No-Arbitrage Approach to Pricing Credit Spread Derivatives

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No arbitrage in the case of pricing credit spread derivatives refers to determination of the time-dependent drift terms in the mean reversion stochastic processes of the instantaneous spot rate and spot spread by fitting the current term structures of default-free and defaultable bond prices. The riskless rate and the credit spread of a reference entity are taken to be correlated stochastic state variables in this pricing model.

When the spot rate and spot spread both follow the Hull and White model, one can derive an analytic representation for the time-dependent drift terms and analytic price formulas for credit spread options. Algorithms for the numerical valuation of credit spread derivatives are developed, and the pricing behaviors of credit spread options are examined.

Credit spread derivatives are credit rate-sensitive financial instruments designed to hedge against or capitalize on changes in the credit spread of a reference entity such as a risky bond. The credit spread of a risky bond is the difference between the current yield of the risky bond and a benchmark rate of comparable maturity. The spread represents the risk premium the market demands for holding the risky bond.

Unlike default swaps, credit spread derivatives do not depend upon any specific credit event occurring. The buyer of a credit spread option pays an up-front premium, and in return the writer agrees to pay an amount should the actual spot spread breach some strike level. Alternatively, the payoff may depend on the difference between the spot spreads of two reference entities. Credit spread options may be used either to earn income from the option premium, to bet one's view on credit spread change, or to target the purchase of a reference entity on a forward basis at a favored price.

By following a reduced-form approach that models the occasion of default as a point process, Jarrow, Lando, and Turnbull [1997] construct a Markov chain model for valuation of risky debt and credit derivatives that incorporates the credit ratings of a firm as an indicator of the likelihood of default. The Markov model provides the evolution of an arbitrage-free term structure of the credit spread.

Problems arise in numerical implementation of the model because bonds with high credit ratings may experience no default within the sample period, so the risk premium adjustments become ill-defined as the estimated default probabilities are zero. Kijima and Komoribayashi [1998] propose a technique of risk premium adjustment to overcome this difficulty with high-rated bonds.

In related work, Duffie and Singleton [1999] and Schönbucher [1998] use intensity models to analyze the term structure of defaultable bonds and compute the price of credit derivatives. Schönbucher [1999] constructs a two-factor tree model for fitting both defaultable and default-free term structures of bond prices, following the Hull and White framework with time-dependent drift terms. He uses the Hull and White [1990] model for the dynamics of the risk-free interest rate and default intensity. Garcia, van Gindersen, and Garcia [2001] use a two-factor tree algorithm that is composed of a Hull and White tree for the risk-free interest rate dynamics and a Black and Karasinski [1991] tree for the intensity dynamics.

Das and Sundaram [2000] propose a discrete-time Heath-Jarrow-Morton [1992] model for valuation of credit derivatives. They derive risk-neutral drifts for the processes of the risk-free forward rate and forward spread. Combining a recovery of market value condition, they obtain a recursive representation of drifts that are path-dependent.

This formulation has the advantage that it can easily handle the American exercise feature in the pricing of the credit derivatives, and the corresponding algorithm also includes information about default probabilities inferred from market data. Its major disadvantage is that its computational complexity grows exponentially with the number of time steps, since the corresponding binomial tree is non-recombining.

A structural approach derives the term structure of the credit spread of a credit entity by examining the credit quality of the issuer. In structural models, the credit spread is not one of the state variables in the model, but rather is a derived quantity. Default occurs when the issuer firm value falls below some threshold value. The structural approach has been used by Das [1995] to price credit risk derivatives.

Longstaff and Schwartz [1995] develop a valuation model for pricing credit spread derivatives by assuming that the riskless interest rate and the spread follow correlated mean reversion diffusion processes. From empirical analysis of observed credit spreads, they argue that the logarithm of the credit spread follows a mean reversion process. They assume the parameters in the characterization of the processes to be constant, and derive closed-form price formulas for the credit spread options. A disadvantage of the process is that their model does not incorporate information about the current term structures of bond prices.

Chacko and Das [2002] have since then derived price formulas for credit spread options based on the uncorrelated Cox-Ingersoll-Ross [1985] processes for the spot rate and spot spread, and again the parameters in the stochastic processes are assumed to be constant. Tahani [2000] models the process of the logarithm of credit spread using the GARCH model and obtains closed-form price formulas for credit spread options. Mougeot [2000] extends the two-factor Longstaff-Schwartz model to a three-factor model for pricing options on the spread of the values of two reference assets.

We extend the Longstaff and Schwartz model by assuming time-dependent drift functions in the mean reversion diffusion processes of the riskless interest rate and credit spread. The spot riskless interest rate is taken to follow the Hull-White model, while the spot credit spread can be taken to follow either the Hull-White model [1990] or the Black-Karasinski model [1991]. When both state variables follow the Hull-White model, we can derive analytic expressions for the time-dependent drift functions that fit the current term structures of default-free and defaultable bond prices. Also, we can obtain analytic pricing formulas for credit spread options.

Unfortunately, the time-dependent drift function in the spot spread process is not analytically tractable when the logarithm of the spot spread is assumed to follow a mean-reverting diffusion (this is the assumption in the Black-Karasinski model). By extending the Hull-White trinomial tree fitting methodologies [1994], we extend the fitting algorithms to allow numerical valuation of credit spread options.

I. CONTINUOUS MODELS

We begin with the pricing of credit spread derivatives under the Hull-White [1990] model. This is an extension of the one-factor Hull-White model to the two-factor version, where both the instantaneous spot riskless interest rate and the credit spread follow a mean-reverting model with time-dependent drift terms. Like the one-factor version, the two-factor model exhibits nice analytical tractability and ease of numerical implementation.

Pricing Framework Under Two-Factor Hull-White Model

We specify the processes of the instantaneous spot riskless interest rate r and the spot spread s to follow the extended Vasicek [1977] model, where the drift terms are assumed to be time-dependent functions. Under a risk-neutral valuation framework, we assume that the stochastic processes followed by r and s are given by

$$dr = [\theta(t) - ar]dt + \sigma_r dW_r, \tag{1-A}$$

$$ds = [\phi(t) - bs]dt + \sigma_s dW_s, \tag{1-B}$$

where $\theta(t)$ and $\phi(t)$ are time-dependent functions, and a and b are positive constant parameters that give the rates of mean reversion of r and s, respectively. Further, σ_r and σ_s are the constant volatility parameters for the spot rate r and spot spread s, respectively, and dW_r and dW_s are standard Brownian motions under the risk-neutral measure $\mathbf{P}$. Let ρ denote the correlation coefficient between dW_r and dW_s; that is, $dW_r dW_s = \rho dt$.

Determination of the Time-Dependent Drift Terms. Hull and White [1990] demonstrate how to determine the time-dependent function $\theta(t)$ in the drift term of the default-free spot rate by fitting the current term structure of default-free bond prices. Their procedure can be summarized as follows. Let $P(t, T)$ denote the price at time t of a discount bond maturing at time T, which is given by:

$$P(t,T) = E^{\mathbf{P}}[\exp(-\int_t^T r(u)\ du)], \tag{2}$$

where $E^{\mathbf{P}}$ denotes the expectation under the risk-neutral measure $\mathbf{P}$. By taking the risk-neutral expectation in Equation (2), where r is defined in Equation (1-A), one obtains:

$$P(t,T) = \exp\left(-\frac{r(t)}{a}[1 - e^{-a(T-t)}] - \int_t^T \theta(u)\frac{1 - e^{-a(T-u)}}{a}\ du + \int_t^T \frac{\sigma_r^2}{2a^2}[1 - e^{-a(T-u)}]^2\ du\right) \tag{3}$$

Differentiating Equation (3) with respect to T twice, we obtain:

$$\theta(T) = \frac{\sigma_r^2}{2a}[1 - e^{-2a(T-t)}] - \frac{\partial^2}{\partial T^2}\ln P(t,T) - a\frac{\partial}{\partial T}\ln P(t,T) \tag{4}$$

This equation gives the dependence of the drift term $\theta(t)$ using the information of the current term structure of the default-free bond price $P(t, T)$ over the period $[t, T]$. Suppose we write the process followed by $P(t, T)$ as:

$$\frac{dP(t,T)}{P(t,T)} = rdt + \sigma_P(t,T)dW_r, \tag{5}$$

then

$$\sigma_P(t,T) = -\frac{\sigma_r}{a}[1 - e^{-a(T-t)}]. \tag{6}$$

By following a similar procedure, it is possible to determine the other time-dependent function $\phi(t)$ in the mean reversion process of s by fitting the current term structure of defaultable bond prices. This is achieved by changing the probability measure to the forward-adjusted measure $\mathbf{Q^T}$ with delivery at future time T. Let $D(t, T)$ denote the price at time t of a defaultable bond maturing at time T. By the Girsanov theorem, we have:

$$\begin{aligned} D(t,T) &= E^{\mathbf{P}}\left[\exp\left(-\int_t^T [r(u) + s(u)]\ du\right)\right] \\ &= P(t,T)E^{\mathbf{Q^T}}\left[\exp\left(-\int_t^T s(u)\ du\right)\right] \end{aligned} \tag{7}$$

Accordingly, the process followed by the spot spread s under the forward measure $\mathbf{Q^T}$ is given by

$$ds = [\phi(t) - bs + \rho\sigma_s\sigma_P(t,T)]dt + \sigma_s dW^T, \tag{8}$$

where dW^T is a standard Brownian motion under $\mathbf{Q^T}$. Under the process defined in Equation (8), the quantity $\int_t^T s(u)\ du$ follows the normal distribution with mean:

$$E^{\mathbf{Q^T}}\left[\int_t^T s(u)\ du\right] = \frac{s(t)}{b}\left[1 - e^{-b(T-t)}\right] + \int_t^T [\phi(u) + \rho\sigma_s\sigma_P(u,T)]\frac{1 - e^{-b(T-u)}}{b}\ du \tag{9-A}$$

and variance

$$\text{var}\left[\int_t^T s(u)\ du\right] = \int_t^T \frac{\sigma_s^2}{b^2}[1 - e^{-b(T-u)}]^2\ du. \tag{9-B}$$

Since

$$E^{\mathbf{Q^T}}\left[\exp\left(-\int_t^T s(u)\ du\right)\right] =$$

$$\exp\left(-E^{\mathbf{Q^T}}\left[\int_t^T s(u)\ du\right] + \frac{1}{2}\mathrm{var}\left[\int_t^T s(u)\ du\right]\right)$$

(10-A)

we obtain

$$\frac{D(t,T)}{P(t,T)} = \exp\left(-\frac{s(t)}{b}[1-e^{-b(T-t)}] - \int_t^T [\phi(u) + \rho\sigma_s\sigma_P(u,T)]\left(\frac{1-e^{-b(T-u)}}{b}\right) du + \int_t^T \frac{\sigma_s^2}{2b^2}[1-e^{-b(T-u)}]^2\ du\right).$$

(10-B)

Again, by differentiating Equation (10-B) with respect to T twice and observing Equation (6), we obtain:

$$\phi(T) = \frac{\sigma_s^2}{2b}[1-e^{-2b(T-t)}] - b\frac{\partial}{\partial T}\left[\ln\frac{D(t,T)}{P(t,T)}\right] - \frac{\partial^2}{\partial T^2}\left[\ln\frac{D(t,T)}{P(t,T)}\right] + \rho\sigma_s\sigma_r\left[\frac{1-e^{-a(T-t)}}{a} + e^{-a(T-t)}\frac{1-e^{-b(T-t)}}{b}\right].$$

(11)

Credit Spread Put Option. Let $p_{sp}(r, s, t)$ denote the price of the credit spread put option, whose terminal payoff is given by

$$p_{sp}(r,s,T) = \max(s-K, 0). \qquad (12)$$

Here K is the strike spread at which the option holder has the right to put the risky bond to the option writer. It is called a credit spread put option although the terminal payoff resembles a call payoff. When the spot credit spread is higher than the strike spread, this signifies that the price of the underlying risky bond drops below the strike price of the bond corresponding to the strike spread. Since the terminal payoff depends only on s, we may use the default free bond price as the numeraire and obtain:

$$p_{sp}(r,s,t) = P(t,T)E^{\mathbf{Q^T}}[\max(s-K,0)]. \qquad (13)$$

Suppose we define $\xi(t) = s(t)\exp(-b[T-t])$. Then ξ follows the stochastic process:

$$d\xi = [\phi(t) + \rho\sigma_s\sigma_P(t,T)]s(t)(-b(T-t))dt + \sigma_s s(t)(-b(T-t))dW^T. \qquad (14)$$

Note that ξ_T is conditionally normally distributed with mean μ_ξ and variance σ_ξ^2, where:

$$\mu_\xi = s(t)e^{-b(T-t)} + \frac{\rho\sigma_s\sigma_r}{a}\left[\frac{1-e^{-(a+b)(T-t)}}{a+b} - \frac{1-e^{-b(T-t)}}{b}\right] + \int_t^T \phi(u)e^{-b(T-u)}\ du$$

$$= \frac{\sigma_s^2}{2b^2}[1-e^{-b(T-t)}][1-e^{-2bt}e^{-b(T-t)}] - \frac{\partial}{\partial T}\left[\ln\frac{D(t,T)}{P(t,T)}\right] + \frac{\rho\sigma_s\sigma_r}{a}\left\{\frac{1-e^{-(a+b)(T-t)}}{a+b} - \frac{1-e^{-b(T-t)}}{b} - \frac{e^{-at}[e^{-a(T-t)}-e^{-b(T-t)}]}{b} - \frac{e^{-(a+b)t}e^{-b(T-t)}[1-e^{-a(T-t)}]}{b}\right\}, \qquad (15\text{-A})$$

$$\sigma_\xi^2 = \frac{\sigma_s^2}{2b}[1-e^{-2b(T-t)}]. \qquad (15\text{-B})$$

Note that μ_ξ depends not only on the time to expiration $T-t$ but also on the current time t. Once the mean and variance of ξ_T are available, the price formula for the credit spread put is given by:

$$p_{sp}(r,s,t) = P(t,T) \times \left[\frac{\sigma_\xi}{\sqrt{2\pi}}e^{-(K-\mu_\xi)^2/2\sigma_\xi^2} + (\mu_\xi - K)N\left(\frac{\mu_\xi - K}{\sigma_\xi}\right)\right]$$

(16)

Combined Hull-White and Black-Karasinski Model

Longstaff and Schwartz [1995)] propose that the interest rate and the logarithm of the credit spread follow mean-reverting Vasicek processes. We extend their model by assuming time-dependent functions in the drift terms; that is, we choose the Hull-White model for the interest rate process and the Black-Karasinski model for the credit spread process. Assume that $y = \ln s$, and the stochastic processes followed by r and y are governed by:

$$dr = [\theta(t) - ar]dt + \sigma_r dW_r, \qquad (17\text{-A})$$

$$dy = [\psi(t) - cy]dt + \sigma_y dW_y, \tag{17-B}$$

where c is the rate of mean reversion of y, σ_y is the constant volatility parameter for y, and $dW_r dW_y = \hat{\rho}dt$. Unfortunately, the Black-Karasinski model lacks the level of analytic tractability exhibited by the Hull-White model. Under the Black-Karasinski assumption of the credit spread process, we find that Equation (7) has to be modified as:

$$\frac{D(t,T)}{P(t,T)} = E^{\mathbf{Q^T}}\left[\exp\left(-\int_t^T \exp(y(u))\ du\right)\right]. \tag{18}$$

Unlike $\int_t^T s(u)\ du$, however, the quantity $\int_t^T \exp[y(u)]du$ is no longer normally distributed. Hence, there is not a simple relation between $[D(t,\ T)]/[P(t,\ T)]$ and the time-dependent drift function $\psi(t)$, like that shown in Equation (11).

If $\theta(t)$ and $\psi(t)$ are known functions, we may derive the price formula for the credit spread put option. In terms of y, the terminal payoff of the credit spread put is given by $\max(e^y - K, 0)$. The put price formula takes the usual Black-Scholes form:

$$p_{sp}(s,r,t) = P(t,T)\left[e^{\mu_\zeta + \sigma_\zeta^2/2}N(d) - KN(d - \sigma_\zeta)\right] \tag{19-A}$$

where

$$d = \frac{-\ln K + \mu_\zeta + \sigma_\zeta^2}{\sigma_\zeta}. \tag{19-B}$$

Similar to μ_ξ and σ_ξ^2 as given in Equations (15-A) and (15-B), the mean μ_ξ and variance σ_ξ^2 are given by:

$$\mu_\zeta = y(t)e^{-c(T-t)} + \frac{\widehat{\rho}\sigma_y\sigma_r}{a} \times \left[\frac{1 - e^{-(a+c)(T-t)}}{a+c} - \frac{1 - e^{-c(T-t)}}{c}\right] + \int_t^T \psi(u)e^{-c(T-u)}\ du, \tag{20-A}$$

$$\sigma_\zeta^2 = \frac{\sigma_y^2}{2c}[1 - e^{-2c(T-t)}]. \tag{20-B}$$

Heath-Jarrow-Morton Model

The two-factor extended Vasicek model is a special case of the Heath-Jarrow-Morton (HJM) model corresponding to a specific choice of the volatility structures. First, we present the general HJM formulation of the pricing model with the two state variables r and s.

Let $f(t,\ T)$ denote the riskless instantaneous forward rate. Following the HJM framework, the dynamic of $f(t,\ T)$ is assumed to be

$$df(t,T) = \alpha_f(t,T)dt + \sigma_f(t,T)dW_1, \tag{21}$$

where α_f is the drift, and σ_f is the volatility. Also, dW_1 is a standard Brownian motion under the risk-neutral measure $\mathbf{P}$. Under the no-arbitrage condition, the drift α_f and volatility σ_f must observe:

$$\alpha_f(t,T) = \sigma_f(t,T)\int_t^T \sigma_f(t,u)\ du. \tag{22}$$

From the dynamics of the risky bond prices, we can define the risky instantaneous forward rate $f_d(t,\ T)$ in the same manner as the risk-free instantaneous forward rate. Next, the instantaneous forward spread $h(t,\ T)$ is defined to be $f_d(t,\ T) - f(t,\ T)$; accordingly, the spot spread $s(t) = f_d(t,\ t) - f(t,\ t)$. Assume that $h(t,\ T)$ follows the stochastic process:

$$dh(t,T) = \alpha_h(t,T)dt + \sigma_{h_1}(t,T)dW_1 + \sigma_{h_2}(t,T)dW_2, \tag{23}$$

where α_h is the drift, σ_{h1} and σ_{h2} are volatilities, and dW_1 and dW_2 are independent standard Brownian motions under the risk-neutral measure $\mathbf{P}$. Again, the no-arbitrage condition dictates that α_h, σ_{h1}, and σ_{h2}, satisfy (see Schönbucher [1998]):

$$\alpha_h(t,T) = \sum_{i=1}^{2} \sigma_{h_i}(t,T) \int_t^T \sigma_{h_i}(t,u)\ du + \sigma_{h_1}(t,T) \int_t^T \sigma_f(t,u)\ du\ +$$

$$\sigma_f(t,T) \int_t^T \sigma_{h_1}(t,u)\ du. \qquad (24)$$

By recalling that $s(t) = h(t, t)$, one can show that the process followed by s is governed by:

$$ds(T) = \left\{ \frac{\partial h(t,T)}{\partial T} + \int_t^T \left[\sum_{i=1}^{2} \sigma_{h_i}^2(u,T) + 2\sigma_{h_i}(u,T)\sigma_f(u,T) \right] du\ + \right.$$

$$\int_t^T \int_u^T \left[\sum_{i=1}^{2} \sigma_{h_i}(u,v) \frac{\partial \sigma_{h_i}(u,T)}{\partial T} + \sigma_{h_1}(u,v) \frac{\partial \sigma_f(u,T)}{\partial T}\ + \right.$$

$$\left. \left. \sigma_f(u,v) \frac{\partial \sigma_{h_1}(u,T)}{\partial T} \right]\ dv\ du + \sum_{i=1}^{2} \int_t^T \frac{\partial \sigma_{h_i}(u,T)}{\partial T} dW_i(u) \right\}\ dT\ +$$

$$\sum_{i=1}^{2} \sigma_{h_i}(T,T) dW_i(T). \qquad (25)$$

Equation (25) reveals that the drift term of s is in general path-dependent. If instead we choose the volatility structures of the riskless instantaneous forward rate and the instantaneous forward spread to be:

$$\sigma_f(t,T) = \sigma_r e^{-a(T-t)}, \qquad (26\text{–A})$$

$$\sigma_{h_1}(t,T) = \rho\sigma_s e^{-b(T-t)}, \qquad (26\text{–B})$$

$$\sigma_{h_2}(t,T) = \sqrt{1-\rho^2}\sigma_s e^{-b(T-t)}, \qquad (26\text{–C})$$

then the process of s becomes Markovian. By comparing the resulting expression for ds and that given in Equation (1–B), we obtain:

$$\phi(T) = \frac{\sigma_s^2}{2b} \left[1 - e^{-2b(T-t)} \right] + bh(t,T) + \frac{\partial h(t,T)}{\partial T}\ +$$

$$\rho\sigma_s\sigma_r \left[\frac{1-e^{-a(T-t)}}{a} + e^{-a(T-t)} \frac{1-e^{-b(T-t)}}{b} \right] \qquad (27)$$

This is consistent with the expression given in Equation (11) by noting that:

$$h(t,T) = -\frac{\partial}{\partial T} \left[\ln \frac{D(t,T)}{P(t,T)} \right]. \qquad (28)$$

II. CONSTRUCTION OF NUMERICAL ALGORITHMS

Now we can extend the Hull and White [1994] methodology to fitting the current term structures of default-free and defaultable bond prices.

Two-Factor Algorithms

Let $x = x(r)$ and $z = z(s)$ be some specified functions of the spot interest rate and spot credit spread, respectively. Assume that both x and z follow mean reversion processes:

$$dx = [\tilde{\theta}(t) - \tilde{a}x]dt + \sigma_x dW_x, \qquad (29\text{–A})$$

$$dz = [\tilde{\phi}(t) - \tilde{b}z]dt + \sigma_z dW_z, \qquad (29\text{–B})$$

where $dW_x dW_z = \tilde{\rho}dt$. We would like to construct a two-factor tree that fits the current term structures of prices of default-free and defaultable bonds through numerical procedures. For example, the combined Hull-White and Black-Karasinski model corresponds to $x(r) = r$ and $z(s) = \ln s$.

We first briefly review some of the key steps in the Hull-White fitting procedure for the one-factor tree, and follow by generalization to the two-factor tree. In the one-factor Hull-White tree, we first build the trinomial tree for the process x^* with $x_0 = 0$ that corresponds to $\tilde{\theta}(t) = 0$, where:

$$dx^* = -\tilde{a}x^* dt + \sigma_x dW_x, \qquad x^*(0) = 0. \qquad (30\text{–A})$$

Note that $[x^*(t + \Delta t) - x^*(t)]$ is normally distributed with mean M_x and variance V_x as given by

$$M_x = -[1 - e^{-\tilde{a}\Delta t}]x^*(t)$$

and

$$V_x = \frac{\sigma_x^2}{2\tilde{a}}[1 - e^{-2\tilde{a}\Delta t}]. \qquad (30\text{–B})$$

We first construct a trinomial tree that simulates the process x^*. The node probabilities are obtained by equating the mean and variance of the discrete tree process and the continuous process as given by Equation (30-B).

A key step in the Hull-White fitting procedure is determination of the time-varying drift α_m at time t_m so that the lattice tree can price default-free bonds at different maturities as observed in the market at the current time. Let Y_m be the current yield of a default-free bond maturing at time t_m; $Q_{m,j}$ be the state price of node (m, j), which is the present value of a security that pays off \$ 1 if node (m, j) is reached and zero otherwise; and (m, j) indicates the node at time t_m at which $x = \alpha_m + j\Delta x$. Initially, we have $Q_{0,0} = 1$, $\alpha_0 = x(Y_1)$. Also, we define $r_{m,j} = x^{-1}(\alpha_m + j\Delta x)$, where x^{-1} denotes the inverse function of $x(r)$.

If we have obtained α_{m-1} and $Q_{m-1,j}$ for all values of j, the state price at time t_m is given by:

$$Q_{m,j} = \sum_{j^*} Q_{m-1,j^*} p_{j^*}^{j} e^{-r_{m-1,j^*}\Delta t}, \tag{31}$$

where $p_{j^*}^{j}$ is the node probability of moving from node x_{j^*} at time t_{m-1} to node x_j at time t_m, and the summation is taken over all values of j^* for which the nodes x_j^* are connected to the node x_j.

From the relation in Equation (2), we deduce that:

$$e^{-Y_{m+1}(m+1)\Delta t} = \sum_{j=-n_m^{(j)}}^{n_m^{(j)}} Q_{m,j} e^{-x^{-1}(\alpha_m + j\Delta x)\Delta t}, \tag{32}$$

where $n_m^{(j)}$ is the number of nodes on each side of the central node at time t_m. The time-varying drift α_m can then be determined numerically in a recursive manner.

When $x = r$, α_m can be solved explicitly by:

$$\alpha_m = \frac{\ln\left(\sum_{j=-n_m^{(j)}}^{n_m^{(j)}} Q_{m,j} e^{-j\Delta r \Delta t}\right)}{\Delta t} + (m+1)Y_{m+1}. \tag{33}$$

Moreover, the value of the time-dependent drift $\tilde{\theta}(t_m)$ can be estimated by:

$$\widehat{\theta}_m = \frac{\alpha_m - \alpha_{m-1}}{\Delta t} + \tilde{a}\alpha_m, \tag{34}$$

where $\tilde{\alpha}$ is the mean reversion rate of x [see Equation (29-A)].

In the combined two-factor tree that simulates the correlated evolution of r and s, there are nine branches emanating from the (m, j, k) node at $t_m = m\Delta t$, $x_j = j\Delta x$, and $z_k = k\Delta z$. We use the Hull-White approach of constructing probability matrices that give the node probabilities of moving from a node at t_m to the different nine nodes at t_{m+1}. We let β_m denote the time-varying drift for the discrete credit spread process. Similarly, we try to determine β_m numerically by fitting the current term structure of defaultable bond prices.

Let $Q_{m,j,k}$ denote the state price of the (m, j, k) node and $\tilde{Y}_m$ be the current yield of a defaultable bond maturing at time t_m. At initiation, $\beta_0 = z(\tilde{Y}_1 - Y_1)$; and the successive β_m, $m \geq 1$, are obtained recursively by the relationship:

$$e^{-\tilde{Y}_{m+1}(m+1)\Delta t} = \sum_{j=-n_m^{(j)}}^{n_m^{(j)}} \sum_{k=-n_m^{(k)}}^{n_m^{(k)}} Q_{m,j,k} e^{-(r_{m,j} + z^{-1}(\beta_m + k\Delta z))\Delta t}, \tag{35}$$

where $n_m^{(k)}$ is defined similarly as $n_m^{(j)}$. When $z(s) = s$, β_m can be solved explicitly by the formula:

$$\beta_m = \frac{\ln\left(\sum_{j=-n_m^{(j)}}^{n_m^{(j)}} \sum_{k=-n_m^{(k)}}^{n_m^{(k)}} Q_{m,j,k} e^{-(r_{m,j} + k\Delta s)\Delta t}\right)}{\Delta t} + (m+1)\tilde{Y}_{m+1}. \tag{36}$$

Once β_m is known, the value of $\tilde{\phi}(t_m)$ can be estimated by:

$$\widehat{\phi}_m = \frac{\beta_m - \beta_{m-1}}{\Delta t} + \tilde{b}\beta_m, \tag{37}$$

where $\tilde{b}$ is the mean reversion rate of z [see Equation (29-B)].

When the state prices are known at the nodes, the value of any European-style credit derivative at time t_0 whose value depends on r and s can be obtained by:

$$p_{sp}(r_0, s_0, t_0) = \sum_j \sum_k Q_{M,j,k} p_{sp}(r_{M,j}, s_{M,k}, t_M), \tag{38}$$

where $p_{sp}(r_{M,j}, s_{M,k}, t_M)$ is the terminal payoff at maturity date t_M, and the summation is taken over all terminal nodes.

It is quite computationally demanding to calculate the state prices since this two-factor algorithm requires a search process at each node. When $x(r) = r$ and $z(s) = s$, one can derive explicit formulas for the time drifts α_m and β_m without calculating the state prices. Further, when the terminal payoff of a credit derivative is independent of the interest rate, we can develop a one-factor fitting algorithm via the use of valuation in the forward-adjusted risk-neutral measure. This reduced one-factor fitting algorithm can be considered an extension of the Grant-Vora [2001] approach to derive an explicit analytic expression for the time drift function of the one-factor Hull-White model.

One-Factor Tree for Joint Extended Vasicek Processes

The no-arbitrage fitting procedure developed by Grant and Vora [2001] is indeed the discrete analogue of the analytic procedure for deriving the time-dependent drift $\theta(t)$. The discrete analogue of the extended Vasicek process for r as defined in Equation (1-A) can be represented by:

$$\Delta r_m = r_{m+1} - r_m = M_r r_m + \delta_m + \sqrt{V_r} Z_m, \quad (39)$$

where the mean M_r and variance V_r are defined much the same as in Equation (30-B), Z_m is a standard normal variable, and δ_m is an adjustment parameter for the no-arbitrage fitting procedure. Note that α_m is equal to $E^{\mathbf{P}}[r_m]$.

Deductively, we obtain:

$$r_m = r_0(1+M_r)^m + \sum_{j=0}^{m-1}(1+M_r)^{m-1-j}\delta_j + \sum_{j=0}^{m-1}\sqrt{V_r}(1+M_r)^{m-1-j}Z_j, \quad (40)$$

where the normal variables Z_j, $j = 0, \ldots, m-1$, are independently and identically distributed.

From Equation (39), one can deduce that:

$$\delta_m = E^{\mathbf{P}}[r_{m+1}] - (1+M_r)E^{\mathbf{P}}[r_m]. \quad (41)$$

Let $\gamma_m^2 \Delta t$ be the variance of $\sum_{j=0}^{m} r_j \Delta t$. It can be shown that

$$\gamma_m^2 = \frac{\sigma_r^2}{2a(1-e^{-a\Delta t})^2}[e^{-2a\Delta t}(1-e^{-2am\Delta t}) - 2e^{-a\Delta t}(1-e^{-am\Delta t})(1+e^{-a\Delta t}) + m(1-e^{-2a\Delta t})]. \quad (42)$$

Since the price at time t_0 of a default-free bond maturing at time t_{m+1} is given by:

$$P(t_0, t_{m+1}) = E^{\mathbf{P}}\left[\exp\left(-\sum_{j=0}^{m} r_j \Delta t\right)\right] \quad (43\text{-A})$$

so that

$$E^{\mathbf{P}}[r_m] = -\frac{\ln P(t_0, t_{m+1})}{\Delta t} - \sum_{j=0}^{m-1} E^{\mathbf{P}}[r_j] + \frac{\gamma_m^2}{2}, \quad (43\text{-B})$$

we then have

$$E^{\mathbf{P}}[r_m] = -\frac{1}{\Delta t}\ln\frac{P(t_0, t_{m+1})}{P(t_0, t_m)} + \frac{\gamma_m^2 - \gamma_{m-1}^2}{2}, \quad (44)$$

which gives the time-varying drift α_m.

As in the continuous model, we can reduce the two-factor algorithm to a one-factor version by choosing the default-free bond price as the numeraire. First, we would like to find the time-varying drift β_m of the spot spread process under the risk-neutral measure $\mathbf{P}$ and the forward measure $\mathbf{Q^T}$. Note from Equations (1-B) and (8) that the two measures are related by

$$E^{\mathbf{Q^T}}[s(t_m)] = E^{\mathbf{P}}[s(t_m)] + \mathcal{X}_{0,m,M}, \quad (45)$$

where the last term

$$\mathcal{X}_{j,m,M} = \int_{t_j}^{t_m} \rho\sigma_s\sigma_P(u, t_M)e^{-b(t_m-u)}\, du \quad (46)$$

represents the drift adjustment required for effecting the change of measure.

Using the bond volatility $\sigma_P(t, T)$ defined in Equation (6), and assuming constant values for ρ and σ_s, we have

$$\mathcal{X}_{j,m,M} = \frac{\rho\sigma_s\sigma_r}{a}\left[\frac{e^{-a(t_M-t_m)} - e^{-at_M-bt_m+(a+b)t_j}}{a+b} - \frac{1-e^{-b(t_m-t_j)}}{b}\right] \tag{47}$$

The discrete analogue of the extended Vasicek process followed by s can be represented by

$$\Delta s_m = s_{m+1} - s_m = M_s s_m + \xi_m + \mathcal{X}_{m,m+1,M} + \sqrt{V_s}Z_m, \tag{48}$$

where M_s, V_s, and ξ_m are defined in an analogous manner as in Equation (39). Similarly, the adjustment parameter ξ_m is determined by fitting the term structure of defaultable bond prices.

Solution of the recursive relation (48) gives

$$s_m = s_0(1+M_s)^m + \sum_{j=0}^{m-1}(1+M_s)^{m-1-j}(\xi_j + \mathcal{X}_{j,j+1,M}) + \sum_{j=0}^{m-1}\sqrt{V_s}(1+M_s)^{m-1-j}Z_j. \tag{49}$$

The discrete version of Equation (7) is seen to be

$$D(t_0,t_m) = P(t_0,t_m)E^{\mathbf{Q}^{t_m}}\left[\exp\left(-\sum_{j=0}^{m-1}s_j\Delta t\right)\right] \tag{50}$$

from which we deduce the formula for $E^{\mathbf{P}}[s_m]$ (this is equal to β_m, the time-varying drift for the credit spread process):

$$E^{\mathbf{P}}[s_m] = -\frac{1}{\Delta t}\ln\frac{D(t_0,t_{m+1})}{P(t_0,t_{m+1})} - \sum_{j=0}^{m-1}E^P[s_j] - \sum_{j=0}^{m-1}\frac{\mathcal{X}_{j,j+1,m}}{M_s}[(1+M_s)^{m-j}-1] + \frac{\widetilde{\gamma}_m^2}{2}, \tag{51}$$

where $\widetilde{\gamma}_m^2$, is defined similarly as γ_m^2, except that r_m, a, and σ_r are replaced by s_m, b, and σ_s, respectively.

Under the forward measure, the time t_0 price of a credit spread derivative is given by:

$$p_{sp}(r_0,s_0,t_0) = P(t_0,t_M)\sum_k U_{M,k}p_{sp}(s_{M,k},t_M), \tag{52}$$

where $p_{sp}(s_{M,k}, t_M)$ is the terminal payoff on maturity date t_M, and $U_{M,k}$ is the cumulative probability under the forward measure reaching the (M, k) node from the starting (0, 0) node. Note that $U_{0,0} = 1$, $s_{m,k} = \beta_m + k\Delta s$, and $\beta_m = \xi_m + \chi_{0,m,m}$.

III. NUMERICAL RESULTS

We implement the two-factor and one-factor fitting algorithms using current term structures of default-free and defaultable yield curves so as to check the accuracy and the effectiveness of the numerical algorithms against the analytic price formulas. In our numerical experiments, we generate the current yield of the default-free Treasury bond by:

$$Y(T) = 0.08 - 0.05e^{-1.8T}. \tag{53}$$

We use the Briys and de Varenne [1997] intertemporal default model to generate the yield for a high-rated bond and a low-rated bond. The spread curves are obtained by subtracting the yield of the default-free Treasury bonds from the yield curves of the defaultable bonds. Note that the yield curves of the default-free Treasury bond and the high-rated bond increase monotonically with time to maturity, while the yield curve of the low-rated bond exhibits the typical hump shape.

These yield curves (shown in Exhibit 1) are used as input in all subsequent calculations.

In Exhibits 2-A and 2-B, we plot the time-dependent drift function $\phi(t)$ in the mean reversion of the spot spread process, obtained by fitting the yield curves of the high-rated bond and the low-rated bond, respectively. The drift $\phi(t)$ is shown to be an increasing function of the spot spread volatility σ_s for both high-rated and low-rated bonds.

Since the spread approaches zero as time approaches maturity, the drift goes to zero as t tends to zero. The drift functions for both high-rated and low-rated bonds reach some peak value, then decline at a higher value of t and settle at some asymptotic value at infinite t.

The drift is greater for the low-rated bond since the corresponding spot spread has a higher value. Also, the drift shows less sensitivity to the spot spread volatility for the low-rated bond.

EXHIBIT 1
Yield Curve Plots

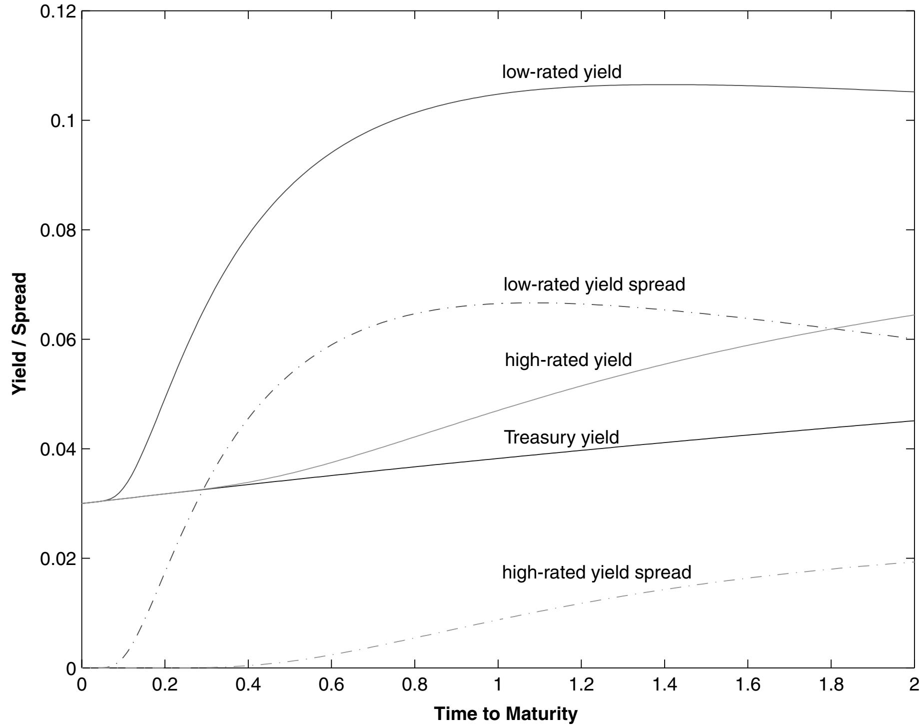


EXHIBIT 2-A
Time-Dependent Drift Function for High-Rated Bond

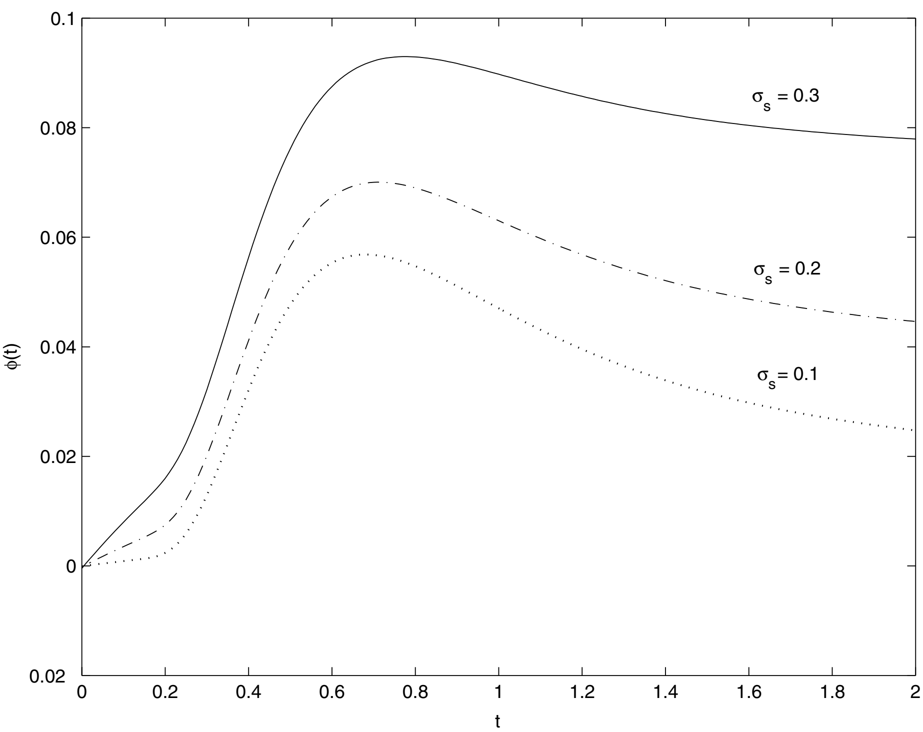


Exhibit 2-B

Time-Dependent Drift Function for Low-Rated Bond

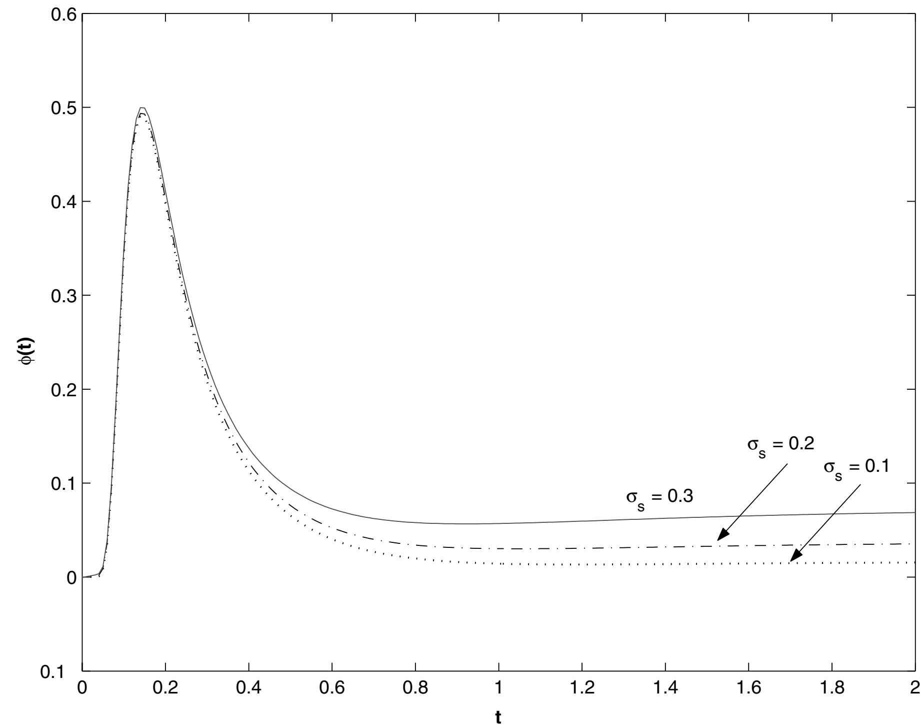


Exhibit 3

Comparison of One-Factor and Two-Factor Tree Calculations

Number of Time Steps	Average Relative Errors in Two-Factor Tree Calculations	Average Relative Errors in One-Factor Tree Calculations
8	3.18%	2.92%
16	1.175%	1.63%
32	1.10%	1.03%

To test the convergence of the two-factor and one-factor fitting algorithms, we apply the algorithms to price a credit spread put with payoff as given in Equation (12). We use different combinations of parameter values in the pricing model, and compare the numerical results with the results obtained from the analytic price formula in Equation (16) with varying numbers of time steps.

Exhibit 3 shows the convergence behaviors of the two fitting algorithms in terms of average relative errors (in percentages). The relative errors have relatively low values that decline in a roughly linear manner with increasing numbers of time steps.

The one-factor algorithm provides better accuracy. In terms of computational complexity, the one-factor algorithm is $O(M^2)$, while the two-factor algorithm is $O(M^3)$, where M is the number of time steps. This property of a polynomial order of complexity is highly desirable, as numerical implementation does not take an unreasonable amount of computation time.

We also investigate the pricing behavior of the credit spread put option with respect to time to expiration and spot spread volatility. In Exhibit 4-A, we plot the spread put option function against time to expiration with varying spot spread volatility where the underlying is a high-rated bond. The spread put price increases with longer time to expiration and higher spot spread volatility.

When the underlying is a low-rated bond, however, the time-dependent behavior of the spread put price exhibits a humped shape (*see Exhibit 4-B*). This may be due to a declining credit spread at very long maturities for

EXHIBIT 4-A
Price of Credit Spread Put Against Time to Expiration—High-Rated Bond

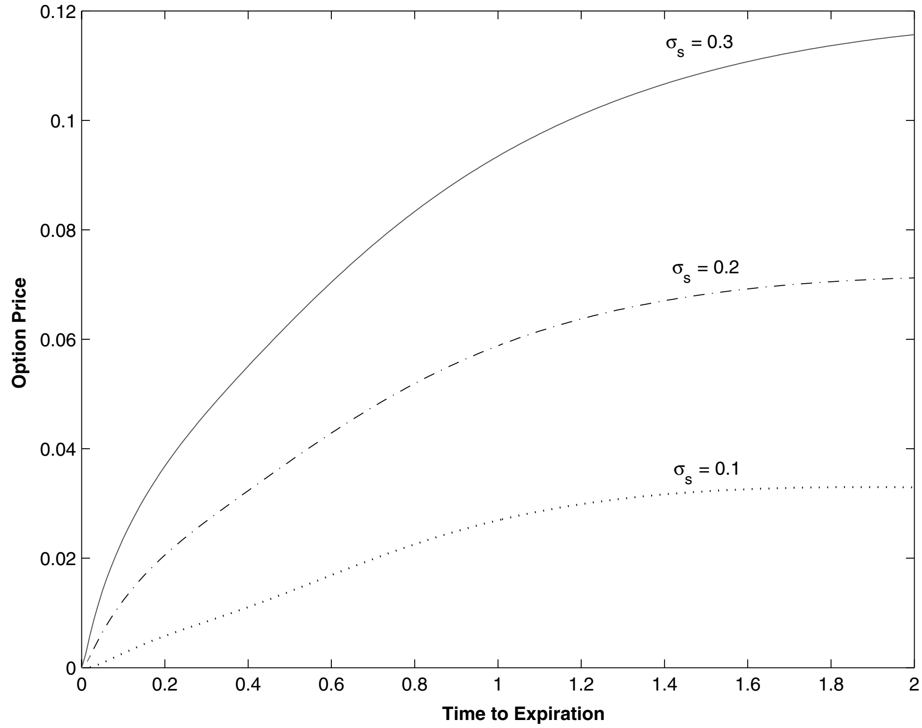
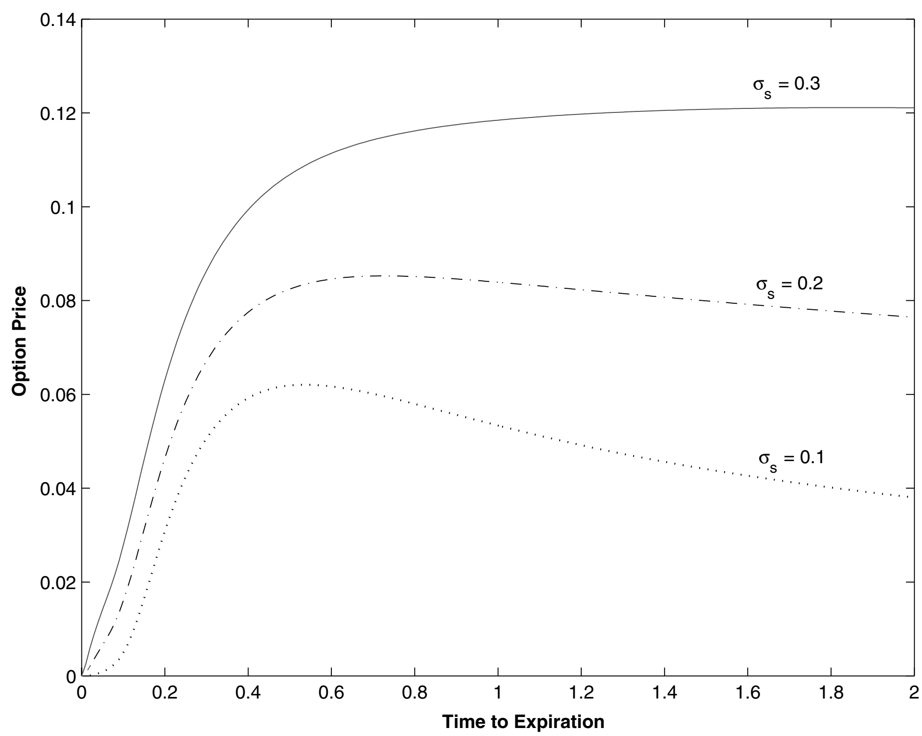


EXHIBIT 4-B
Price of Credit Spread Put Against Time to Expiration—Low-Rated Bond



low-rated bonds. Spread put prices are always seen to be increasing functions of volatility, regardless of the credit quality of the underlying bond.

IV. CONCLUSION

The credit spread is a common state variable in pricing models for credit spread derivatives. By assuming that the instantaneous spot riskless interest rate and the credit spread follow correlated stochastic processes, we have developed a no-arbitrage approach for pricing credit spread derivatives.

Our pricing model assumes the riskless interest rate follows a mean-reverting process with time-dependent drift, and the credit spread (or its logarithm) follows a similar process. The time-dependent drift functions in the mean reversion processes are determined by fitting current term structures of default-free and defaultable bond prices.

When both the spot rate and the spot spread follow an extended Vasicek process, the time-dependent drift functions can be obtained in closed form. Also, we take an expectation under the risk-adjusted forward measure to derive analytic price formulas for credit spread derivatives.

When the logarithm of the spread follows the extended Vasicek process, the Black-Karasinki model does not exhibit the same level of analytic tractability, but one can always rely on numerical fitting algorithms to determine the drift term and compute the price of the credit spread option. We also show that the two-factor extended Vasicek model is a special case of the Heath-Jarrow-Morton model corresponding to some specific choice of the volatility structures.

By following the Hull-White tree-fitting algorithm, we have developed two-factor and one-factor fitting algorithms for pricing credit spread derivatives. Since our trinomial tree construction has recombining properties, a polynomial order of computational complexity is observed in our numerical algorithm. The polynomial order property is an overriding advantage over the Heath-Jarrow-Morton (HJM) framework, since the non-recombining tree in the HJM model leads to an exponential order of complexity.

The validity of our analytic price formulas and the versatility of the numerical schemes are verified through numerical comparison of the computed results and the results obtained by analytic formulas. The algorithms are numerically accurate even with relatively few time steps.

Assuming that both the spot rate and the spot spread follow extended Vasicek processes, we observe that the price of the credit spread put is an increasing function of the volatility of the spread process. This result holds for both high-rated and low-rated underlying risky bonds. When spread volatility is relatively low and time to expiration is relatively long, the credit spread put price function may become a decreasing function of time to expiration for lower-rated underlying bonds. This property contrasts with the behavior of most other equity option products, but it is consistent with the decline of the spread of a long-term lower-rated bond under low spread volatility.

Our work extends pricing models for credit derivatives in several respects. Under the assumption of extended Vasicek processes for the underlying state variables, one can obtain analytic price formulas for credit spread derivatives and closed-form representations of the time-dependent drift terms using a no-arbitrage approach that fits the current term structures of bond prices. Our one-factor and two-factor fitting algorithms for pricing credit spread derivatives exhibit only a polynomial order of computational complexity. And we show how to choose volatility structures so that the Heath-Jarrow-Morton model reduces to the Hull and White model.

REFERENCES

Black, F., and P. Karasinski. "Bond and Option Pricing when the Short Rates are Lognormal." *Financial Analysts Journal*, July/August 1991, pp. 52-59.

Briys, E., and F. de Varenne. "Valuing Risky Fixed Rate Debt: An Extension." *Journal of Financial and Quantitative Analysis*, 32 (1997), pp. 239-248.

Chacko, G., and S. Das. "Pricing Interest Rate Derivatives: A General Approach." *Review of Financial Studies*, 15 (2002), pp. 195-241.

Cox, J.C., J.E. Ingersoll, Jr., and S.A. Ross. "A Theory of the Term Structure of Interest Rates." *Econometrica,* 53 (1985), pp. 385-407.

Das, S.R. "Credit Risk Derivatives." *The Journal of Derivatives*, 2, 3 (Spring 1995), pp. 7-23.

Das, S.R., and R.K. Sundaram. "A Discrete-Time Approach to Arbitrage-Free Pricing of Credit Derivatives." *Management Science*, 46 (2000), pp. 46-62.

Duffie, D., and K. Singleton. "Modeling Term Structures of Defaultable Bonds." *Review of Financial Studies*, 12 (1999), pp. 687-720.

Garcia, J., H. van Gindersen, and R. Garcia. "On the Pricing of Credit Spread Options: A Two-Factor HW-BK Algorithm." Working paper, Artesia BC and University of California at Berkeley, 2001.

Grant, D. and G. Vora. "An Analytical Implementation of the Hull and White Model." *The Journal of Derivatives*, Winter 2001, pp. 54-60.

Heath, D., R. Jarrow, and A. Morton. "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation." *Econometrica,* 60 (1992), pp. 77-105.

Hull, J., and A. White. "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models." *The Journal of Derivatives*, Winter 1994, pp. 37-48.

——. "Pricing Interest-Rate-Derivative Securities." *Review of Financial Studies*, 3, 4 (1990), pp. 573-592.

Jarrow, R., D. Lando, and S.M. Turnbull. "A Markov Model for the Term Structure of Credit Risk Spreads." *Review of Financial Studies*, 10 (1997), pp. 481-523.

Kijima, M., and K. Komoribayashi. "A Markov Chain Model for Valuing Credit Risk Derivatives." *The Journal of Derivatives*, Fall 1998, pp. 97-108.

Longstaff, F.A., and E.S. Schwartz. "Valuing Credit Derivatives." *The Journal of Fixed Income*, June 1995, pp. 6-12.

Mougeot, N. "Credit Spread Specification and the Pricing of Spread Options." Working paper, Ecole des HEC, 2000.

Schönbucher, P.J. "Term Structure Modeling of Defaultable Bonds." *Review of Derivatives Research*, 2 (1998), pp. 161-192.

——. "A Tree Implementation of a Credit Spread Model for Credit Derivatives." Working paper, Bonn University, 1999.

Tahani, N. "Estimating and Valuing Credit Spread Options with GARCH Models." Working paper, HEC Montreal, 2000.

Vasicek, O. "An Equilibrium Characterization of the Term Structure." *Journal of Financial Economics,* 5 (1977), pp. 177-188.