CHARACTERIZATION OF OPTIMAL STOPPING REGIONS OF AMERICAN PATH DEPENDENT OPTIONS

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A general framework is developed to analyze the optimal stopping (exercise) regions of American path dependent options with either Asian feature or lookback feature. We examine the monotone properties of the option values and stopping regions with respect to the interest rate, dividend yield and time. From the ordering properties of the values of American lookback options and American Asian options, we deduce the corresponding nesting relations between the exercise regions of these American options. We illustrate how some properties of the exercise regions of the American Asian options can be inferred from those of the American lookback options.

Key Words: American options, optimal stopping, Asian feature, lookback feature, monotone properties

1 INTRODUCTION

The path dependence in an option payoff is signified by the dependence of the payoff function on the path history of the underlying asset price. The two most common path dependent features involve either taking some form of averaging or recording the extremum value of the asset price path. Options with averaging payoff are called Asian options while those options whose payoff depends on some extremum value are called lookback options. The American feature entitles the right to the option holders to exercise the option prematurely. These added contractual features help option holders
capitalize their view on the future movement of the underlying asset price or hedge against certain type of risk exposure.

The price function of an American path dependent option is known to be governed by parabolic variational inequalities. Besides option value, the solution of the variational inequalities involves the determination of the time dependent free boundary at which optimal stopping occurs, that is, the holder should exercise the option optimally at some threshold asset price level. The domain of the option pricing model is divided into two regions by the optimal stopping boundary, namely, the continuation region where the holder should continue to hold the option and the stopping region (exercise region) where the option should be optimally exercised. In this paper, we develop a general framework using partial differential equation theory to analyze the monotone properties of the price functions and the optimal exercise policies of American path dependent options. In particular, we examine the characterization of the exercise regions of American options with different path dependent payoffs.

There have been numerous papers on the valuation of American Asian options (Wu et al., 1999; Hansen and Jorgensen, 2000; Ben-Ameur et al., 2002; Marcozzi, 2003; Wu and Fu, 2003) and American lookback options (Dai, 2001; Yu et al., 2001; Lai and Lim, 2004; Dai and Kwok, 2005). Most of these papers deal with the construction of algorithms for numerical valuation of the values and exercise boundaries of the American path dependent option models. The numerical approaches range from the enhanced lattice tree methods, simulation based algorithms, iterative solution of integral equation and Bernoulli random walk approximation. In addition, a collection of interesting results on the analytic properties of the values and exercise boundaries have been reported. For example, Hansen and Jorgensen (2000) observe the non-monotone property of the exercise boundaries of American Asian call and put options with respect to the calendar time. Wu and Fu (2003) prove the convexity property of the exercise boundary under certain assumption of the asset price process.

As the averaging payoff structure looks very differently from the lookback payoff, it appears that there is little resemblance between the exercise policies of American lookback options and American Asian options. The objective of our work is to provide a general framework of deriving analytic properties of the option value and exercise region of an American option with either averaging or lookback payoff. We show how some analytic properties of the exercise regions of the American Asian options can be inferred from those of the American lookback counterparts. This paper is organized as follows. In
the next section, we present the setup of our pricing models and notations. We then consider the existence properties of the exercise regions of American path dependent options and the nesting relations between the option values and exercise regions. Some analytic properties of the optimal stopping boundaries of the American floating strike and fixed strike Asian options are derived. In Section 3, we provide theoretical proofs for the monotone properties of the option values and exercise boundaries of American path dependent options with respect to the calendar time, interest rate and dividend yield. Sample calculations of the values and exercise regions of American path dependent options are performed to verify the monotone properties. The last section presents our conclusive remarks on the characterization of optimal stopping policies of American path dependent options.

2 EXERCISE REGIONS OF AMERICAN PATH DEPENDENT OPTIONS

We consider the usual Black Scholes economy with a risky asset and a money market account. We assume the existence of a risk neutral probability $Q$ under which discounted asset prices are martingales, implying non-existence of arbitrage. Under the pricing measure $Q$, the asset price is assumed to follow

$$\frac{dS_t}{S_t} = (r - q) \, dt + \sigma \, dZ_t,$$

(2.1)

where $r$ and $q$ are the constant riskless interest rate and dividend yield, respectively, $\sigma$ is the constant instantaneous volatility of asset return and $Z$ is a $Q$-Brownian motion. Let $t$ and $T$ denote the current time and date of expiration of the option, respectively. The path dependent variable $J_t$ may take some form of averaging or record an extremum value of the asset price path over the time period $[0, t]$. We assume that the option’s path dependent payoff structure depends on $J_t$ and possibly $S_t$ also. Let $\Lambda(S_t, J_t; K)$ denote the payoff function of an American path dependent option. The common forms of $\Lambda(S_t, J_t; K)$ are

$$\Lambda(S_t, J_t; K) = \begin{cases} 
(S_t - J_t)^+ & \text{floating call} \\
(J_t - S_t)^+ & \text{floating put} \\
(J_t - K)^+ & \text{fixed call} \\
(K - J_t)^+ & \text{fixed put}
\end{cases},$$

(2.2)
where $K$ is the fixed strike and

$$x^+ = \begin{cases} 
x & \text{if } x \geq 0 \\
0 & \text{if } x < 0 
\end{cases}.$$

Suppose continuous monitoring of the asset price path is assumed, we define

(i) arithmetic averaging $A_t$

$$A_t = \frac{1}{t} \int_0^t S_u \, du$$

(ii) minimum value $m_t$ and maximum value $M_t$

$$m_t = \min_{u \in [0,t]} S_u \quad \text{and} \quad M_t = \max_{u \in [0,t]} S_u.$$ (2.3b)

By choosing $J_t$ to be either $A_t$, $m_t$ or $M_t$, a wide variety of path dependent option payoffs can be structured. Some examples are

(i) floating strike lookback call

$$\Lambda(S_t, J_t; K) = (S_t - J_t)^+, \quad \text{where } J_t = m_t;$$

(ii) fixed strike lookback put

$$\Lambda(S_t, J_t; K) = (K - J_t)^+, \quad \text{where } J_t = m_t;$$

(iii) floating strike Asian put

$$\Lambda(S_t, J_t; K) = (J_t - S_t)^+, \quad \text{where } J_t = A_t;$$

(iv) fixed strike Asian call

$$\Lambda(S_t, J_t; K) = (J_t - K)^+, \quad \text{where } J_t = A_t.$$

Consider an American path dependent option with payoff function $\Lambda(S_t, J_t; K)$, its no-arbitrage price is given by

$$V(S, J, t) = \sup_{t^* \in T_{t,T}} E_Q[e^{-r(t^* - t)}\Lambda(S_{t^*}, J_{t^*}; K)|S_t = S, J_t = J],$$

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where $E_Q$ denotes the expectation under the risk neutral measure $Q$, $t^*$ is the optimal stopping time and $\mathcal{T}_{t,T}$ is the set of stopping times taking values between $t$ and $T$. The price function $V(S,J,t)$ is defined in the domain $D \times [0,T]$, where $D$ takes the form

$$D = \begin{cases} 
\{0 < S < \infty, 0 < J < \infty\} & \text{for Asian option} \\
\{0 < S \leq J < \infty\} & \text{for lookback maximum option} \\
\{0 < J \leq S < \infty\} & \text{for lookback minimum option}
\end{cases}$$

The variational inequalities that govern $V(S,J,t)$ are given by

$$\frac{\partial V}{\partial t} + \mathcal{L}V \geq 0 \quad \text{in} \quad D \times (0,T), \quad V \geq \Lambda \quad \text{on} \quad \overline{D} \times [0,T),$$

$$\left(\frac{\partial V}{\partial t} + \mathcal{L}V\right)(V - \Lambda) = 0 \quad \text{in} \quad D \times (0,T), \quad (2.4)$$

with terminal payoff

$$V(S,J,T) = \Lambda(S,J) \quad \text{in} \quad D.$$ 

The differential operator $\mathcal{L}$ takes the form

$$\mathcal{L} = \begin{cases} 
\frac{\sigma^2}{2}S^2 \frac{\partial^2}{\partial S^2} + (r - q)S \frac{\partial}{\partial S} + \frac{S - J}{t} \frac{\partial}{\partial J} - r & \text{for Asian option} \\
\frac{\sigma^2}{2}S^2 \frac{\partial^2}{\partial S^2} + (r - q)S \frac{\partial}{\partial S} - r & \text{for lookback option}
\end{cases}$$

In the exercise region $E$, the price function equals the payoff so that

$$E = \{(S,J,t) \in D \times [0,T) : \ V(S,J,t) = \Lambda(S,J)\}. \quad (2.5a)$$

To characterize the exercise region more precisely, we define $E^t$ to be the exercise region at time $t$, where

$$E^t = \{(S,J) \in D : \ V(S,J,t) = \Lambda(S,J)\}, \quad t \in [0,T). \quad (2.5b)$$

Even when the exercise region $E$ is non-empty, $E^t$ can be empty for some value of $t$. For notational convenience, we denote the exercise region at time $t$ of an American floating strike lookback call and an American fixed strike Asian put by $E^t_{L,fc}$ and $E^t_{A, xp}$, respectively, and similar notations for other types of American path dependent options.
For finite-lived American lookback options, the characterization of the exercise region $E_t^L$ has been analyzed in our earlier paper (Dai and Kwok, 2005). The main results on the existence of $E_t^L$ are summarized in Lemma 1.

**Lemma 1**
Let $E_t^L$ denote the exercise region of a finite-lived American lookback option.

(i) When $q \neq 0$, $E_t^L \neq \emptyset$.

(ii) When $q = 0$, $E_t^L = \emptyset$ for the American floating strike lookback call but $E_t^L \neq \emptyset$ for all other types of American lookback options.

The financial intuition behind the forfeiture of the early exercise right in an American floating strike lookback call when $q = 0$ is quite obvious. It is well known that it is never optimal to exercise an usual American call option when the underlying asset is non-dividend paying. The American floating strike lookback call has the additional privilege of resetting the strike to the newly realized minimum value of the asset price, so $E_{t,fc}^L = \emptyset$ when $q = 0$.

**Ordering properties of price functions**
The price functions of the American lookback option and its Asian option counterpart can be deduced to satisfy some ordering properties. Let $V_{L,fc}(S, J, t)$ [$V_{A,fc}(S, J, t)$] denote the price function of the American floating strike lookback (Asian) call option, where

$$
V_{L,fc}(S, J, t) = \sup_{t^* \in \mathcal{T}_{t,T}} E_Q[e^{-r(t-t^*)}(S - m_{t^*})|m_t = J, S_t = S]
$$

$$
V_{A,fc}(S, J, t) = \sup_{t^* \in \mathcal{T}_{t,T}} E_Q[e^{-r(t-t^*)}(S - A_{t^*})^+|A_t = J, S_t = S].
$$

The following ordering property [for its proof, see Jiang and Dai’s paper (2004)]

$$
V_{A,fc}(S, J, t) \leq V_{L,fc}(S, J, t), \quad \text{for } S \geq J,
$$

can be deduced intuitively from the property that

$$
m_{t^*} \leq A_{t^*} \quad \text{when } m_t = A_t = J, t^* \geq t.
$$

In a similar manner, we can deduce the following ordering properties of the price functions:

$$
V_{A,fp}(S, J, t) \leq V_{L,fp}(S, J, t) \quad \text{for } S \leq J;
$$

$$
V_{A,xc}(S, J, t) \leq V_{L,xc}(S, J, t) \quad \text{for } S \leq J;
$$

$$
V_{A,xp}(S, J, t) \leq V_{L,xp}(S, J, t) \quad \text{for } S \geq J.
$$
Nesting relations of the exercise regions
Based on the above ordering properties of the price functions of the American lookback options and their Asian option counterparts, we can deduce the corresponding nesting relations of the exercise regions of these options. The results are summarized in Lemma 2.

Lemma 2
The exercise regions of various types of American lookback options and their Asian option counterparts observe the following nesting relations:

\[ E_{A,fc}^t \supseteq E_{L,fc}^t \]
\[ E_{A,fp}^t \supseteq E_{L,fp}^t \]
\[ E_{A,xc}^t \supseteq E_{L,xc}^t \]
\[ E_{A,xp}^t \supseteq E_{L,xp}^t. \] (2.6)

The validity of the nesting relations in Lemma 2 can be shown by financial argument as follows. For a given set of values of \( S_t \) and \( J_t \) at time \( t \), suppose it is optimal to exercise an American floating strike lookback call, then it must be optimal to exercise its Asian option counterpart. This is because the exercise payoffs are the same for both options, and the Asian call has a lower value if it stays unexercised. Such policy of continued holding of the Asian call is non-optimal.

Existence of exercise regions
From the results in Lemmas 1 and 2, we can deduce that \( E_A^t \neq \phi \) when \( q \neq 0 \). Also, when \( q = 0 \), \( E_A^t \neq \phi \) for American Asian options other than the floating strike call. We shall show by theoretical argument that the exercise boundary of an American floating strike Asian call option exists at least for times close to expiry, so we can claim that \( E_{A,fc}^t \neq \phi \) though it is not sufficient to claim conclusively that \( E_{A,fc}^t \neq \phi \) for all \( t \in [0, T) \). We performed extensive numerical experiments to search for the possibility of \( E_{A,fc}^t \) being empty. So far, our sample calculations have always revealed the non-emptiness of \( E_{A,fc}^t \) (see Figure 6).

Optimal stopping boundaries of floating strike Asian options
For Asian options with floating strike payoff structure, we can achieve dimension reduction of the pricing model by defining the following similarity variables:

\[ x = A/S \quad \text{and} \quad W(x, \tau) = V(S, A, t)/S \quad \text{where} \quad \tau = T - t. \]
The variational inequalities for an American floating strike Asian option can be reformulated as

\[
\frac{\partial W}{\partial \tau} - \frac{\sigma^2}{2} x^2 \frac{\partial^2 W}{\partial x^2} + \left[ (r - q)x - \frac{1 - x}{T - \tau} \right] \frac{\partial W}{\partial x} + qW \geq 0
\]

in \((0, \infty) \times (0, T)\)

\[
W \geq \eta(1 - x)
\]

in \([0, \infty) \times [0, T]\)

\[
\left\{ \frac{\partial W}{\partial \tau} - \frac{\sigma^2}{2} x^2 \frac{\partial^2 W}{\partial x^2} + \left[ (r - q)x - \frac{1 - x}{T - \tau} \right] \frac{\partial W}{\partial x} + qW \right\} [W - \eta(1 - x)] = 0
\]

in \((0, \infty) \times (0, T)\) (2.7)

with auxiliary condition: \(W(x, 0) = [\eta(1 - x)]^+\). Here, \(\eta = 1\) for the call and \(\eta = -1\) for the put. The \(x\)-derivative of the obstacle function \(\eta(1 - x)\) in the above parabolic variational inequalities is negative (positive) for the call (put). By applying the comparison principle, we deduce that

(i) \(\frac{\partial W}{\partial x} \leq 0\) for an American floating strike Asian call, \quad (2.8a)

(ii) \(\frac{\partial W}{\partial x} \geq 0\) for an American floating strike Asian put. \quad (2.8b)

Let \(x^*(\tau)\) denote the optimal stopping boundary associated with the parabolic variational inequalities (2.7). We would like to show that the optimal stopping region of an American floating strike call (put) lies on the left (right) hand side of the optimal stopping boundary. Also, we examine the behavior of \(x^*(\tau)\) as \(\tau \to 0^+\). The results are summarized in the following lemmas.

Lemma 3
For the American floating strike Asian call, the optimal stopping region is

\[
E_{A,fc} = \{ (x, \tau) : x \leq x^*(\tau), \tau > 0 \} \quad (2.9a)
\]

while that of the put option counterpart is

\[
E_{A,fp} = \{ (x, \tau) : x \geq x^*(\tau), \tau > 0 \}. \quad (2.9b)
\]

Remark
The American floating strike lookback call and put options possess similar properties of the optimal stopping regions (see Dai and Kwok, 2005).

**Proposition 4**

Let \( x^*(0^+) \) denote the value of the optimal stopping boundary at time close to expiry. We have

(i) floating strike Asian call

\[
\begin{align*}
x^*(0^+) &= \min \left( \frac{1 + qt}{1 + rt}, 1 \right); \\
\end{align*}
\]

(ii) floating strike Asian put

\[
\begin{align*}
x^*(0^+) &= \max \left( \frac{1 + qt}{1 + rt}, 1 \right).
\end{align*}
\]

**Remark**

In terms of the similarity variable \( x = J/S \), where \( J \) can be the realized maximum value \( M \) or realized minimum value \( m \), the asymptotic behavior of the optimal stopping boundary at time close to expiry for the American floating strike lookback call and put are \( x^*(0^+) = \min \left( \frac{q}{r}, 1 \right) \) and \( x^*(0^+) = \max \left( \frac{q}{r}, 1 \right) \), respectively (see Dai and Kwok, 2005).

The proof of Lemmas 3 and 4 are presented in the Appendix.

**Optimal stopping boundaries of fixed strike Asian options**

Consider the American fixed strike Asian call option whose payoff is \((A - K)^+\).

In the exercise region, the option value \( V \) satisfies \( V = A - K \) and

\[
\frac{\partial V}{\partial \tau} - \frac{S - A}{T - \tau} \frac{\partial V}{\partial A} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - (r - q) S \frac{\partial V}{\partial S} + rV \geq 0.
\]

In addition, we have \( \frac{\partial V}{\partial S} \geq 0 \). Substituting \( V = A - K \) into the above inequality, we obtain

\[
A - S + rTA - rTK \geq 0 \quad \text{or} \quad A > \frac{S + rTK}{1 + rT}.
\]

(2.11)
This would imply that any point \((S, A, t)\) inside the exercise region must observe the above inequality. Now, for a given value of \(\tau\), suppose \(V(S_1, A, \tau) > A - X\), then \(V(S_2, A, \tau) > A - X\) for \(S_2 > S_1\). The surface of the exercise region can then be parameterized by \(A^*(S, \tau)\), provided that the exercise region exists. Also, we can derive some interesting properties on \(A^*(S, 0^+)\), \(A^*(0^+, \tau)\) and \(\lim_{S \to \infty} \frac{A^*(S, \tau; K)}{S}\). For the American fixed strike Asian call option, these properties are summarized in Proposition 5.

**Proposition 5**

Let \(A^*(S, \tau)\) represent the exercise boundary surface of an American fixed strike Asian call option. The asymptotic properties of \(A^*(S, \tau)\) at \(\tau \to 0^+, S \to 0^+\) and \(S \to \infty\) are given by

\[
A^*(S, 0^+) = \max \left( K, \frac{S + rTK}{1 + rT} \right), \quad A^*(0^+, \tau) = K, \tag{2.12a}
\]

and

\[
\lim_{S \to \infty} \frac{A^*(S, \tau; K)}{S} = \frac{A^*(S, \tau; 0)}{S} = \xi(\tau), \tag{2.12b}
\]

where \(\xi(\tau)\) is some function of \(\tau\).

**Remarks**

1. The exercise boundary surface \(M^*(S, \tau)\) of the American fixed strike lookback call option with payoff function \((M - K)^+\) possesses similar asymptotic behaviors at \(\tau \to 0^+, S \to 0^+\) and \(S \to \infty\) (see Dai and Kwok, 2005), where

\[
M^*(S, 0^+) = K, \quad M^*(0^+, \tau) = K, \tag{2.13a}
\]

and

\[
\lim_{S \to \infty} \frac{M^*(S, \tau; K)}{S} = \frac{M^*(S, \tau; 0)}{S}. \tag{2.13b}
\]

2. Similarly, we can deduce that the exercise boundary surface \(A^*(S, \tau)\) of an American fixed strike Asian put satisfies

\[
A^*(S, 0^+) = \min \left( K, \frac{S + rTK}{1 + rT} \right) \quad \text{and} \quad A^*(\infty, \tau) = K. \tag{2.14}
\]
3. The exercise boundary surface $m^*(S, \tau)$ of the American fixed strike lookback put option with payoff function $(K - m)^+$ observes

$$m^*(S, 0^+) = K \quad \text{and} \quad m^*(\infty, \tau) = K. \quad (2.15)$$

The proof of Proposition 5 is presented in the Appendix.

3. MONOTONE PROPERTIES WITH RESPECT TO DIVIDEND YIELD, INTEREST RATE AND TIME

First, we would like to recall some well known results on the monotone properties of American vanilla option values with respect to the dividend yield $q$ and riskless interest rate $r$. Since the dividend yield tends to reduce the risk neutralized drift rate of the underlying asset price, it is intuitively clear that the value of vanilla call (put) decreases (increases) with increasing $q$. However, the impact of interest rate $r$ on the option value is slightly more complicated since $r$ appears both in the discount factor and the risk neutralized drift rate. For vanilla put options, an increase in $r$ would decrease both the discount factor and expected payoff so that the put value decreases with an increase in $r$. However, the effect of $r$ on the discount factor and expected payoff are counteracting in call options. Therefore, it becomes difficult to determine the effect of interest rate on call option value by simple financial intuition. Fortunately, thanks to the put-call symmetry relation for the prices of American vanilla call and put options:

$$C(S, t; r, q, X) = P(X, t; q, r, S), \quad (3.1)$$

we can conclude that the vanilla call value $C(S, t; r, q, X)$ is an increasing function of $r$. This is because the vanilla put price function $P(S, t; r, q, X)$ is an increasing function of $q$ so that the modified put price function $P(X, t; q, r, S)$ is an increasing function of $r$.

3.1 Monotone properties with respect to dividend yield and interest rate

For fixed strike path dependent options, the above intuitive arguments also hold. That is, the fixed strike call (put) value is monotonically decreasing
(increasing) with respect to \( q \) and the fixed strike put value is monotonically decreasing with respect to \( r \). Unfortunately, we do not have the put-call symmetry relation for the fixed strike call and put. Hence, we cannot infer the monotone properties with respect to \( r \) of the fixed strike call value, like those of the vanilla option counterparts. Indeed such monotone property does not hold. We shall show mathematically the rationale for the lack of such monotonicity and verify the phenomenon through numerical examples. For floating strike path dependent options, it is even more difficult to deduce the monotone properties from financial intuition. We shall show that the monotone properties with respect to \( r \) exist for all floating strike path dependent options. However, only the floating strike call value has monotone property with respect to \( q \). We first state the monotone properties for the values of European path dependent options, then generalize the results to their American counterparts.

*Price functions of European path dependent options*

The comparison principle in partial differential equation theory can be used to prove the monotone properties with respect to \( q \) and \( r \) for European path dependent options. The results for fixed strike and floating strike path dependent options are summarized in the following two lemmas.

**Lemma 6**

The price function of a European fixed strike path dependent call (put) option is monotonically decreasing (increasing) with respect to the dividend yield \( q \) of the underlying asset. With an increase of the riskless interest rate \( r \), the value of a European fixed strike put decreases. However, no monotone properties with respect to \( r \) holds for a European fixed strike call.

**Lemma 7**

The price function of a European floating strike path dependent call (put) is monotonically increasing (decreasing) with respect to the riskless interest rate \( r \). For European floating strike options, only the call value has monotone (decreasing) property with respect to the dividend yield \( q \).

The proof of Lemma 6 is presented in the Appendix. For Lemma 7, the proof of the ambiguity of monotone property with respect to \( q \) of the floating strike put value is presented in the Appendix.

*Price functions of American path dependent options*
The above monotone properties with respect to $r$ and $q$ for the values of European path dependent options also hold for their American counterparts. To prove these claims, one may apply the penalty technique in partial differential equation theory (Jiang, 2002). Here, we would like to provide a simpler approach by proving the results under the discrete binomial tree model (Cox et al., 1979). Since the option values derived from the binomial tree models for American path dependent options have been shown to converge to their continuous counterparts (Jiang and Dai, 2004), we can infer the validity of the monotone properties for the option values of the continuous models from those of the discrete models.

Let us take the American Asian option as an example. The underlying asset price is assumed to follow the discrete binomial process. Let $\Delta t$ be the time step, $u$ and $d$ denote the proportional upward and downward jumps in the binomial process and $p$ denotes the probability of upward jump. The parameter values are given by (Cox et al., 1979)

$$u = \frac{1}{d} = e^{\sigma \sqrt{\Delta t}}, \quad p = \frac{e^{(r-q)\Delta t} - d}{u - d}. \tag{3.2}$$

Let $V^n(S, A)$ denote the option value at the $n^{th}$ time step, with asset price $S$ and average asset value $A$. The discrete binomial pricing model for the American Asian option is given by (Jiang and Dai, 2004)

$$V^n(S, A) = \max \left( e^{-r\Delta t} \left[ pV^{n+1}(Su, \frac{nA + Su}{n + 1}) + (1-p)V^{n+1}(Sd, \frac{nA + Sd}{n + 1}), \Lambda(S, A) \right] \right)$$

with terminal conditions:

$$V^N(S, A) = \Lambda(S, A), \tag{3.3}$$

where $\Lambda(S, A)$ is the payoff function and $N\Delta t = T$. Here, $N$ is the total number of time steps.

In the following exposition, we illustrate the proof on the monotone property with respect to $q$ for the American floating strike Asian call value. The proofs for other types of American path dependent options can be performed in a similar manner. We apply the transformation

$$W^n(x) = \frac{V^n(S, A)}{S} \quad \text{and} \quad x = \frac{S}{A}$$
so that the binomial pricing scheme (3.3) can be reformulated as

\[
W^n(x) = \max \left( e^{-r\Delta t} \left[ p_u W^{n+1} \left( \frac{nxd + 1}{n + 1} \right) + (1 - p) d W^{n+1} \left( \frac{nxu + 1}{n + 1} \right), \phi(x) \right] \right)
\]

\[
W^N(x) = \phi(x) = (1 - x)^+.
\]  

(3.4)

To prove the monotonically decreasing property with respect to \(q\) of the American floating strike Asian call, it suffices to show that

\[
W^n(x; q_1) \geq W^n(x; q_2) \quad \text{for} \quad q_1 \leq q_2.
\]  

(3.5)

The detailed proof of the above result is presented in the Appendix.

**Exercise regions of American path dependent options**

Once we have established the monotone properties with respect to \(r\) and \(q\) of the values of American path dependent options, we can deduce the corresponding monotone properties of the exercise regions of these options. These monotone properties are summarized in Proposition 8.

**Proposition 8**

We use \(E_{fc}(r)[E_{fc}(q)]\) to denote the exercise region of the American floating strike call option with interest rate \(r\) (dividend yield \(q\)), and similar notations for other types of American path dependent options. The exercise regions of various types of American path dependent options observe the following monotone properties with respect to the interest rate \(r\) and dividend yield \(q\).

1. Monotone properties with respect to \(r\)

Let \(r_1 \leq r_2\), we have

\[
E_{fc}(r_2) \subseteq E_{fc}(r_1)
\]

\[
E_{fp}(r_1) \subseteq E_{fp}(r_2)
\]

\[
E_{xp}(r_1) \subseteq E_{xp}(r_2).
\]

There is no monotone property for the American fixed strike call options.
2. Monotone properties with respect to $q$

Let $q_1 \leq q_2$, we have

$$E_{xc}(q_1) \subseteq E_{xc}(q_2)$$

$$E_{xp}(q_2) \subseteq E_{xp}(q_1)$$

$$E_{fc}(q_1) \subseteq E_{fc}(q_2).$$

There is no monotone property for the American floating strike put options.

3.2 Monotone properties with respect to calendar time and maturity

A longer-lived American option must be worth at least its shorter-lived counterpart since the longer-lived option has the additional right to exercise between the two expiration dates. Hence, the American option value is always monotonically increasing with respect to $T$ so that $\frac{\partial V}{\partial T} \geq 0$. Let $E(T)$ denote the exercise region of an American path dependent option with maturity date $T$. The monotone property of the option price function with respect to $T$ dictates that if $T_1 \leq T_2$ then

$$E(T_1) \supseteq E(T_2)$$

for both American Asian and lookback options.

For American lookback options, their values depend only on time to expiry $\tau = T - t$ so that

$$\frac{\partial V}{\partial t} = -\frac{\partial V}{\partial T} \leq 0.$$  

From the time-monotone property of the lookback option value, we can deduce the monotonicity of the exercise boundary with respect to the calendar time $t$. However, for American Asian options, their values depend on both $t$ and $T$ so that the monotonicity with respect to $t$ of the option value and exercise boundary do not hold [see the numerical examples demonstrated by Hansen and Jorgensen (2000), and Figures 3, 6 and 7].

Perpetual American path dependent options

While most perpetual options do not exhibit dependence on the calendar time, we observe that the values of American Asian options have dependence on $t$. Otherwise, other analytic properties of the perpetual American...
Asian options are similar to those of the lookback counterparts. For example, it is known that perpetual American lookback option values are finite when \( q \neq 0 \). We can deduce that perpetual American Asian option values are also finite since they are bounded by the values of the lookback counterparts. Furthermore, since the exercise regions of perpetual American lookback options are non-empty when \( q \neq 0 \), so those of the perpetual American Asian options are non-empty also. Interestingly, when \( q = 0 \), the perpetual American floating strike Asian and lookback calls have the same value as that of the non-dividend paying underlying asset. We then deduce that it is never optimal to exercise these American call options prematurely, so their exercise regions are empty. To prove the claim, we use the put-call parity relation of the finite-lived European floating strike Asian options. Let \( v_{A,fc}(S, J, t; T) \) denote the price function of the finite-lived European floating strike call (put) option with maturity \( T \). By establishing the following results

\[
S \geq V_{L,fc}(S, J, t; \infty) \geq V_{A,fc}(S, J, t; \infty) \geq v_{A,fc}(S, J, t; T) = v_{A,fp}(S, J, t; T) + \left[ 1 - \frac{1}{rT} \right] S + \frac{S}{rT} e^{-r(T-t)} - \frac{t}{T} J e^{-r(T-t)} - tT J e^{-r(T-t)} \rightarrow S \text{ as } T \rightarrow \infty,
\]

we can deduce that

\[
V_{L,fc}(S, J, t; \infty) = V_{A,fc}(S, J, t; \infty) = S. \tag{3.6}
\]

### 3.3 Sample calculations

We performed sample calculations to verify the claims on the monotone properties with respect to \( r \) and \( q \) of the option values and exercise regions of various types of American path dependent options. Figure 1 shows that the American fixed strike Asian put value is monotonically decreasing with respect to \( r \) while its call option counterpart exhibits no such monotonicity. Also, Figure 2 verifies that the American floating strike lookback call value is monotonically decreasing with respect to \( q \) while its put option counterpart has no such monotone property.

Recall that the exercise region and continuation region of an American path dependent option are separated by a time-dependent early exercise
boundary. In this subsection, we use $x^*(t)$ to denote the exercise boundary with the calendar time $t$ as the time variable (instead of time to expiry $\tau$ as in previous sections). The American lookback options and their Asian counterparts show distinctive differences on the monotone properties with respect to the calendar time $t$ of the exercise boundary $x^*(t)$. While Asian option values depend on $t$ and $T$ separately, the lookback values depend only on $\tau = T - t$, hence the exercise boundary $x^*(t)$ for any American lookback option is always monotonically decreasing with respect to $t$. As shown in Figure 3, the exercise boundary $x^*(t)$ of the American floating strike Asian put does not possess monotonicity with respect to $t$. Also, the same figure shows that the exercise region of the American lookback put option is contained inside that of its Asian counterpart, a result as predicted by the nesting relations (2.6).

We have deduced the non-monotone property with respect to $q$ for the American floating strike put option values, and accordingly, the exercise regions also do not exhibit monotonicity with respect to $q$. These properties are verified by the plot of the exercise boundaries $x^*(t)$ of the American floating strike lookback put options with varying values of $q$. A simple intuitive argument can be presented to explain the loss of monotone property with respect to $q$ of the exercise regions. When $q = 0$, it is known that the Russian option (perpetual American option with realized maximum asset value as its payoff) will never be exercised prematurely, thus implying that the perpetual floating strike lookback put will never be exercised prematurely too. When $q > 0$, $x^*(t)$ starts with a finite value and tends to $\max\left(\frac{q}{r}, 1\right)$ as time is approaching expiry. When $q = 0$, $x^*(t)$ is infinite at infinite time to expiry but tends to 1 as time is approaching expiry. Therefore, at least for $q > r$, we argue that the corresponding $x^*(t)$ of the American lookback put will intersect $x^*(t)$ of the counterpart with $q = 0$. Hence, the exercise regions of the American lookback put options do not have monotone property with respect to $q$ (see Figure 4). On the other hand, Figure 5 shows that the exercise regions do observe monotone property with respect to $r$, agreeing with the result in Proposition 8. Figures 6 and 7 illustrate that the exercise regions of the American floating strike Asian call options are monotonically increasing (decreasing) with respect to dividend yield (interest rate). Unlike the American lookback options, the early exercise boundaries of the American Asian call options do not have monotone property with respect to the calendar time.
4 CONCLUSIONS

We have developed a general framework of applying partial differential equation theory to analyze the monotone properties of the price functions and the optimal stopping regions of American path dependent options with either Asian or lookback feature. From the ordering properties of the price functions of the American lookback options and their Asian option counterparts, we can deduce that the exercise region of an American lookback option is always contained in the exercise region of its Asian option counterpart. When the underlying asset pays a continuous dividend yield, we show that the exercise region exists for all times for any American Asian option. When the underlying asset is non-dividend paying, the exercise region also exists for all finite-lived American Asian option except that it is inconclusive to claim whether the exercise region of the American floating strike Asian call option exists for all times. Also, we have established some interesting monotone properties of the price functions and exercise regions of European and American path dependent options with respect to dividend yield, riskless interest rate and time. With the floating strike put as an exception, the price functions of all path dependent options are monotone with respect to the dividend yield of the underlying asset. Except for the fixed strike call, the values of path dependent options also exhibit monotone properties with respect to the interest rate. We also find some interesting differences of optimal exercise policies of the two classes of American path dependent option models. The exercise boundary of an American Asian option may not observe the monotone property with respect to the calendar time while the American lookback option counterpart always does. The exercise region of some types of perpetual American path dependent options on a non-dividend paying asset may not exist. The perpetual American floating strike Asian and lookback calls have the same value as that of the non-dividend paying asset, so it is never optimal to exercise these call options prematurely.

APPENDIX

Proof of Lemma 3

To show that the stopping region lies on the left hand side of the stopping boundary for the American floating strike Asian call, it suffices to show that
$W - (1 - x)$ is a non-decreasing function of $x$. That is, it is necessary to establish
\[
\frac{\partial}{\partial x}[W - (1 - x)] = \frac{\partial W}{\partial x} + 1 \geq 0.
\]
Since $\frac{\partial W}{\partial x} = \frac{\partial V}{\partial A}$, it is equivalent to prove
\[
\frac{\partial V}{\partial A} \geq -1.
\]
It suffices to show
\[
\frac{V(S, A + \delta, \tau) - V(S, A, \tau)}{\delta} \geq -1 \quad \text{for any } \delta > 0,
\]
or
\[
V(S, A + \delta, \tau) + \delta \geq V(S, A, \tau) \quad \text{for any } \delta > 0.
\]
Let
\[
\overline{V}(S, A, \tau; \delta) = V(S, A + \delta, \tau) + \delta,
\]
then $\overline{V}(S, A, \tau)$ satisfies the following variational inequalities:
\[
\frac{\partial \overline{V}}{\partial \tau} - \frac{S - A}{T - \tau} \frac{\partial \overline{V}}{\partial A} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 \overline{V}}{\partial S^2} - (r - q) S \frac{\partial \overline{V}}{\partial S} + r \overline{V} \geq r \delta - \frac{\delta}{T - \tau} \frac{\partial V}{\partial A}(S, A + \delta, \tau)
\]
\[
\overline{V} \geq S - (A + \delta) + \delta = S - A
\]
\[
\left[\frac{\partial \overline{V}}{\partial \tau} - \frac{S - A}{T - \tau} \frac{\partial \overline{V}}{\partial A} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 \overline{V}}{\partial S^2} - (r - q) \delta \frac{\partial \overline{V}}{\partial S} + r \overline{V} - r \delta + \frac{\delta}{T - \tau} \frac{\partial V}{\partial A}(S, A + \delta, \tau)\right]
\]
\[
[\overline{V} - (S - A)] = 0
\]
with terminal condition:
\[
V(S, A, T) = (S - A - \delta)^+ + \delta = \max(\delta, S - A).
\]
Since $\frac{\partial V}{\partial A} \leq 0$ for the American floating strike call, we have
\[
r \delta - \frac{\delta}{T - \tau} \frac{\partial V}{\partial A}(S, A + \delta, \tau) > 0.
\]
By applying the comparison principle, we can deduce that
\[
\overline{V}(S, A, \tau) \geq V(S, A, \tau),
\]
19
which gives the desired result.

**Proof of Proposition 4**

We only consider the American floating strike Asian call. The proof for the put option counterpart is similar. In the exercise region, $W = 1 - x$. By substituting $W = 1 - x$ into the first variational inequality for $W$ [see Eq. (2.7)], we obtain

$$
\frac{\partial}{\partial \tau} (1 - x) - \frac{\sigma^2}{2} x^2 \frac{\partial^2}{\partial x^2} (1 - x) + \left[ (r - q)x - \frac{1 - x}{T - \tau} \right] \frac{\partial}{\partial x} (1 - x) + q(1 - x)
$$

$$
= (q - r)x + \frac{1 - x}{T - \tau} + q(1 - x) \geq 0,
$$

so that the optimal stopping region must be contained in the region

$$\left\{ (x, \tau) : -rx + q + \frac{1 - x}{T - \tau} \geq 0, \tau > 0 \right\}.$$

This leads to

$$x^*(0^+) \leq \frac{1 + qT}{1 + rT}.$$

Together with the requirement of non-negativity of the exercise payoff: $1 - x^*(0^+) \geq 0$, we obtain

$$x^*(0^+) \leq \min \left( \frac{1 + qT}{1 + rT}, 1 \right).$$

To show that the above inequality reduces to an equality, it suffices to show that $x^*(0^+) \geq \min \left( \frac{1 + qT}{1 + rT}, 1 \right)$. Assume the contrary, suppose there exists $(x, 0^+)$ in the continuation region where $x < \min \left( \frac{1 + qT}{1 + rT}, 1 \right)$. In the continuation region, $W(x, 0^+) = 1 - x$ and

$$
\left. \frac{\partial W}{\partial \tau} \right|_{\tau=0^+} = - \left[ (q - r)x + \frac{1 - x}{T - \tau} + q(1 - x) \right] < 0
$$

when $x < \min \left( \frac{1 + qT}{1 + rT}, 1 \right)$. This leads to a contradiction with $\left. \frac{\partial W}{\partial \tau} \right|_{\tau=0^+} > 0$ since $W(x, \tau) \geq (1 - x)^+$ and $W(x, 0) = (1 - x)^+$. Hence, we deduce that

$$x^*(0^+) = \min \left( \frac{1 + qT}{1 + rT}, 1 \right).$$
Proof of Proposition 5

(a) We have shown that \( A^*(S, \tau) > \frac{S + rTK}{1 + rT}, \tau > 0 \) [see Eq. (2.11)]. Also, from the non-negativity of the exercise payoff, we have \( A^*(S, \tau) \geq K, \tau > 0 \) so that \( A^*(S, 0^+) \geq \max \left( K, \frac{S + rTK}{1 + rT} \right) \). To show

\[
A^*(S, 0^+) = \max \left( K, \frac{S + rTK}{1 + rT} \right),
\]

it suffices to show that \( A^*(S, 0^+) \leq \max \left( K, \frac{S + rTK}{1 + rT} \right) \). Assume the contrary, suppose \( A^*(S, 0^+) > \max \left( K, \frac{S + rTK}{1 + rT} \right) \), then there exists a point \((S, A, 0^+)\) in the continuation region where \( A > \max \left( K, \frac{S + rTK}{1 + rT} \right) \). At \( \tau \to 0^+ \), \( V(S, A, 0^+) = A - K \), so we obtain

\[
\left. \frac{\partial V}{\partial \tau} \right|_{\tau=0^+} = -A + S - rTA + rTK < 0 \quad \text{since} \quad A > \frac{S + rTK}{1 + rT}.
\]

This is a violation of the condition \( \left. \frac{\partial V}{\partial \tau} \right|_{\tau=0^+} > 0 \) (condition for the option to remain alive in the continuation region). Combining all these arguments, we obtain the result in Eq. (i).

(b) From the non-negativity of the payoff of the fixed strike call, we have \( A^*(0^+, \tau) \geq K, \tau > 0 \). Dai and Kwok (2005) show that \( M^*(0^+, \tau) = K \) for the fixed strike lookback call. From the nesting property of the exercise regions (see Lemma 2), we have \( M^*(0^+, \tau) \geq A^*(0^+, \tau), \tau > 0 \). Combining all the results, we obtain

\[ A^*(0^+, \tau) = K. \] (ii)

(c) When \( K = 0 \), the payoff of the American fixed strike Asian call becomes linear homogeneous in \( S \), like that of an American floating strike Asian option. Hence, we expect that \( A^*(S, \tau; 0) \) is also linear homogeneous in \( S \) so that the ratio \( \frac{A^*(S, \tau; 0)}{S} \) is some function of \( \tau \) [call it \( \xi(\tau) \)].
Furthermore, by observing that $A^*(S, \tau; K)$ is linear homogeneous in $K$, we obtain

$$
\lim_{S \to \infty} \frac{A^*(S, \tau; K)}{S} = \lim_{S \to \infty} \frac{A^*(\frac{S}{K}, \tau; 1)}{\frac{S}{K}} = \lim_{K \to 0} \frac{A^*(S, \tau; K)}{S} = \frac{A^*(S, \tau; 0)}{S}.
$$

(iii)

Proof of Lemma 6

Here, the European Asian option model is chosen as an illustration. Similar arguments of proof can be applied to European lookback option models. First, we show the monotone properties with respect to $q$. Let $U(S, A, \tau) = V(S, A, \tau; q_1) - V(S, A, \tau; q_2), q_1 \leq q_2$, where $V(S, A, \tau; q)$ is the price function of a European fixed strike Asian option with dividend yield $q$. Since $V(S, A, \tau; q)$ satisfies the following governing equation

$$\frac{\partial V}{\partial \tau} - \frac{S - A}{T - \tau} \frac{\partial V}{\partial A} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - (r - q) S \frac{\partial V}{\partial S} + r V = 0,$$

then $U(S, A, \tau)$ satisfies

$$\frac{\partial U}{\partial \tau} - \frac{S - A}{T - \tau} \frac{\partial U}{\partial A} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 U}{\partial S^2} - (r - q_1) S \frac{\partial U}{\partial S} + r U = (q_2 - q_1) S \Delta(q_2)$$

with initial condition: $U(S, A, 0) = 0$. Here, $\Delta(q_2)$ denotes the option delta $\frac{\partial V}{\partial S}(S, A, \tau; q_2)$. It is well known that $\Delta(q_2)$ is non-negative (non-positive) for a fixed strike call (put) so that

$$\begin{cases} 
(q_2 - q_1) S \Delta(q_2) \geq 0 & \text{for a fixed strike call} \\
(q_2 - q_1) S \Delta(q_2) \leq 0 & \text{for a fixed strike put}.
\end{cases}$$

Hence, we conclude from the comparison principle that

$$\begin{cases} 
U(S, A, \tau) \geq 0 & \text{for a fixed strike call} \\
U(S, A, \tau) \leq 0 & \text{for a fixed strike put}.
\end{cases}$$

To show the monotone properties with respect to $r$, we consider $\hat{U}(S, A, \tau) = V(S, A, \tau; r_1) - V(S, A, \tau; r_2), r_1 \leq r_2$, where $V(S, A, \tau; r)$ is the price function of a European fixed strike option with riskless interest rate $r$. It is seen
that \( \hat{U}(S, A, \tau) \) satisfies

\[
\frac{\partial \hat{U}}{\partial \tau} - \frac{S - A}{T - \tau} \frac{\partial \hat{U}}{\partial A} - \frac{\sigma^2}{2} S \frac{\partial^2 \hat{U}}{\partial S^2} - (r_1 - q) S \frac{\partial \hat{U}}{\partial S} + r_1 \hat{U} \\
= (r_2 - r_1) [-S \Delta(r_2) + V(S, A, \tau; r_2)]
\]

with zero initial condition. Here, \( \Delta(r_2) \) denotes the option delta \( \frac{\partial V}{\partial S}(S, A, \tau; r_2) \).

(i) For a fixed strike Asian put, we have \( \Delta(r_2) \leq 0 \) so that

\[-S \Delta(r_2) + V(S, A, \tau; r_2) \geq 0.\]

Hence, we can deduce that

\[V(S, A, \tau; r_1) \geq V(S, A, \tau; r_2), \quad r_1 \leq r_2.\]

(ii) For a fixed strike Asian call, however, the sign of \(-S \Delta(r_2) + V(S, A, \tau; r_2)\) is ambiguous. Therefore, no monotone property with respect to \( r \) can be deduced for the European fixed strike Asian call.

**Proof of Lemma 7**

The pricing model of a floating strike path dependent option can be reduced to an one-dimensional model by normalizing the price function by the asset price \( S \) and using the similarity variable \( x = A/S \) as the independent variable. To show the monotone property with respect to \( q \) of the floating strike path dependent option, we let

\[\tilde{U}(x, \tau) = W(x, \tau; q_1) - W(x, \tau; q_2), \quad q_1 \leq q_2,\]

where \( W(x, \tau; q) \) is the price function of a floating strike Asian option with dividend yield \( q \). We write \( W'(q_2) = \frac{\partial W}{\partial x}(x, \tau; q_2) \). It is seen that \( \tilde{U} \) satisfies

\[
\frac{\partial \tilde{U}}{\partial \tau} - \frac{\sigma^2}{2} x^2 \frac{\partial^2 \tilde{U}}{\partial x^2} - \left[(q_1 - r)x + \frac{1 - x}{T - \tau} \right] \frac{\partial \tilde{U}}{\partial x} + q_1 \tilde{U} \\
= (q_2 - q_1) [-xW'(q_2) + W(x, \tau; q_2)]
\]

with initial condition: \( \tilde{U}(x, 0) = 0 \). Recall that \( \frac{\partial W}{\partial x} \leq 0 \) for a floating strike Asian call while \( \frac{\partial W}{\partial x} \geq 0 \) for the put option counterpart. For a floating strike
Asian call, we have 

$$-xW'(q_2) + W(x, \tau; q_2) \geq 0$$

so that the non-homogeneous term in the differential equation for $\tilde{U}$ is always positive. By applying the comparison principle, we deduce that $\tilde{U} \geq 0$ so that

$$W(x, \tau; q_1) \geq W(x, \tau; q_2), \quad q_1 \leq q_2.$$ 

However, for a European floating strike Asian put, the sign of $-xW'(q_2) + W(x, \tau; q_2)$ is ambiguous so that no monotone property with respect to $q$ can be deduced for the floating strike Asian put.

**Proof of inequality (3.5)**

Suppose we let

$$p_i = \frac{e^{(r-q_i)\Delta t} - d}{u - d}, \quad i = 1, 2,$$

we can rewrite $W^n(x; q_1)$ as follows:

$$W^n(x; q_1)$$

$$= \max \left\{ \frac{1}{\rho} \left[ p_1 u W^{n+1} \left( \frac{nxd + 1}{n + 1}; q_1 \right) + (1 - p_1) d W^{n+1} \left( \frac{nxu + 1}{n + 1}; q_1 \right) \right], \phi(x) \right\}$$

$$= \max \left\{ \frac{e^{-q_1\Delta t}}{u - d} \left[ u W^{n+1} \left( \frac{nxd + 1}{n + 1}; q_1 \right) - d W^{n+1} \left( \frac{nxu + 1}{n + 1}; q_1 \right) \right] + \frac{e^{-r\Delta t}}{u - d} \left[ W^{n+1} \left( \frac{nxu + 1}{n + 1}; q_1 \right) - W^{n+1} \left( \frac{nxd + 1}{n + 1}; q_1 \right) \right], \phi(x) \right\}.$$ 

By observing that $W^{n+1}(x_1) \geq W^{n+1}(x_2)$ if $x_1 \leq x_2$, we can establish

$$u W^{n+1} \left( \frac{nxd + 1}{n + 1}; q_1 \right) - d W^{n+1} \left( \frac{nxu + 1}{n + 1}; q_1 \right) \geq 0.$$ 

We apply the induction argument. Suppose inequality (3.5) is valid for $n+1,$
then by the assumption of induction, we obtain

\[
W^n(x; q_1) \geq \max \left\{ e^{-q_2 \Delta t} \left[ uW^{n+1} \left( \frac{nx + 1}{n+1}; q_1 \right) - dW^{n+1} \left( \frac{nx + 1}{n+1}; q_1 \right) \right] + e^{-r \Delta t} \left[ W^{n+1} \left( \frac{nu + 1}{n+1}; q_1 \right) - W^{n+1} \left( \frac{nx + 1}{n+1}; q_1 \right) \right], \phi(x) \right\}
\]

\[
= \max \left\{ \frac{1}{\rho} \left[ p_2uW^{n+1} \left( \frac{nx + 1}{n+1}; q_1 \right) + (1 - p_2)dW^{n+1} \left( \frac{nx + 1}{n+1}; q_1 \right) \right], \phi(x) \right\}
\]

\[
= \max \left\{ \frac{1}{\rho} \left[ p_2uW^{n+1} \left( \frac{nx + 1}{n+1}; q_2 \right) + (1 - p_2)dW^{n+1} \left( \frac{nx + 1}{n+1}; q_2 \right) \right], \phi(x) \right\}
\]

\[
= W^n(x; q_2)
\]

where the last inequality is due to the assumption of induction, \( p_2 > 0 \) and \((1 - p_2) > 0 \). Therefore, we see that \( W^n(x) \) decreases as \( q \) increases for the American floating strike Asian call. Following similar argument, we can show the monotonicity with respect to \( r \) for the American fixed strike Asian option values.

Remark
For the American floating strike Asian put option, we have

\[ W^{n+1}(x_1) \leq W^{n+1}(x_2) \quad \text{if} \quad x_1 \leq x_2. \]

As a result, there is ambiguity about the sign of

\[ uW^{n+1} \left( \frac{nx + 1}{n+1} \right) - dW^{n+1} \left( \frac{nx + 1}{n+1} \right). \]

Hence, we cannot deduce monotonicity with respect to \( q \) for the American floating strike Asian put option.

REFERENCES


Figure 1 The American fixed strike Asian put value is monotonically decreasing with respect to the riskless interest rate $r$ while its call option counterpart exhibits no such monotone property. The parameter values of the option models are: $t = 0, T = 20, S = A = K = 1, \sigma = 0.4$ and $q = 0$. 
Figure 2 The American floating strike lookback call value is monotonically decreasing with respect to the dividend yield $q$ while its put option counterpart has no such monotone property. The parameter values of the option models are: $\tau = 10, S = M = m = 1, \sigma = 0.2$ and $r = 0.035$. 
Figure 3  The solid (dotted) line shows the early exercise boundary $x^*(t)$ separating the exercise region and continuation region of an American floating strike lookback (Asian) put option. The parameter values of the models are: $T = 1, \sigma = 0.3, r = 0.04, q = 0.02$. The critical threshold $x^*(t)$ is monotonically decreasing with respect to the calendar time $t$ for the lookback put while that of the Asian put has no such time-monotone property. The exercise region of the lookback option is contained inside that of the Asian option.
Figure 4 Plot of the early exercise boundaries of the American floating strike lookback put options against the calendar time $t$ with varying values of dividend yield $q$. The exercise regions do not exhibit monotone property with respect to $q$. Other parameters in the option models are: $\sigma = 0.2$ and $r = 0.04$. 

\[ q = 0.04 \quad \text{exercise region} \]
\[ q = 0.06 \]
\[ q = 0 \quad \text{continuation region} \]
Figure 5  Plot of the early exercise boundaries of American floating strike lookback put options against the calendar time $t$ with varying values of riskless interest rate $r$. The exercise regions observe monotone property with respect to $r$. Other parameters in the option models are: $T = 50$, $\sigma = 0.2$ and $q = 0.04$. 
Figure 6  Plot of the early exercise boundary of the American floating strike Asian call options against the calendar time $t$ with varying values of dividend yield $q$. The exercise regions are monotonically increasing with respect to $q$ while $x^*(t)$ does not observe time-monotone property. Other parameters in the option models are: $T = 50, \sigma = 0.2$ and $r = 0.04$. 
Figure 7  Plot of the early exercise boundary of the American floating strike Asian call options against the calendar time $t$ with varying values of riskless interest rate $r$. The exercise regions are monotonically decreasing with respect to $r$. Other parameters in the option models are: $T = 50, \sigma = 0.2$ and $q = 0.04$. 