Pricing participating policies with rate guarantees and bonuses

Chi Chiu Chu and Yue Kuen Kwok[†]

Department of Mathematics,
Hong Kong University of Science and Technology,
Clear Water Bay, Hong Kong, China

Abstract

We construct the contingent claims models that price participating policies with rate guarantees, bonuses and default risk. These policies are characterized by the sharing of profits from an investment portfolio between the insurer and the policy holders. A certain surplus distribution mechanism (reversionary bonus) is employed to credit interest at or above certain specified guaranteed rate periodically to the policy holders. Besides the reversionary bonus, terminal bonus is also paid to the policy holder if the terminal surplus is positive. However, the insurer may default at maturity and the policy holders can only receive the residual assets. By neglecting market frictions, mortality risk and surrender option, and under certain assumptions on the bonus distribution mechanism, we are able to find analytic approximation solution to the pricing model. We also develop effective finite difference algorithms for the numerical solution of the contingent claims models. Pricing behaviors of these participating policies with respect to various parameters in the pricing models are examined.

1. Introduction

Participating life insurance policies have been very popular in the past decades since they provide a low risk but yet competitive return compared to other equity-linked products. In these policies, the insurer's profits from an investment portfolio are shared with the policy

Keywords: Participating policies; Bonus distribution; Contingent claims valuation; Perturbation techniques

JEL classification: G13; G22; G23

[†] Corresponding author. Tel: 852-2358-7418; fax: 852-2358-1643. Email address: maykwok@ust.hk

holders. At the initiation of the contract, the policy holder pays a single lump sum deposit to the insurance company. The insurer then manages the trusted funds by investing in a well diversified and specified reference portfolio. A certain bonus distribution mechanism is employed to credit interest on the policy's account balance, the amount of which is linked to the annual market return of the investment portfolio. The credit interest rate is usually ensured not to fall below some specified guaranteed level. The net difference between the market value of the asset portfolio and the book value of the policy holder's account is called the bonus reserve or buffer. The bonus reserve is used to provide stable and smooth returns to policy holders in the future and protect against solvency. Besides the interest rate guarantee and bonus distribution mechanism, these participating policies also contain other embedded features. For example, the maturity guarantee promises to pay the holder some guaranteed amount at maturity, and the surrender option entitles the holder the right to terminate the contract prior to maturity. More detailed discussion on the product nature of these policies can be found in Grosen and Jørgensen's paper (2000) and Ballotta *et al.*'s paper (2003). Also, Consiglio *et al.* (2001) discuss asset and liability modeling for participating policies with guarantees.

Accurate pricing of the fair value of the participating policy using the contingent claims approach requires the careful modeling of all of the embedded features in such policy. In recent years, the accurate valuation of these guarantees has received much attention since the guarantees become quite valuable due to falling equity returns and interest rate. The rule of the bonus distribution mechanism obviously plays a crucial role in the determination of the value of the participating policy. Wilkie (1987) pioneers the use of modern option pricing approach to analyze the embedded options in with-profits policies. Grosen and Jørgensen (2000) analyze the minimum rate guarantee and bonus distribution mechanism and model the surrender risk as American early exercise feature in their contingent claims model. They examine the pricing behaviors of the policy value with different levels of interest rates, bonus policy parameters and volatility of asset portfolio value. Prieul *et al.* (2001) construct the continuous contingent claims model that include the path dependence associated with the bonus distribution mechanism. They apply appropriate similarity transformation of variables to reduce the dimension of the pricing model, and perturbation techniques are then used to obtain an asymptotic solution of the policy value. They also construct numerical scheme to solve for the optimal surrender policy. In a series of papers, Bacinello (2001, 2003a,b) constructs pricing models of participating policies with different types of embedded features and devises binomial schemes for the numerical solution of the models. Jensen *et al.* (2001) construct an implicit finite difference scheme for numerical valuation of participating life insurance policies with interest rate guarantee, bonus and surrender options. Willder (2003) apply option pricing techniques to analyze the effects of different bonus strategies in unitized with-profits policies. To model insolvency and default risks, Grosen and Jørgensen (2002) construct the contingent claims model that takes into account the seniority of claims on the company's asset at maturity by various parties in the policies. Also, they study the impact of regulatory intervention rules for reducing

the insolvency risk of the policies using a barrier option framework. Ballotta *et al.* (2003) propose valuation techniques for participating policies that incorporate reversionary bonus, terminal bonus and default option.

In this paper, we construct continuous contingent claims model for pricing participating policies with interest rate guarantee, reversionary bonus, terminal bonus distribution and default option at maturity. We neglect market frictions, mortality risk and surrender option in our model. For certain type of bonus distribution mechanism, we manage to obtain analytic approximation solution of the pricing model. Since the coefficients in the governing equation are state dependent, the numerical solution by the usual binomial scheme may suffer numerical instabilities and oscillations. We construct an implicit finite difference scheme that provides more effective numerical valuation of the pricing model. The paper is organized as follows. In the next section, we construct the continuous contingent claims pricing model for participating policies with rate guarantee, bonus distribution and default option. We then present details of the implicit finite difference scheme for the numerical solution of the pricing models under general bonus distribution mechanism. In Section 3, we derive analytic approximation solution to the pricing model. In Section 4, computational results of the price functions of participating policies are presented and their pricing behaviors are examined. Conclusive remarks are summarized in the last section.

2. Contingent claims models

We follow a similar approach as adopted by Grosen and Jørgensen (2000) and Ballotta *et al.* (2003) to derive the continuous contingent claims model for finding the fair value of a participating policy. We assume that the balance sheet of a life insurance company consists of a homogeneous block of participating policies so that the whole business is modeled as a single contract. At initiation of the contract, a single premium is invested in an asset portfolio. Let $A(t)$ denote the market value of the asset, whose dynamics is modeled by

$$\frac{dA(t)}{A(t)} = \mu\, dt + \sigma\, dZ(t). \tag{1}$$

Here, μ is the expected growth rate, σ is the volatility and $Z(t)$ is the standard Wiener process. Let $P(t)$ denote the book value of the policy reserve and $B(t)$ be the bonus reserve. The sum of the policy reserve and bonus reserve observes the following accounting identity

$$A(t) = P(t) + B(t), \tag{2}$$

where $P(0) = \alpha A(0)$, $0 < \alpha \leq 1$, and $P(0)$ is the single premium paid by the policy holder at initiation of the contract. Here, α is called the cost allocation parameter. That is, the policy holder finances an α-portion of the initial asset portfolio.

Let $\mu_P(A, P)$ denote the interest rate credited to the policy reserve, that is,

$$\frac{dP(t)}{P(t)} = \mu_P\, dt. \tag{3}$$

The bonus distribution rule is determined by the management of the insurance company, thus provides the specification of μ_P. The actual process of deciding μ_P is highly subtle. Grogen and Jørgensen (2000) propose that there is a long term constant target ratio β (say, $10-15\%$) specified by the management. The insurance company normally would distribute to the policy holder certain fraction δ of the excess of the ratio of bonus reserve $B(t)$ to the policy reserve $P(t)$ over the target ratio β. We call δ to be the reversionary bonus distribution rate, $0 < \delta \leq 1$. In addition, the imposition of the interest rate guarantee means that μ_P cannot fall below some specified guarantee rate r_g. We assume that the interest rate crediting scheme is prescribed as[1]

$$\mu_P = \max\left(r_g, \delta\left(\ln\frac{A(t)}{P(t)} - \beta\right)\right). \tag{4}$$

In summary, the interest rate credited to the policy holder's account includes both the guaranteed rate and the reversionary bonus.

Let $V(A, P, t)$ denote the fair value of the participating policy. We would like to formulate the contingent claims model with the inclusion of bonus distribution rule, terminal bonus and default option for pricing the participating policy. By following the usual Black-Scholes continuous riskless hedging argument, we consider a portfolio that consists of long position of one unit of the policy and short position of Δ units of the underlying asset $A(t)$. Over the time interval $(t, t+dt]$, the infinitesimal change of the portfolio value $\Pi(t)$ is given by

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}A^2\frac{\partial^2 V}{\partial A^2} + \mu_P P\frac{\partial V}{\partial P}\right)dt, \tag{5}$$

where Δ has been chosen to be $\dfrac{\partial V}{\partial A}$ in order to hedge against the financial risk. By the no-arbitrage principle, the rate of return from the portfolio should be equal to the riskfree interest rate r. We then have

$$d\Pi = r\Pi\,dt = r\left(V - \frac{\partial V}{\partial A}A\right)dt. \tag{6}$$

[1] Grosen and Jørgensen (2000) and Prieul *et al.* (2001) both choose the following form

$$\mu_P = \max\left(r_g, \delta\left(\frac{B(t)}{P(t)} - \beta\right)\right).$$

Here, we replace the ratio $\dfrac{B(t)}{P(t)}$ by its continuous compounding version, $\ln\left(1 + \dfrac{B(t)}{P(t)}\right)$ $= \ln\dfrac{A(t)}{P(t)}$. Our functional form of μ_P is consistent with the lognormal assumption of the stochastic process for $A(t)$.

By rearranging the terms, we obtain the following governing equation for the value of the participating policy

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} A^2 \frac{\partial^2 V}{\partial A^2} + rA\frac{\partial V}{\partial A} + \mu_P P \frac{\partial V}{\partial P} - rV = 0, \quad A > 0, P > 0, t < T. \tag{7}$$

The auxiliary condition of the pricing model is given by the payoff on policy maturity date T. One feasible form of terminal payoff is given by (Ballotta *et al.*, 2003)

$$\begin{aligned} V(A,P,T) &= \begin{cases} A & \text{if } A < P \\ P & \text{if } P \le A \le P/\alpha \\ P + \gamma B & \text{if } A > P/\alpha \end{cases} \\ &= P + \gamma B - D. \end{aligned} \tag{8}$$

Here, $B = (\alpha A - P)^+$ represents the terminal bonus option and $D = (P - A)^+$ represents the terminal default option. The notation x^+ is defined by $x^+ = \max(x, 0)$. The parameter γ is the terminal bonus distribution rate. The bonus option is a call option granted to the policy holder in the sense that the policy holder has the right to pay the policy as strike to receive α-portion of the asset portfolio. On the other hand, the policy holder grants a put option to the insurer so that the insurer has the right to put the asset for the policy value when the asset value falls below the policy value.

Dimension reduction of the model formulation via similarity transformations

The governing equation is a two-dimensional degenerate diffusion equation, similar to that of an Asian option model. The path dependent feature of the pricing model is exhibited by the term $\mu_P P \dfrac{\partial V}{\partial P}$, which represents the impact of the interest rate crediting scheme. When $A(t)$ is well in excess of $P(t)$, the crediting effect due to $\mu_P P \dfrac{\partial V}{\partial P}$ dominates over the diffusion effect $\dfrac{\sigma^2}{2} A^2 \dfrac{\partial^2 V}{\partial A^2}$ due to volatility of the asset value. Provided that the crediting scheme μ_P and the ratio of terminal payoff to policy reserve value $V(A,P,T)/P$ are expressible in terms of the similarity variable A/P, the governing equation can be reduced to an one-dimensional equation by taking P as the numeraire. Suppose we let

$$x = \ln A/P \quad \text{and} \quad U(x,t) = V(A,P,t)/P, \tag{9a}$$

and provided that

$$\mu_P(A,P) = \mu_P(x) \quad \text{and} \quad V(A,P,T)/P = H(x), \tag{9b}$$

then Eq. (7) can be simplified into the following one-dimensional equation

$$\begin{aligned} \frac{\partial U}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 U}{\partial x^2} + \left[r - \mu_P(x) - \frac{\sigma^2}{2} \right] \frac{\partial U}{\partial x} - [r - \mu_P(x)]U = 0, \\ -\infty < x < -\infty, t < T, \end{aligned} \tag{10}$$

with auxiliary condition: $U(x,T) = H(x)$.

Assume that μ_P takes the form as shown in Eq. (4). We define the following parameters

$$\epsilon = \frac{\sigma^2}{2\delta}, \quad \widehat{r} = \frac{2r}{\sigma^2}, \quad \widehat{r}_g = \frac{2r_g}{\sigma^2}, \quad \widehat{\beta} = \frac{2\beta\delta}{\sigma^2}$$

and independent variables

$$\tau = \delta(T-t) \quad \text{and} \quad y = x - (\widehat{r}_g + \widehat{\beta})\epsilon,$$

then the governing equation for $U(y,\tau)$ can be further reduced to

$$\frac{\partial U}{\partial \tau} = \epsilon \mathcal{L} U + y^+ \left(U - \frac{\partial U}{\partial y} \right), \quad -\infty < y < \infty, \tau > 0, \tag{11a}$$

where $\mathcal{L}$ denotes the linear operator

$$\mathcal{L} = \frac{\partial^2}{\partial y^2} + (\widehat{r} - \widehat{r}_g - 1)\frac{\partial}{\partial y} - (\widehat{r} - \widehat{r}_g).$$

Furthermore, we define

$$K_D = e^{-(\widehat{\beta}+\widehat{r}_g)\epsilon} \quad \text{and} \quad K_B = \frac{K_D}{\alpha},$$

then the terminal payoff can be expressed as

$$U(y,0) = \widehat{H}(y) = 1 + \frac{\gamma}{K_B}(e^y - K_B)^+ - \frac{1}{K_D}(K_D - e^y)^+. \tag{11b}$$

In the differential equation (11a), the term $y\left(U - \dfrac{\partial U}{\partial y} \right)$ appears only when $y > 0$. Also, the terminal payoff is expressible in terms of y only. For the special forms of crediting scheme and terminal payoff chosen in our model, we are able to achieve dimension reduction of the model formulation via the similarity transformations defined in Eq. (9a). In the left region $y \leq 0$, the equation is a parabolic convection-diffusion equation. In the right region $y > 0$, the equation has a strongly dominated convective term when the value of y is sufficiently large.

In general, no analytic solution can be found for Eqs. (11a,b). One may obtain numerical solution to the pricing model via the finite difference method. Since the coefficients in the differential equation contain the factor y, it is advisable to use an implicit finite difference scheme to avoid plausible numerical instabilities and oscillations in the finite difference calculations.

Implicit finite difference scheme

While the continuous version of the pricing model has infinite domain, the discretized computational domain must be limited by finite boundaries. Let $[-L, L] \times [0, T]$ denote the finite computational domain, where the width of the spatial interval $2L$ is chosen to be sufficiently large. The computational domain is discretized into a finite difference mesh, where Δy and $\Delta\tau$ are the stepwidth and time step, respectively. Let U_j^m denote the numerical approximation to $U(j\Delta y, m\Delta\tau)$, where $m = 0, 1, 2, \cdots, j = -J, -J+1, \cdots, -1, 0, 1, \cdots, J$, and $J\Delta y = L$. Instead of prescribing the boundary conditions along the numerical boundaries, corresponding to $j = -J$ and $j = J$, we enforce the satisfaction of the discretized version of the governing equation along the boundaries. This is done by using one-sided difference operators to approximate the differential operators in the differential equation so that fictitious mesh points outside the computational domain are avoided. Say, at the left boundary $j = -J$, we approximate the differential terms in the differential equation by the difference terms as follows:

$$\frac{\partial^2 U}{\partial y^2}(-J\Delta y, m\Delta\tau) \approx \frac{2U_{-J}^m - 5U_{-J+1}^m + 4U_{-J+2}^m - U_{-J+3}^m}{\Delta y^2}$$
$$\frac{\partial U}{\partial y}(-J\Delta y, m\Delta\tau) \approx \frac{-3U_{-J}^m + 4U_{-J+1}^m - U_{-J+2}^m}{2\Delta y}.$$

We apply fully implicit discretization in the construction of our finite difference scheme, that is, all spatial differential terms in the differential equation are discretized at the new time level. We define the following set of parameters

$$a = \frac{\epsilon\Delta\tau}{\Delta y^2}, \quad b_j = \frac{\epsilon\Delta\tau}{\Delta y}(\widehat{r} - \widehat{r}_g - 1) - j^+\Delta\tau,$$
$$c_j = j^+\Delta y\Delta\tau - (\widehat{r} - \widehat{r}_g)\epsilon\Delta\tau.$$

In terms of the above parameters, the implicit finite difference scheme that relates numerical solution values at the $(m+1)^{th}$ and m^{th} time levels is given by

(i) at the interior points, $j = -J+1, \cdots, 0, J-1$,

$$(b_j - a)U_{j-1}^{m+1} + (1 + 2a - b_j - c_j)U_j^{m+1} - aU_{j+1}^{m+1} = U_j^m, \tag{12a}$$

(ii) along the left boundary, $j = -J$,

$$\left(1 - 2a + \frac{3}{2}b_{-J} - c_{-J}\right)U_{-J}^{m+1} + (5a - 2b_{-J})U_{-J+1}^{m+1}$$
$$+ \left(\frac{1}{2}b_{-J} - 4a\right)U_{-J+2}^{m+1} + aU_{-J+3}^{m+1} = U_{-J}^m, \tag{12b}$$

(iii) along the right boundary, $j = J$,

$$\left(1 - 2a - \frac{3}{2}b_J - c_J\right) U_J^{m+1} + (5a + 2b_J)U_{J-1}^{m+1}$$
$$- \left(\frac{1}{2}b_J + 4a\right) U_{J-2}^{m+1} + aU_{J-3}^{m+1} = U_J^m. \tag{12c}$$

3. Analytic approximation solutions

The governing equation (11a) consists of two components: the diffusion component $\epsilon\mathcal{L}U$ and the crediting component $y^+\left(U - \dfrac{\partial U}{\partial y}\right)$. The diffusion term is pre-multiplied by the parameter ϵ which is a small quantity. Say, we take $\sigma = 0.2, \delta = 0.8$, then $\epsilon = 0.025$. If we treat ϵ as a perturbation parameter, then the pricing model resembles a singular perturbation problem since the perturbation parameter appears in the highest order derivative term. Here, the diffusion component will be dominated by the crediting term when $y \gg \epsilon$. We let $U^+(y,\tau)$ to be an analytic approximation to the pricing model over the right half domain: $y > y_0$. We may take y_0 to be some multiple k of the perturbation quantity ϵ, that is, $y_0 = k\epsilon$. For convenience, we choose k such that

$$k \geq \left(\frac{\ln K_B}{\epsilon}\right)^+ = \left[-\frac{2}{\sigma^2}(r_g + \beta\delta + \delta\ln\alpha)\right]^+ \tag{13}$$

so that the terminal payoff can be simplified into

$$\widehat{H}(y) = 1 - \gamma + \frac{\gamma}{K_B}e^y, \quad y \geq k\epsilon. \tag{14}$$

Upon expanding the terminal payoff $\widehat{H}(y)$ in powers of ϵ, we obtain

$$\widehat{H}(y) = \widehat{H}_0(y) + \epsilon\widehat{H}_1(y) + O(\epsilon^2) \tag{15a}$$

where

$$\widehat{H}_0(y) = 1 - \gamma + \alpha\gamma e^y, \quad \widehat{H}_1(y) = \alpha\gamma(\widehat{\beta} + \widehat{r}_g)e^y. \tag{15b}$$

Solution in the right region, $y \geq k\epsilon$

We would like to find an approximate solution to $U^+(y,\tau)$ using the perturbation method in partial differential equation theory. Suppose we seek the perturbation expansion of $U^+(y,\tau)$ in powers of the perturbation parameter ϵ, that is,

$$U^+(y,\tau) = U_0^+(y,\tau) + \epsilon U_1^+(y,\tau) + \cdots, \tag{16}$$

then the governing equation for the zeroth order term $U_0^+(y,\tau)$ is given by the following linear hyperbolic equation

$$\frac{\partial U_0^+}{\partial \tau} = y\left(U_0^+ - \frac{\partial U_0^+}{\partial y}\right), \quad y \geq k\epsilon, \tau > 0,$$
$$U_0^+(y,0) = \widehat{H}_0(y) = 1 - \gamma + \alpha\gamma e^y. \tag{17}$$

The solution to the above hyperbolic equation is found to be (detailed derivation is presented in Appendix A)

$$U_0^+(y,\tau) = \widehat{H}_0(ye^{-\tau})\exp(y(1-e^{-\tau})). \tag{18}$$

Next, the governing equation for the first order term $U_1^+(y,\tau)$ is given by

$$\frac{\partial U_1^+}{\partial \tau} = y\left(U_1^+ - \frac{\partial U_1^+}{\partial y}\right) + \mathcal{L}U_0^+$$
$$U_1^+(y,0) = \widehat{H}_1(y) = \alpha\gamma(\widehat{\beta} + \widehat{r}_g)e^y. \tag{19}$$

Here, $\mathcal{L}U_0^+$ can be considered as a known source term in the hyperbolic equation. Suppose we write $f(y,\tau) = \mathcal{L}U_0^+$, then the solution to $U_1^+(y,\tau)$ can be deduced to be

$$\begin{aligned} U_1^+(y,\tau) = {} & \widehat{H}_1(ye^{-\tau})\exp(y(1-e^{-\tau})) \\ & + \int_0^\tau \widetilde{\mathcal{L}} Z_0(ye^{-\tau}, u)\exp(y(1-e^{-(\tau-u)}))\,du, \end{aligned}$$

where

$$\widetilde{\mathcal{L}} = e^{-2\eta}\frac{\partial^2}{\partial \xi^2} + (\widehat{r} - \widehat{r}_g - 1)\,e^{-\eta}\frac{\partial}{\partial \xi} - (\widehat{r} - \widehat{r}_g)$$

and

$$Z_0(\xi,\eta) = \widehat{H}_0(\xi)\exp(\xi(e^\eta - 1)).$$

After some manipulation, we obtain

$$\begin{aligned} U_1^+(y,\tau) = {} & \exp(y(1-e^{-\tau}))\{\widehat{H}_1(ye^{-\tau}) \\ & + \frac{1-e^{-2\tau}}{2}[\widehat{H}_0''(ye^{-\tau}) - 2\widehat{H}_0'(ye^{-\tau}) + \widehat{H}_0(ye^{-\tau})] \\ & + (\widehat{r} - \widehat{r}_g + 1)(1-e^{-\tau})[\widehat{H}_0'(ye^{-\tau}) - \widehat{H}_0(ye^{-\tau})]\}. \end{aligned} \tag{20}$$

To the first order approximation, the solution to $U^+(y,\tau)$ can be expressed as

$$\begin{aligned} U^+(y,\tau) = {} & \exp(y(1-e^{-\tau}))\left\{\left[\alpha\gamma\exp\left(ye^{-\tau} + \beta + \frac{r_g}{\delta}\right) + (1-\gamma)\right]\right. \\ & \left. + \epsilon(1-\gamma)\left[\frac{1-e^{-2\tau}}{2} - (\widehat{r} - \widehat{r}_g + 1)(1-e^{-\tau})\right]\right\} + O(\epsilon^2), \\ & y \geq k\epsilon, \tau > 0. \end{aligned} \tag{21}$$

Solution in the left region, $y < k\epsilon$

For $y < k\epsilon$, we approximate the solution to $U(y,\tau)$ by $U^-(y,\tau)$, whose governing equation is given by

$$\frac{\partial U^-}{\partial \tau} = \epsilon \mathcal{L} U^- + y^+ \left(U^+ - \frac{\partial U^+}{\partial y} \right), \quad y < k\epsilon, \tau > 0, \tag{22}$$

while the auxiliary conditions are:

$$U^-(y,0) = \widehat{H}(y) \quad \text{and} \quad U^-(k\epsilon,\tau) = U^+(k\epsilon,\tau).$$

Here, we make the approximation that for $y \in (0, k\epsilon)$, the crediting component terms is replaced by the known source term $y\left(U^+ - \frac{\partial U^+}{\partial y} \right)$. This is well justified since the difference $y\left[\left(U - \frac{\partial U}{\partial y} \right) - \left(U^+ - \frac{\partial U^+}{\partial y} \right) \right]$ should be small for $0 < y < k\epsilon$. The matching of value of the solutions on the two sides is imposed at $y = k\epsilon$ by setting

$$U^-(k\epsilon,\tau) = U^+(k\epsilon,\tau),$$

but the smooth pasting of the two solutions at $y = k\epsilon$ is not enforced.

In Appendix B, we show how to derive the solution of a linear diffusion equation with a source term in a semi-infinite domain. Using this known form of solution, the solution to $U^-(y,\tau)$ can be deduced to be

$$\begin{aligned} U^-(y,\tau) = U^+(k\epsilon,\tau) - \widehat{H}(k\epsilon) G(y,\tau;k\epsilon) + \int_{-\infty}^{k\epsilon} \widehat{H}(\xi) g(y,\tau;\xi)\, d\xi \\ - \int_0^\tau \left[(\widehat{r} - \widehat{r}_g)\epsilon U^+(k\epsilon,\tau-u) + \frac{\partial U^+}{\partial \tau}(k\epsilon,\tau-u) \right] G(y,u;k\epsilon)\, du \\ + \int_0^\tau \int_0^{k\epsilon} \xi \left[U^+(\xi,\tau-u) - \frac{\partial U^+}{\partial \xi}(\xi,\tau-u) \right] g(y,u;\xi)\, d\xi du, \end{aligned} \tag{23}$$

where $g(y,\tau;\xi)$ is the Green function defined in Eq. (B.1) and $G(y,\tau;\xi)$ is an integral of $g(y,\tau;\xi)$ defined in Eq. (B.2). We define

$$y_1 = -(\widehat{\beta} + \widehat{r}_g)\epsilon \quad \text{and} \quad y_2 = y_1 - \ln \alpha,$$

then the initial condition can be expressed as

$$U^-(y,0) = \widehat{H}(y) = \begin{cases} e^{y-y_1} & y < y_1 \\ 1 & y_1 \le y \le y_2 \\ 1 - \gamma + \alpha\gamma e^{y-y_1} & y > y_2 \end{cases}.$$

After simplifying the first three terms in Eq. (23), the analytic form of $U^-(y,\tau)$ can be reduced to

$$\begin{aligned} U^-(y,\tau) = & U^+(k\epsilon,\tau) - \alpha\gamma e^{k\epsilon - y_1} G(y,\tau;k\epsilon) \\ & + \gamma G(y,\tau;y_2) - G(y,\tau;y_1) \\ & + \alpha\gamma[\widehat{G}(y,\tau;k\epsilon) - \widehat{G}(y,\tau;y_2)] + \widehat{G}(y,\tau;y_1) \\ & - \int_0^\tau \left[\epsilon(\widehat{r}-\widehat{r}_g)U^+(k\epsilon,\tau-u) + \frac{\partial U^+}{\partial\tau}(k\epsilon,\tau-u)\right] G(y,u;k\epsilon)\,du \\ & + \int_0^\tau \int_0^{k\epsilon} \xi \left[U^+(\xi,\tau-u) - \frac{\partial U^+}{\partial\xi}(\xi,\tau-u)\right] g(y,u;\xi)\,d\xi du, \end{aligned} \tag{24}$$

where

$$\begin{aligned} \widehat{G}(y,\tau;\xi) = & \int_{-\infty}^{\xi} e^{x-y_1} g(y,\tau;x)\,dx \\ & = e^{y-y_1}\left[N\left(\frac{\xi - y - (\widehat{r}-\widehat{r}_g+1)\epsilon\tau}{\sqrt{2\epsilon\tau}}\right)\right. \\ & \left. - e^{(\widehat{r}-\widehat{r}_g+1)(k\epsilon-y)} N\left(\frac{\xi + y - 2k\epsilon - (\widehat{r}-\widehat{r}_g+1)\epsilon\tau}{\sqrt{2\epsilon\tau}}\right)\right]. \end{aligned}$$

Though the last two integrals in Eq. (24) can be evaluated in closed form, we leave the numerical evaluation of the two integrals by numerical integration.

4. Behaviors of the pricing functions of participating policies

We performed numerical experiments to examine the accuracy of the analytic approximation solution by comparing with the numerical solution obtained from finite difference calculations. While U^+ is presented in closed form [see Eq. (21)], the valuation of U^- is done via numerical integration of the integrals in Eq. (24). The finite difference solution is obtained by performing numerical calculations based on the implicit scheme presented in Eqs. (12a-c). In the following sample calculations, the time to maturity $T-t$ is set to be 10. Also, the value of k defined in Eq. (13) is set to be $\max\left(1, \frac{\ln K_B}{\epsilon}\right)$. In Figure 1, we show the comparison of the finite difference solution and analytic approximation solution to $U(x,\tau), -1 \le x \le 1$. The parameters used in the pricing model are: $r = 0.05, r_g = 0.03, \sigma = 0.15, \delta = 0.3, \beta = 0.1, \gamma = 0.9$ and $\alpha = 0.9$, and 1000 time steps are used in the finite difference calculations. Very good agreement of the two solutions is exhibited over a reasonable range of values of x.

We also explored how different bonus distribution schemes may affect the solution $U(x,\tau)$. Three bonus distribution rules $\mu_P(x)$ are used in our sample calculations, namely (i) $x - \beta = \ln\frac{A(t)}{P(t)} - \beta$, (ii) $e^x - 1 - \beta = \frac{B(t)}{P(t)} - \beta$, (iii) $2(e^{x/2}-1) - \beta = 2\left(\sqrt{\frac{A(t)}{P(t)}} - 1\right) - \beta$.

The second rule considers the ratio $\frac{B(t)}{P(t)}$ based on annual compounding while the first and third rules consider the ratio based on continuous compounding and semi-annual compounding, respectively. Parameter values used in the calculations are $r = 0.05, r_g = 0.03, \sigma = 0.15, \delta = 0.8, \beta = 0.1, \gamma = 0.75$ and $\alpha = 0.7$. For the range of values $x \in [-0.5, 0.5]$, Figure 2 shows that the difference of the solution values under different crediting schemes is not significant.

There are several basic parameters in the participating policy pricing model: $r_g, \sigma, \delta, \beta$, γ and α. In Figures 3-5, we show how the solution $U(x, \tau)$ at $\tau = 5$ and $x = -0.2, x = 0$ and $x = 0.2$ depend on some of these parameters. The basic set of parameter values used in our sample calculations are chosen to be $r = 0.05, r_g = 0.03, \sigma = 0.15, \delta = 0.5, \beta = 0.1, \gamma = 0.5$ and $\alpha = 0.5$. In Figure 3, we show how $U(x, \tau)$ may depend on σ with varying values of x. When $x = -0.2, A(t)$ is below $P(t)$ and U is seen to be a decreasing function of σ. This is not surprising since the market value of the participating policy is the difference of the bonus option and default option. Both options are increasing function of σ, but the default option has stronger influence on U when $A(t) < P(t)$. Depending on the choices of the parameter values, the plot of U against σ can be hump-shaped or a strictly increasing function of σ [see Figure 9 in Ballotta *et al.*'s paper (2003)]. We also explored the dependence of the policy value on the reversionary bonus distribution rate δ. As revealed by the plots in Figure 4, the policy value is an increasing function of δ. This is intuitively obvious since the crediting scheme $\mu_P(x)$ is an increasing function of δ. When x increases, the rate of increase of U with respect to δ is more significant since a larger value of x means a higher chance that the crediting scheme $\mu_P(x)$ stays above the guarantee rate r_g. In a similar manner, but in the reverse sense, the policy value is a decreasing function of the target ratio β since $\mu_P(x)$ is a decreasing function of β (see Figure 5). Also, a larger value of x means a stronger influence of β on $\mu_P(x)$, hence a more significant drop in policy value with increasing β.

5. Conclusion

We have developed an analytic approximation method using perturbation techniques to solve for contingent claims models that price participating policies with rate guarantees, terminal bonus and default option. By exploring the analytic structure of the contingent claims model, our perturbation approach provides fairly accurate analytic approximation to the policy value. To achieve analytic tractability in the perturbation solution, we have made certain simplifying assumptions in the pricing models, like the neglect of mortality risk and surrender option. Also, the interest rate crediting scheme assumes some specific functional forms. As an alternative solution method, we propose an implicit finite difference scheme for solving the partial differential equation governing the contingent claims model. The pricing behaviors of participating policies with varying values of the parameters in the pricing models have also been explored.

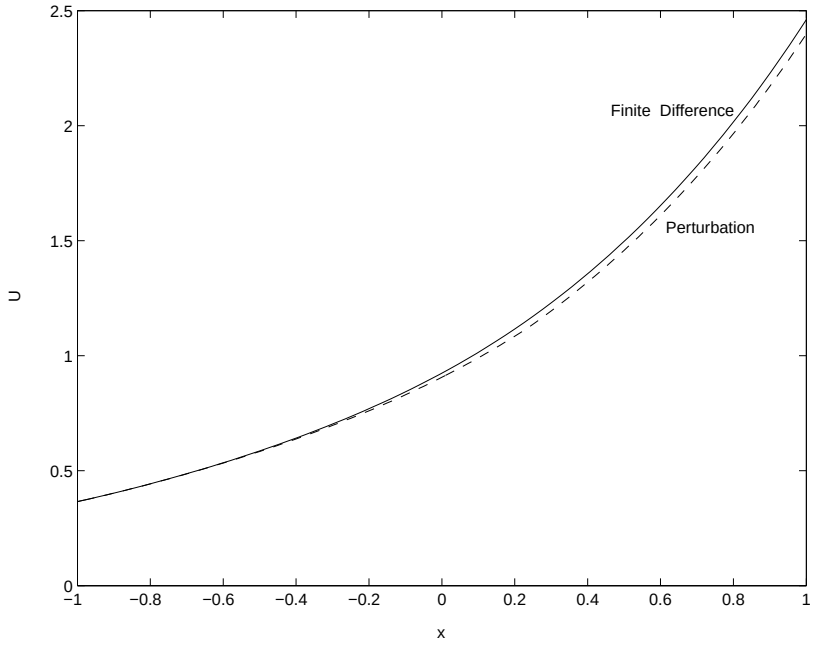

Fig. 1. Comparison of the solution to $U(x, \tau)$ at $\tau = 3$ using the finite difference method and perturbation method.

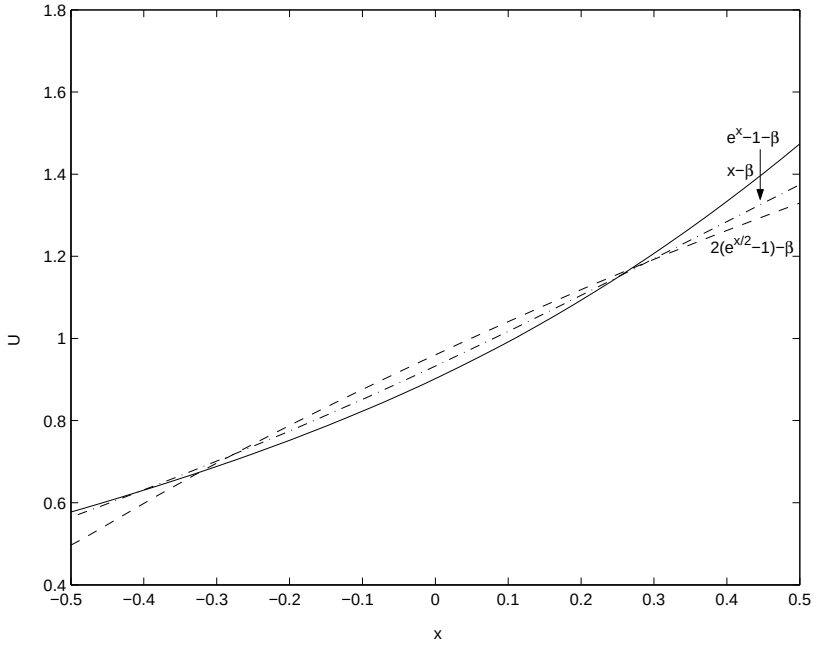

Fig. 2. Behaviors of the solution $U(x, \tau)$ at $\tau = 8$ based on different bonus distribution schemes.

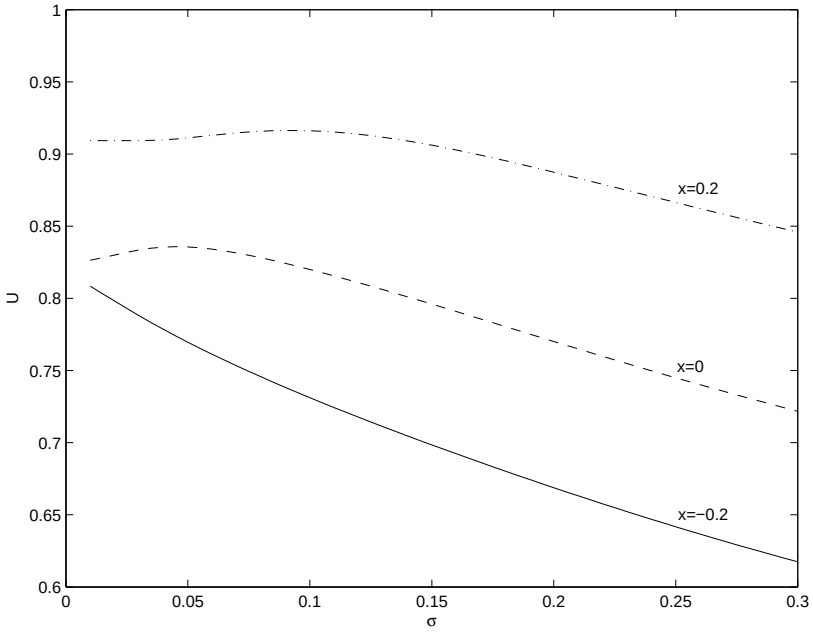

Fig. 3. Plot of $U(x,\tau)$ at $\tau = 5$ against asset volatility σ with varying values of $x = \ln A/P$.

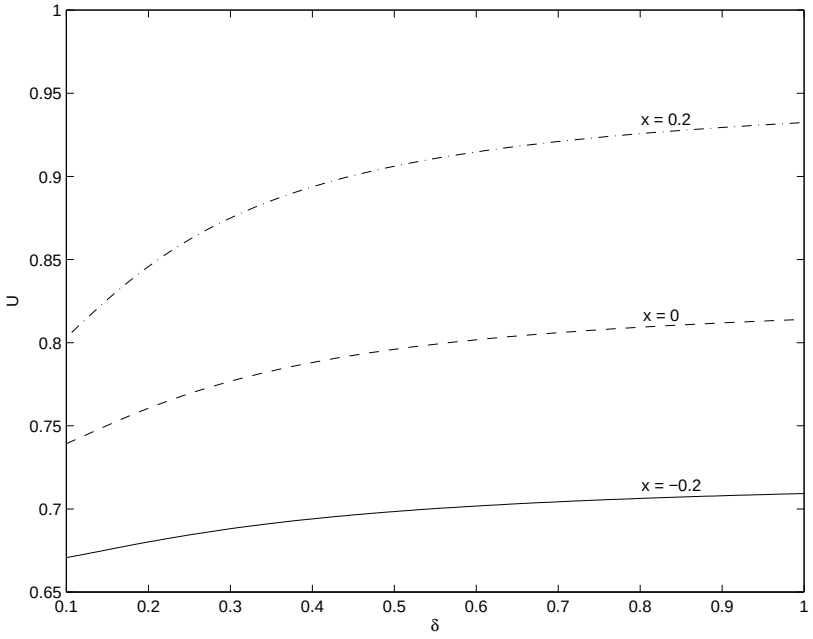

Fig. 4. Plot of $U(x,\tau)$ at $\tau = 5$ against reversionary bonus distribution rate δ with varying value of $x = \ln A/P$.

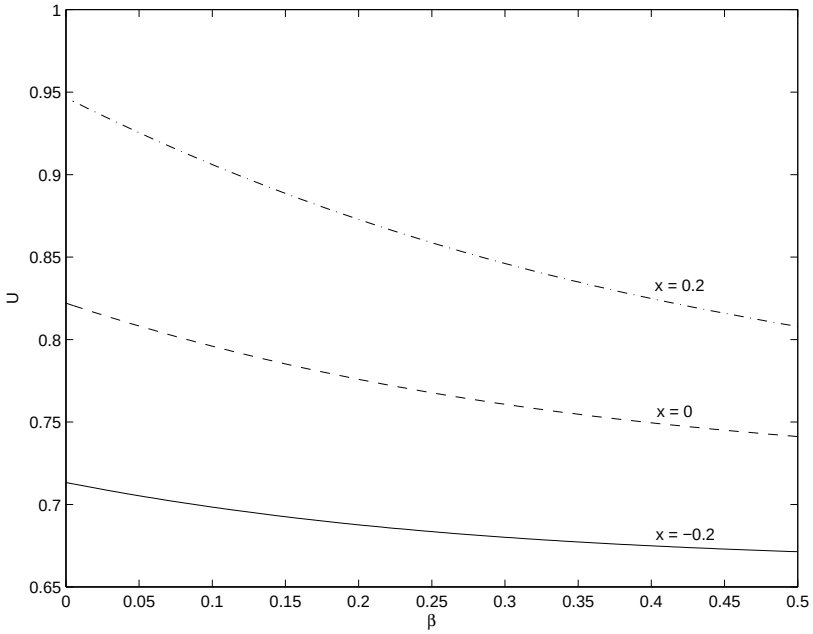

Fig. 5. Plot of $U(x, \tau)$ at $\tau = 5$ against target ratio β with varying values of $x = \ln A/P$.

References

Bacinello, A.R., 2001. Fair pricing of life insurance participating policies with a minimum interest rate guaranteed. ASTIN Bulletin 31(2), 275-297.

Bacinello, A.R., 2003a. Fair valuation of a guaranteed life insurance participating contract embedding a surrender option. Journal of risk and insurance 70(3), 461-487.

Bacinello, A.R., 2003b. Pricing guaranteed life insurance participating policies with annual premiums and surrender option. North American Actuarial Journal 7(3), 1-17.

Ballotta, L., Haberman, S. and Wang, N., 2003. Guarantees in with-profit and unitised with profit life insurance contracts: fair valuation problem in presence of the default option. Working paper of Cass Business School, City University, London.

Consiglio, A., Cocco, F., Zenios, S.A., 2001. Asset and liability modelling for participating policies with guarantees. Working paper of University of Palermo and University of Cyprus.

Grosen, A., J ørgensen, P.L., 2000. Fair valuation of life insurance liabilities: The impact of interest rate guarantees, surrender options, and bonus policies. Insurance: Mathematics and Economics 26, 37-57.

Grosen, A., Jørgensen, P.L., 2002. Life insurance liabilities at market value: an analysis of insolvency risk, bonus policy, and regulatory intervention rules in a barrier option framework. Journal of risk and insurance 69(1), 63-91.

Jensen, B., J ørgensen, P.L., Grosen, A., 2001. A finite difference approach to the valuation of path dependent life insurance liabilities. Geneva Papers on Risk and Insurance Theory 26, 57-84.

Prieul, D., Putyatin, V., Nassar, T., 2001. On pricing and reserving with-profits life insurance contracts. Applied Mathematical Finance 8, 145-166.

Wilkie, A.D., 1987. An option pricing approach to bonus policy. Journal of the Institute of Actuaries. 114, 21-77.

Willder, M., 2003. Application of derivatives to unitised with-profits policies. Working paper of Heriot-Watt University.

Appendix A — Analytic solution of $U_0^+(y,\tau)$

The zeroth order solution $U_0^+(y,\tau)$ in the right half domain, $y \geq y_0$ and $\tau > 0$, is governed by the following first order hyperbolic equation

$$\frac{\partial U_0^+}{\partial \tau} + y\frac{\partial U_0^+}{\partial y} = yU_0^+, \quad y \geq y_0 \quad \text{and} \quad \tau > 0,$$

with the initial condition: $U_0^+(y,0) = H_0(y), y \geq y_0$. Consider the integral surface $z = U_0^+(y,\tau)$ and let $\mathbf{r} = (y,\tau,z)$ be the position vector of a point on the integral surface. From the initial condition, we deduce that the following curve $\mathbf{r}_0(\xi) = (\xi, 0, \widehat{H}_0(\xi))$ lies on the integral surface. We use ξ and η to parameterize the characteristic curve $(Y(\xi,\eta), \Gamma(\xi,\eta), Z(\xi,\eta))$ on the integral surface. The curve $\mathbf{r}_0(\xi)$ corresponds to $\eta = \eta_0$. The evolution equations for the characteristic curves on the integral surface are given by

$$\begin{aligned}
&\frac{dY(\xi,\eta)}{d\eta} = Y(\xi,\eta), && Y(\xi,\eta_0) = \xi,\\
&\frac{d\Gamma(\xi,\eta)}{d\eta} = 1, && \Gamma(\xi,\eta_0) = 0,\\
&\frac{dZ(\xi,\eta)}{d\eta} = Y(\xi,\eta)Z(\xi,\eta), && Z(\xi,\eta_0) = \widehat{H}_0(\xi).
\end{aligned}$$

The solution to (Y,Γ,Z) is given by

$$(Y(\xi,\eta), \Gamma(\xi,\eta), Z(\xi,\eta)) = (\xi e^{\eta-\eta_0}, \eta-\eta_0, \widehat{H}_0(\xi)\exp(\xi(e^{\eta-\eta_0}-1)).$$

We then deduce that $y = \xi e^{\eta-\eta_0}$ and $\tau = \eta - \eta_0$ so that $\xi = ye^{-\tau}$ and $\eta = \tau + \eta_0$. Hence, we obtain

$$U_0^+(y,\tau) = \widehat{H}_0(ye^{-\tau})\exp(y(1-e^{-\tau})).$$

Appendix B — Analytic solution of $U^-(y,\tau)$

Consider the following linear diffusion equation with a known source term $f(y,\tau)$

$$\frac{\partial U^-}{\partial \tau} = \epsilon\mathcal{L}U^- + f(y,\tau), \quad y < k\epsilon, \tau > 0,$$

with auxiliary conditions: $U^-(y,0) = \widehat{H}(y)$ and $U^-(k\epsilon,\tau) = h(\tau)$. Let $V(y,\tau) = U^-(y,\tau)$ $-h(\tau)$, then $V(y,\tau)$ is governed by

$$\frac{\partial V}{\partial \tau} = \epsilon\mathcal{L}V + f(y,\tau) - h'(\tau) - \epsilon(\widehat{r}-\widehat{r}_g)h(\tau), \quad y < k\epsilon, \tau > 0,$$

with auxiliary conditions: $V(y,0) = \widehat{H}(y) - h(0)$ and $V(k\epsilon,\tau) = 0$. Let $g(y,\tau;\xi)$ denote the Green function that satisfies

$$\frac{\partial g}{\partial \tau} = \epsilon \mathcal{L} g, \quad y < k\epsilon, \tau > 0,$$

with auxiliary conditions: $g(y,0;\xi) = \delta(y-\xi)$ and $g(k\epsilon,\tau;\xi) = 0$. The analytic solution of $g(y,\tau;\xi)$ is found to be

$$\begin{aligned} g(y,\tau;\xi) =& \frac{1}{\sqrt{2\epsilon\tau}} e^{-(\widehat{r}-\widehat{r}_g)\epsilon\tau} \left[n\left(\frac{y-\xi+(\widehat{r}-\widehat{r}_g-1)\epsilon\tau}{\sqrt{2\epsilon\tau}} \right) \right. \\ & \left. - e^{(\widehat{r}-\widehat{r}_g-1)(k\epsilon-y)} n\left(\frac{y+\xi-2k\epsilon-(\widehat{r}-\widehat{r}_g-1)\epsilon\tau}{\sqrt{2\epsilon\tau}} \right) \right]. \end{aligned} \tag{B.1}$$

The solution of $V(y,\tau)$ is then given by

$$\begin{aligned} V(y,\tau) = & \int_{-\infty}^{k\epsilon} [\widehat{H}(\xi) - h(0)] g(y,\tau;\xi)\, d\xi - \int_0^\tau [\epsilon(\widehat{r}-\widehat{r}_g)h(\tau-u) + h'(\tau-u)] G(y,u;k\epsilon)\, du \\ & + \int_0^\tau \int_{-\infty}^{k\epsilon} f(\xi,\tau-u) g(y,u;\xi)\, d\xi du. \end{aligned}$$

where

$$\begin{aligned} G(y,\tau;k\epsilon) =& \int_{-\infty}^{k\epsilon} g(y,\tau;\xi)\, d\xi \\ =& e^{-(\widehat{r}-\widehat{r}_g)\epsilon\tau} \left[N\left(\frac{-y+k\epsilon-(\widehat{r}-\widehat{g}-1)\epsilon\tau}{\sqrt{2\epsilon\tau}} \right) \right. \\ & \left. - e^{(\widehat{r}-\widehat{r}_g-1)(k\epsilon-y)} N\left(\frac{y-k\epsilon-(\widehat{r}-\widehat{r}_g-1)\epsilon\tau}{\sqrt{2\epsilon\tau}} \right) \right]. \end{aligned} \tag{B.2}$$