve model, (b) parallel shifts model. The "almost linear" property of G is observed.

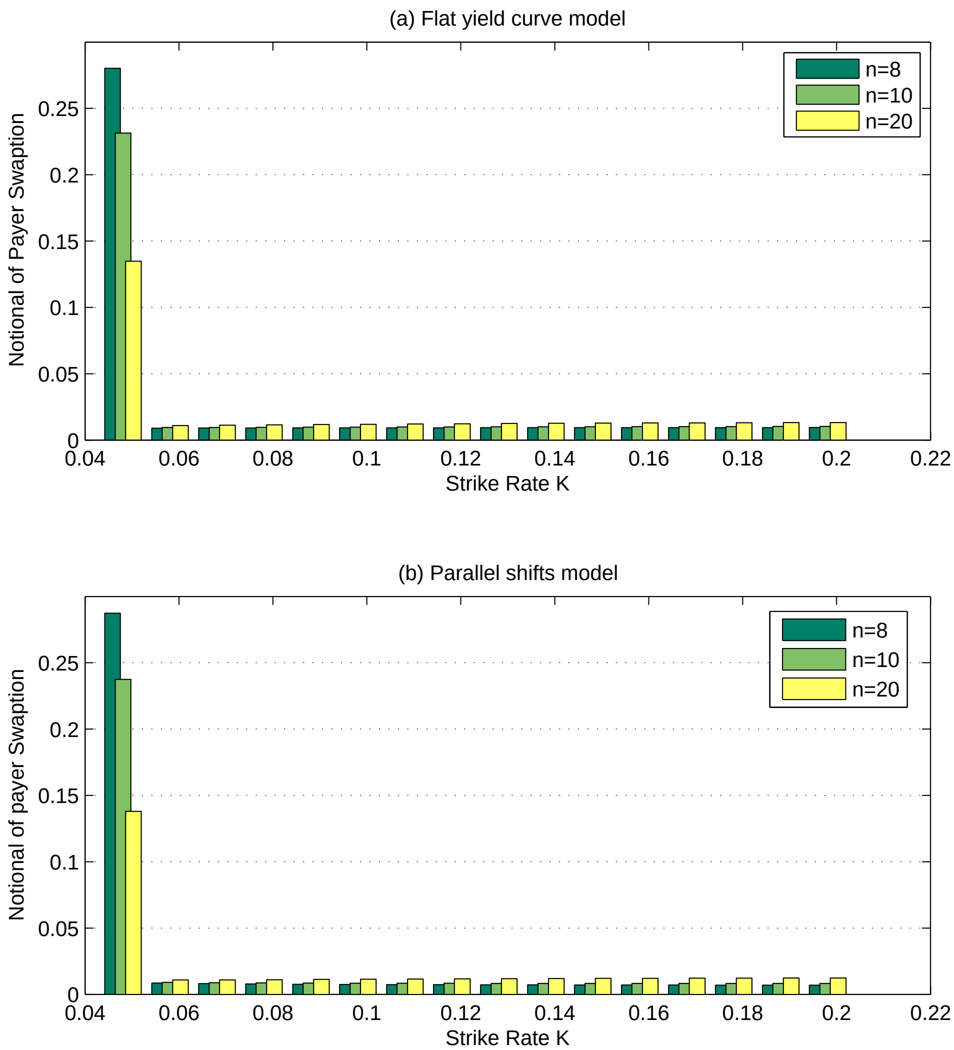

Figure 2: Plot of the notional of the payer swaption against strike rate under the assumption of (a) flat yield curve model, (b) parallel shifts model, and corresponding to different tenor structure (signified by the number of payment dates n in the tenor). The payer swaption with the same strike as that of the CMS caplet is the dominant one.

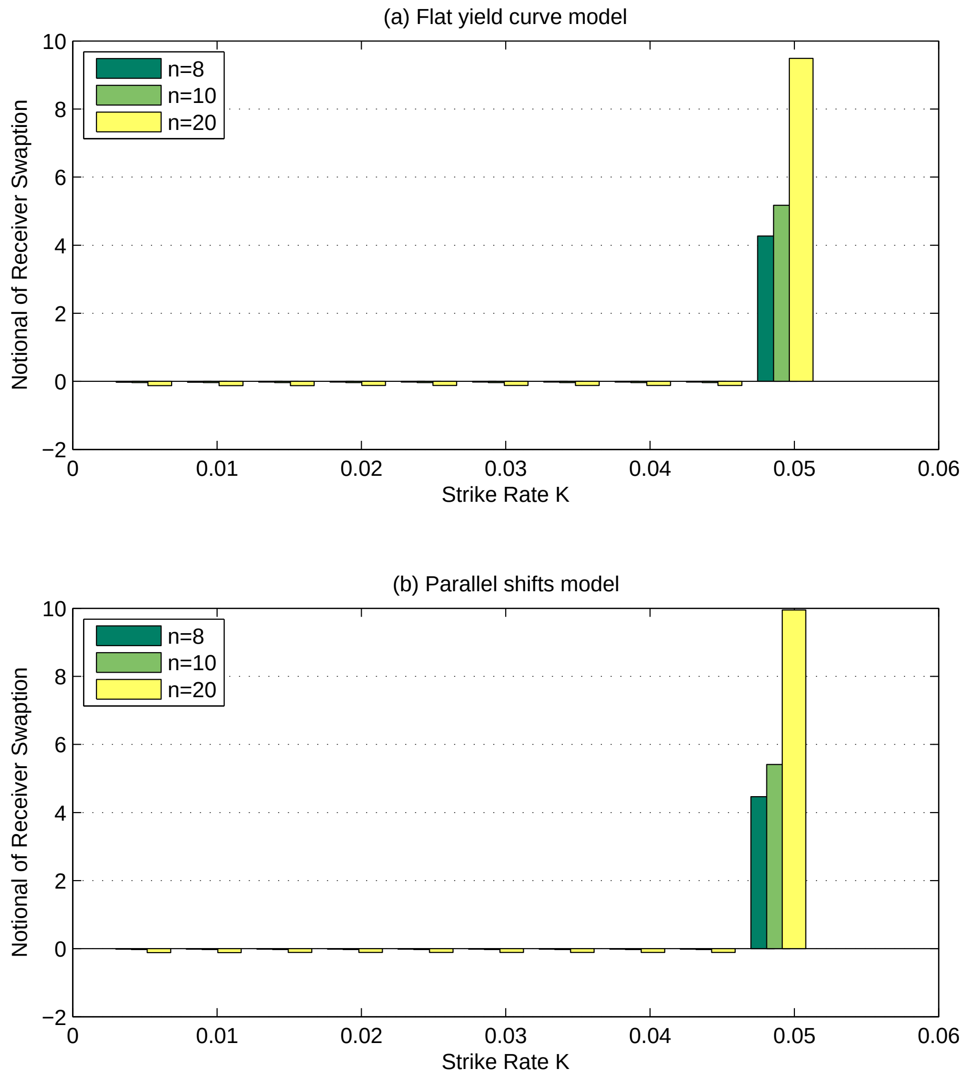

Figure 3: Plot of the notional of the receiver swaption against strike rate under the assumption of (a) flat yield curve model, (b) parallel shifts model, and with varying tenor structure (signified by the number of payment dates n in the tenor). The notional amounts of the receiver swaptions relevant for convexity correction are seen to be very small due to the "almost linear" property of G.