Optimal initiation of Guaranteed Lifelong Withdrawal Benefit with dynamic withdrawals

Yao Tung Huang, * Pingping Zeng, † Yue Kuen Kwok ‡

Abstract

We consider pricing Guaranteed Lifelong Withdrawal Benefit (GLWB) that consists of the early phase of accumulation of benefit base and the later income phase of withdrawals. The most recent form of the GLWB provides flexibility in allowing additional purchases in the accumulation phase, dynamic withdrawals in the income phase, dynamic initiation into the income phase and complete surrender right throughout the life of the contract. The policyholder chooses the initiation of the income phase optimally based on a combination of factors, like the age-dependent scheduled withdrawal rates, penalty charge rate, bonus and ratchet provisions. Using the bang-bang control analysis, we show that the strategy space of the optimal policies is limited to four choices: maximum allowable purchase, zero withdrawal, withdrawal at the contractual amount or complete surrender. We construct the Fourier transform algorithm for effective pricing of GLWB products with policy fund value under the Heston model and complex path dependent features arising from the ratchet and bonus events, dynamic control of withdrawals and additional purchases, together with optimality in the time of initiation of the income phase. We also analyze various pricing properties of the GLWB based on the effective and accurate Fourier transform algorithms. In particular, we examine the impact of various contractual specifications of the GLWB on the optimal decision of initiation of the income phase and optimal withdrawal strategies.

JEL Classification: G22, C50

Keywords: variable annuities, lifelong withdrawal guarantees, optimal initiation, bang-bang analysis, Fourier transform algorithm

1 Introduction

Variable annuities are long-term unit-linked insurance products that offer various types of guarantees. In most common design of variable annuities contracts, policyholders first accumulate assets during the accumulation phase and later receive incomes during the income phase. In 2005, variable annuities with guaranteed withdrawal benefit for life that combines the longevity protection of an income benefit and periodic withdrawal benefits was introduced. By 2016, the Guaranteed Lifelong Withdrawal Benefit (GLWB) rider is structured in about half of new

*Department of Mathematics, Hong Kong University of Science and Technology, Hong Kong, China
†Department of Mathematics, South University of Science and Technology of China, China
‡Correspondence author: maykwok@ust.hk. Department of Mathematics, Hong Kong University of Science and Technology, Hong Kong, China
variable annuities sales in the US markets. These guarantees are funded by the rider charges (proportional fees), which are paid annually by the policyholder from the policy account. Since the embedded guarantees may be too costly to the issuers and they are difficult to be hedged, many insurers of variable annuities faced record levels of breakage of their risk hedging strategies. This problem became more acute in the period after the financial tsunami in 2008.

The types of risks faced by insurers in the actual implementation of hedging strategies include policyholder behavior, basis risk and execution risk from poor liquidity of hedging instruments. In a typical contractual design of GLWB, the policyholder pays an upfront premium and the amount is then invested in her own choice of portfolio of mutual funds. Policyholder can be allowed to place additional purchases of funds during the accumulation phase beyond the initial upfront premium. In the literature, Chi and Lin (2012) once study flexible premium variable annuities that allow additional contribution and Bernard et al. (2016) also analyze the risk of mispricing by insurers that offer guarantees on flexible premiums in variable annuities. The policy account is the ongoing value of mutual funds held by the policyholder. Besides changes due to investment returns and withdrawal amounts, it is also subject to periodic deduction for payment of the rider charges and increment in value due to additional purchases of funds during the accumulation phase. Rider charges and contractual withdrawal amounts are calculated based on certain fixed proportions of the benefit base. The GLWB’s benefit base is not the same as the policy account value, except at initiation of the GLWB contract where the benefit base is set equal to the policy account value. The ratchet (step-up) provision increases the benefit base periodically if the policyholder’s risky asset investment account has increased to a value higher than the benefit base. During the accumulation phase, the benefit base is increased under the bonus (roll-up) provision if the policyholder does not withdraw in a given year. Once the income phase is initiated at the discretion of the policyholder, the bonus provision on the benefit base become ineffective as the accumulation phase ends. During the income phase, the policyholder is entitled to receive the guaranteed lifelong annual income calculated based on the product of the benefit base and the scheduled withdrawal rate. The withdrawal rate is dependent on the age of the policyholder in the year of entry into the income phase. The scheduled withdrawal rate is locked in and never changes in the remaining life of the contract. The initiation of the income phase is chosen optimally by the policyholder based on various considerations, like the age-dependent scheduled withdrawal rates, bonus rates and others. It is common to set an upper bound on the period length of the accumulation phase, say, 20 years. Beyond the allowable time period, the income phase will be activated automatically. Throughout the whole life of the policy, the policyholder is allowed to withdraw more than the contractual amount (up to the policy account value) by paying a penalty charge. Indeed the complete withdrawal of the policy account means surrender (early termination) of the contract. Empirical studies have revealed various forms of policyholders’ behaviors that induce surrender. The surrender of the contract incurs various costs to the issuer, like upfront costs of finding new customers, adverse selection problem of keeping only poor-insurability policies and liquidity issues of coping with cash payments.

In the literature, there have been many papers that discuss pricing and hedging of various types of GLWB products. Fung et al. (2014) examine how financial and demographic parameters would affect the fair guaranteed fee charged for a GLWB. Their studies are limited to the plain GLWB that allows static withdrawals and the withdrawals start immediately without any deferment period. Their results show that though financial risk is usually dominant for GLWB, the effect of systematic mortality risk cannot be ignored. Steinorth and Mitchell (2015) adopt an expected utility framework to examine how a risk-averse decision maker would choose her optimal withdrawal policies in a GLWB. They show that the ratchet provision may make the policyholder behavior more predictable. Forsyth and Vetzal (2014) construct finite difference
schemes for solving a coupled system of one-dimensional partial differential equations to compute the hedging cost for a GLWB when the underlying fund value process follows a Markov regime switching process. They also consider the different costs of hedging under various withdrawal policies, like the optimal withdrawal policies that maximize policyholder’s expected value of cash flows and the sub-optimal withdrawal policies that are dependent on moneyness of the surrender option. Though the policyholder may be allowed to withdraw any portion of the account according to the contractual withdrawal guarantees, Azimzadeh and Forsyth (2015) show by using the bang-bang control theory that a holder can maximize the issuer’s costs by only choosing either zero withdrawal, withdrawal at the contractual rate or complete surrender for GLWB products. The success of their bang-bang analysis rests on the choices of contractual features such that the solution to the optimal control model can be formulated as maximizing a convex objective function, together with the satisfaction of the technical condition that the underlying fund value process preserves convexity and monotonicity. Interestingly, these conditions are satisfied for the GLWB products but not the related Guaranteed Minimum Withdrawal Benefits (GMWB) products. Huang and Kwok (2016) develop effective regression-based Monte Carlo simulation algorithms for solving the stochastic control models associated with pricing of GLWB products. With the simplification of the strategy space of optimal withdrawal policies to only three choices, the solution of the stochastic control GLWB model by the regression-based Monte Carlo simulation algorithm becomes feasible. They perform sensitivity analysis of the GLWB price function with respect to different parameter values in the stochastic control models of GLWB. Their numerical calculations show that high bonus rate and short cycle of ratchet event add more value to the GLWB price, and reveal the downward trend in the adoption of the zero withdrawal as the optimal strategy when the policyholder ages. On the other hand, the adoption of the contractual guaranteed withdrawal exhibits an upward trend over the calendar time. They also show how high penalty rate suppresses the propensity of adopting the strategy of complete surrender. Also, there exists significant difference in the GLWB prices under different assumptions of the policyholder’s withdrawal behavior. Huang et al. (2014) consider the optimal initiation of a GLWB, the optimal time that the policyholder should end the accumulation phase and initiate withdrawals. They argue that it is non-optimal to delay the withdrawal phase since the guarantee rider in a GLWB is more valuable when the probability of ruined (zero policy fund value) is higher. Once the policy fund value becomes zero, the rider charges terminate. This may be visualized as a concessionary compensation for the loss of equity participation when ruined.

The contribution of this paper is four-fold. Firstly, we present the full formulation of the stochastic control models for pricing the GLWB products with both the accumulation and income phases, the ratchet and bonus provisions, additional purchases, and the dynamic controls of withdrawals and initiation into the income phase. Secondly, we perform the bang-bang analysis of the set of control policies and show that the strategy space of the optimal withdrawal policies and additional purchases is limited to a finite discrete values from the set of continuous values. Thirdly, we construct efficient and accurate Fourier transform algorithms for solving the stochastic control models associated with pricing of GLWB products. The numerical evaluation of the value function can be performed over successive event dates (typically one year) in single step, without the necessity of performing time-stepping evaluations over successive event dates as in typical finite difference calculations. Lastly, we examine how the optimal initiation regions and optimal choices of withdrawal policies are affected by various structural features in the GLWB, like the age-dependent scheduled withdrawal rate, bonus rate, additional purchase and penalty charge rate.

This paper is organized as follows. In the next section, we present a detailed product description of GLWB, in particular, the different features in the accumulation phase and in-
come phase. We then discuss the model formulation of GLWB under the general framework of two-dimensional Markov process for the underlying policy fund value and its variance process. Special attention is paid to consider the jump conditions on the policy fund value and benefit base across the event dates of additional purchases, initiation of income phase, withdrawals, death payment event, step-up and roll-up provisions. In Section 3, we present the details of the bang-bang analysis of the strategy space of optimal policies. For the optimal polices, we show that there are only four possible choices: maximum allowable purchase, zero withdrawal, withdrawal at the contractual amount and complete surrender. Therefore, during the accumulation phase, when it is optimal to initiate additional purchases, the policyholder should purchase additional amount up to the allowable cap. In Section 4, we discuss the construction of the Fourier transform algorithms for pricing GLWB products under the dynamics of Heston stochastic volatility. In Section 5, we present the numerical studies that analyze how the GLWB price, optimal withdrawal policies and optimal initiation regions are affected by various contractual features and model parameters of the GLWB. The last section contains summary of results and conclusive remarks.

2 Formulation

We start with the product description of the GLWB in a variable annuity contract. At initiation of the contract, the policyholder pays an upfront single premium into her policy account, which is then invested in mutual funds of her own choice. The initial policy account value is set to be the initial premium paid. The rider charges paid by the policyholder throughout the policy life for the provision of the guarantees are calculated based on a fixed proportion of the benefit base and they are taken from the policy account periodically through the cancellation of fund units. The benefit base is set to be the upfront premium initially, which can be adjusted upward via the ratchet provision (step-up) or bonus feature (roll-up). The ratchet mechanism increases the benefit case to the level of the policy account at the time right after the ratchet event date if the current policy account value exceeds the benefit base resulted from the withdrawal. If the policyholder chooses not to withdraw any amount on a withdrawal date in the accumulation phase, then the benefit base is increased proportionally by the bonus rate. The policyholder is allowed to have additional purchases of fund units until the income phase is initiated. In the income phase, the contractual withdrawal amount is a fixed proportion of the benefit base. On the other hand, the contractual withdrawal amount is set to be zero in the accumulation phase. The policyholder is also allowed to withdraw more than the contractual withdrawal amount and the extra amount is subject to a proportional penalty charge. If the policyholder withdraw the whole policy account, then this signifies complete surrender. Indeed, complete surrender is allowed in both the accumulation phase and income phase. Another event that causes the termination of the contract is the death of the policyholder. The value that remains in the policy account will be passed to a beneficiary. We assume that all events of additional purchase, initiation of the income phase, death payment event, withdrawals, surrender, bonus and ratchet provisions are limited to a predetermined set of event dates.

We present the following list of notations used in our later discussion.

**Notations**

\( T \): maximum remaining longevity of the policyholder

\( \mathcal{T} \): set of the annual event dates, where \( \mathcal{T} = \{1, 2, \ldots, T - 1\} \)

\( \mathcal{T}_e \): set of the ratchet event dates, \( \mathcal{T}_e \subseteq \mathcal{T} \)
the last event date on which the policyholder can remain in the accumulation phase, beyond which the income phase will be activated automatically.

\( T_a \): \text{characterized by the vector} \((\gamma_1, \gamma_2, \ldots, \gamma_{T-1})\), \text{where} \( \gamma_i \) \text{is the annual withdrawal amount or additional purchase (considered as negative withdrawal) on the withdrawal date} \( i \).

\( W_t \): \text{time-}t \text{ policy fund value process}

\( A_t \): \text{time-}t \text{ benefit base process}

\( B \): \text{cap multiplier of the benefit base that fixes the upper bound of additional purchase}

\( \eta_b \): \text{percentage of the benefit base charged on the policy fund value as the annual rider fee}

\( \tau_I \): \( \mathcal{F}_t \)-stopping time at which the policyholder activates the income phase

\( \tau_S \): \( \mathcal{F}_t \)-stopping time at which the policyholder chooses to surrender the contract

\( G(\tau_I) \): \text{percentage of the benefit base for calculating the annual contractual withdrawal amounts with dependence on the initiation time of the income phase} \( \tau_I \)

\( k_i \): \text{proportional penalty charge applied on the excess of withdrawal amount over the contractual withdrawal at year} \( i \)

\( b_i \): \text{bonus rate at year} \( i \)

\( x_0p_i \): \text{probability that an} \( x_0 \)-year old policyholder survives in the next} \( i \) \text{ years (written as} \( p_i \) \text{ for notational convenience for fixed} \( x_0 \))

\( q_{x_0+i} \): \text{probability that a policyholder at age} \( x_0+i \) \text{ dies within the next year (written as} \( q_i \) \text{ for notational convenience for fixed} \( x_0 \))

In our pricing model, the joint process of policy fund value and its stochastic variance \( \{(W_t, v_t)\}_{0 \leq t \leq T} \) is assumed to be a two-dimensional càdlàg Markov process defined on the filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{Q})\) between any two consecutive withdrawal dates \( i < t < i+1 \), \( i = 0, 1, \ldots, T-1 \). The technical condition imposed on the joint process \( \{(W_t, v_t)\}_{0 \leq t \leq T} \) to be càdlàg Markov process stems from the convexity analysis of the price function of the GLWB product in the bang-bang analysis. The general two-dimensional càdlàg Markov process nests most popular stochastic models, such as the geometric Brownian motion, Heston’s model, Merton’s jump diffusion model and the double-exponential jump diffusion model.

The mortality risk is assumed to be diversifiable across a large number of policyholders. The optimal complete surrender time is dictated by the optimal choice of the withdrawal amount \( \gamma_i \), where

\[ \tau_S = \inf \{i \in \mathcal{T} | \gamma_i = W_i - \eta_b A_i > 0 \} . \]

Implicitly, the complete surrender amount \( W_i - \eta_b A_i \) should be larger than the contractual withdrawal amount \( G(\tau_I)A_i \) in the income phase for activation of the optimal complete surrender.

**Specification of the accumulation phase and income phase**

The initiation of the GLWB contract starts in the accumulation phase and the policyholder has the right to activate the income phase or stay in the accumulation phase at each of the later
event dates. Once leaving the accumulation phase, the GLWB contract stays in the income phase for the remaining life of the contract. In the accumulation phase, the policyholder is allowed to make additional purchase or withdraw any nonnegative amount up to full depletion of the fund (surrender), otherwise there is no contractual withdrawal. The benefit base enjoys the bonus feature if no withdrawal or additional purchase is made. There is a maximum length of the period of the accumulation phase, where the GLWB contract is mandated to move into the income phase beyond the specified date \( T_a \). In the income phase, additional purchase is not allowed while the policyholder is guaranteed to receive the contractual withdrawal amount even when the policy account value is fully depleted. The updating procedures of the policy fund value and benefit base observe different mechanisms in the two different phases, the details of which are presented later.

**Jump conditions across a withdrawal date in the income phase**

In the income phase, when the withdrawal amount \( \gamma_i \) at year \( i \) taken by the policyholder does not exceed the contractual withdrawal amount \( G(\tau_i)A_i \), then the benefit base would not be reduced and the withdrawal is not subject to penalty charge. When \( \gamma_i \) exceeds \( G(\tau_i)A_i \), the benefit base decreases proportionally according to the ratio of the policy fund value resulted from the actual withdrawal to that resulted from contractual withdrawal. On the other hand, the updated benefit base may benefit from the ratchet provision when the policy fund value after withdrawal together with payment of rider charge exceeds the updated benefit base. The jump conditions of the benefit base and policy fund value across the withdrawal date at year \( i \) are presented below:

\[
W_{i+} = \left( (W_i - \eta_i A_i)^+ - \gamma_i \right)^+ \quad 0 \leq \gamma_i \leq \max(W_i - \eta_i A_i, G(\tau_i)A_i); \\
A_{i+} = \left\{ \begin{array}{ll}
\max \left( A_i, \left( (W_i - \eta_i A_i)^+ - \gamma_i \right)^+ 1_{\{i \in \mathcal{T}_i\}} \right) & \text{if } 0 \leq \gamma_i \leq G(\tau_i)A_i \\
\max \left( \frac{W_i - \eta_i A_i - \gamma_i}{W_i - \eta_i A_i - G(\tau_i)A_i} A_i, \left( (W_i - \eta_i A_i)^+ - \gamma_i \right)^+ 1_{\{i \in \mathcal{T}_i\}} \right) & \text{if } G(\tau_i)A_i < \gamma_i \leq W_i - \eta_i A_i
\end{array} \right. .
\]

Note that \( W_i \) has zero value as the floor and the rider charge \( \eta_i A_i \) is deducted from the policy fund value \( W_i \) before the policyholder makes the withdrawal \( \gamma_i \). Since the excess withdrawal beyond the contractual withdrawal amount \( G(\tau_i)A_i \) is charged at proportional penalty rate \( k_i \), the actual cash amount received by the policyholder in the income phase resulted from the withdrawal amount \( \gamma_i \) is given by

\[
f_i^I(\gamma_i; A_i, G(\tau_i)) = \left\{ \begin{array}{ll}
\gamma_i & \text{if } 0 \leq \gamma_i \leq G(\tau_i)A_i \\
G(\tau_i)A_i + (1 - k_i) [\gamma_i - G(\tau_i)A_i] & \text{if } G(\tau_i)A_i < \gamma_i \leq W_i - \eta_i A_i
\end{array} \right. .
\]

**Jump condition across a withdrawal date in the accumulation phase**

In the accumulation phase at year \( i \), the benefit base \( A_i \) rolls up by predetermined bonus rate \( b_i \) if there is no withdrawal. The policyholder is allowed to have an additional purchase of the fund units, which would increase both the benefit base and policy account value. Otherwise, any positive withdrawal taken by the policyholder reduces both the policy fund value and benefit base, and the withdrawal amount is subject to penalty charge. The jump conditions on the benefit base and policy fund value across year \( i \) are summarized as follows:

\[
W_{i+} = \left( (W_i - \eta_i A_i)^+ - \gamma_i \right)^+ - BA_i \leq \gamma_i \leq (W_i - \eta_i A_i)^+; \\
A_{i+} = \left\{ \begin{array}{ll}
\max \left( A_i (1 + b_i) - \gamma_i, \left( (W_i - \eta_i A_i)^+ - \gamma_i \right)^+ 1_{\{i \in \mathcal{T}_i\}} \right) & \text{if } -BA_i \leq \gamma_i \leq 0 \\
\max \left( \frac{W_i - \eta_i A_i - \gamma_i}{W_i - \eta_i A_i} A_i, \left( (W_i - \eta_i A_i)^+ - \gamma_i \right)^+ 1_{\{i \in \mathcal{T}_i\}} \right) & \text{if } 0 < \gamma_i \leq (W_i - \eta_i A_i)^+
\end{array} \right. .
\]

The cash flow \( f_i^A(\gamma_i; A_i) \) received by the policyholder as resulted from the withdrawal amount \( \gamma_i \) is given by

\[
f_i^A(\gamma_i; A_i) = \left\{ \begin{array}{ll}
\gamma_i & \text{if } -BA_i \leq \gamma_i \leq 0 \\
(1 - k_i) \gamma_i & \text{if } 0 < \gamma_i \leq (W_i - \eta_i A_i)^+
\end{array} \right. .
\]
Note that $\gamma$ may assume negative value up to $-BA_i$, which indicates that additional purchase can be up to the cap multiplier $B$ times the benefit base $A_i$.

The vector functions $(W_i^+, A_i^+) = h_i^A(W_i, A_i, \gamma_i)$ in the accumulation phase and $(W_i^+, A_i^+) = h_i^I(W_i, A_i, \gamma_i; G(\tau_I))$ in the income phase are introduced to characterize the jump conditions of the policy fund value and benefit base associated with the withdrawal amount $\gamma_i$ in the accumulation phase ($i < \tau_I$) and income phase ($i \geq \tau_I$), respectively.

(i) In the accumulation phase where $i < \tau_I$, we have

\[
(W_i^+, A_i^+) = h_i^A(W_i, A_i, \gamma_i) = \begin{cases} 
\left(\left(\left(W_i - \eta_i A_i\right)^+ - \gamma_i\right)^+ \max \left(A_i(1 + b_i) - \gamma_i, (W_i - \eta_i A_i)^+ - \gamma_i\right)^+ 1_{(i \in \mathcal{T}_i)} \right) & \text{if } -BA_i \leq \gamma_i \leq 0 \\
\left(\left(\left(W_i - \eta_i A_i\right)^+ - \gamma_i\right)^+ \max \left(W_i - \eta_i A_i, (W_i - \eta_i A_i)^+ - \gamma_i\right)^+ 1_{(i \in \mathcal{T}_i)} \right) & \text{if } 0 < \gamma_i \leq (W_i - \eta_i A_i)^+.
\end{cases}
\tag{2.5a}
\]

(ii) In the income phase where $i \geq \tau_I$, we have

\[
(W_i^+, A_i^+) = h_i^I(W_i, A_i, \gamma_i; G(\tau_I)) = \begin{cases} 
\left(\left(\left(W_i - \eta_i A_i\right)^+ - \gamma_i\right)^+ \max \left(A_i, (W_i - \eta_i A_i)^+ - \gamma_i\right)^+ 1_{(i \in \mathcal{T}_i)} \right) & \text{if } 0 \leq \gamma_i \leq G(\tau_I)A_i \\
\left(\left(\left(W_i - \eta_i A_i\right)^+ - \gamma_i\right)^+ \max \left(W_i - \eta_i A_i, (W_i - \eta_i A_i)^+ - \gamma_i\right)^+ 1_{(i \in \mathcal{T}_i)} \right) & \text{if } G(\tau_I)A_i < \gamma_i \leq (W_i - \eta_i A_i)^+.
\end{cases}
\tag{2.5b}
\]

The full amount of the policy fund value $W_i$ is given to the beneficiary at year $i$ as the death payment when the policyholder dies within year $i - 1$ and year $i$. The set of the control variables do not include $\tau_S$ since $\tau_S$ is implicitly dictated by the optimal choices of $\Gamma$ and $\tau_I$. Let $\mathcal{E}$ be the admissible strategy set for the pair of control variables $(\Gamma, \tau_I)$. Taking mortality risk into account, the value function $V(W, A, v, 0)$ at initiation is determined by choosing the control variables $(\Gamma, \tau_I)$ so as to maximize the expectation of discounted cash flow. This gives

\[
V(W, A, v, 0) = \sup_{(\Gamma, \tau_I) \in \mathcal{E}} E_Q \left[ \sum_{i=1}^{\tau_S \wedge (T-1)} e^{-r_i} p_{i-1} q_{i-1} W_i + \sum_{i=1}^{(\tau_I-1) \wedge \tau_S} e^{-r_i} p_i f_i^A(\gamma_i; A_i) \right. \\
+ \left. \sum_{i=\tau_I}^{T} e^{-r_i} p_i f_i^I(\gamma_i; A_i, G(\tau_I)) + 1_{\{\tau_S > T-1\}} e^{-r_T} p_{T-1} W_T \right]. \tag{2.6}
\]

The first summation term represents the death payment weighted by the probability of mortality from the first withdrawal date to the complete surrender time $\tau_S$ or $T - 1$, whichever comes earlier. The second summation term gives the sum of discounted withdrawal cash flows from the initiation date of the contract to the last withdrawal date in the accumulation phase or the complete surrender time $\tau_S$, whichever comes earlier. The third summation term gives the sum of discounted withdrawal cash flows from the activation time of the income phase to the complete surrender time $\tau_S$ or $T - 1$, whichever comes earlier. The last single term is the discounted cash flow received by the policyholder at the maximum remaining life $T$ provided that complete surrender has never been adopted throughout the whole life of the policy. In our
subsequent exposition, we drop the subscript $Q$ in the expectation operator $E_Q$ for brevity.

**Dynamic programming procedure**

We write GLWB$^{(A)}$ and GLWB$^{(I)}$ to represent the GLWB rider in the accumulation phase and income phase, respectively. The time-$t$ value function of GLWB$^{(I)}$, denoted by $V^{(I)}(W, A, v, i; G_0)$, is seen to have dependence on the guaranteed withdrawal rate $G(\tau_t)$. Since the contractual withdrawal rate depends on the activation time of the income phase $T$, it is necessary to calculate a set of $V^{(I)}(W, A, v, t; G_0)$ with $G_0$ being set to be $G(i), i = 1, 2, \ldots, T + 1$. For example, suppose the policyholder purchases the GLWB at the age of 50 years old. The GLWB contract sets the contractual withdrawal rate to be 5% if the activation time is between age 70 and age 75; 6% if the activation time is between age 75 and age 80; 6.5% if the activation time is beyond age 80. It is necessary to calculate $V^{(I)}(W, A, v, t; G_0)$ with $G_0$ being equal to the four separate cases: 5%, 5.5%, 6%, and 6.5%, respectively. For notational convenience, we let \{G_{n_1}, \ldots, G_{n_K}\} denote all possible outcomes for $G_0$.

Using the dynamic programming principle of backward induction, we compute $V^{(I)}(W, A, v, i; G_0)$ as follows:

$$V^{(I)}(W, A, v, T; G_0) = p_{T-1}W_T,$$

$$V^{(I)}(W, A, v, i; G_0) = p_{i-1}q_{i-1}W_i + \sup_{\gamma_i \in [0, \max((W_i, -\eta_i, A_i, G_0A), 0)]} \{p_i f^I_i(\gamma_i; A_i, G_0)$$

$$+ e^{-r}E[V^{(I)}(W, A, v, i + 1; G_0)|(W_{i+1}, A_{i+1}) = h^I_i(W_i, A_i, \gamma_i; G_0), v_{i+1} = v_i]\},$$

where $i = 1, 2, \ldots, T - 1$ and $G_0 = G_{n_k}, k = 1, \ldots, K$. Since the GLWB rider is in the accumulation phase at the initiation of the contract, the calculation of the value function of GLWB$^{(I)}$ at time 0 is not required for pricing the GLWB rider. In Section 3, we show the bang-bang analysis for GLWB$^{(I)}$; and in Section 4, we present an efficient Fourier transform algorithm to calculate the values of $V^{(I)}(W, A, v, t; G_{n_k})$ for $k = 1, \ldots, K$.

Similarly, we let $V^{(A)}(W, A, v, t)$ be the time-$t$ value function of GLWB$^{(A)}$. Since $T_a$ is the last withdrawal date on which the GLWB contract may stay in the accumulation phase, we start from the withdrawal date $T_a$ and calculate $V^{(A)}(W, A, v, T_a)$. Let $V^{(A)}(T_a)$ be the continuation value at year $T_a$ conditional on the policyholder chooses to remain in the accumulation phase on $T_a$. Also, we let $V^{(I)}(T_a)$ be the continuation value at year $T_a$ conditional on the policyholder chooses to activate the income phase. We then have

$$V^{(A)}(W, A, v, T_a) = p_{T_a-1}q_{T_a-1}W_{T_a} + \max\{V^{(A)}(T_a), V^{(I)}(T_a)\},$$

where

$$V^{(A)}(T_a) = \sup_{\gamma_{T_a} \in \{-B_{A_{T_a}}(W_{T_a} - \eta_{A_{T_a}})\}^+} \{p_{T_a} f^{A}_{T_a}(\gamma_{T_a}; A_{T_a})$$

$$+ e^{-r}E[V^{(I)}(W, A, v, T_a + 1; G(T_a + 1))|(W_{T_a+1}, A_{T_a+1}) = h^A_{T_a}(W_{T_a}, A_{T_a}, \gamma_{T_a}, v_{T_a} = v)]\},$$

$$V^{(I)}(T_a) = \sup_{\gamma_{T_a} \in [0, \max((W_{T_a} - \eta_{A_{T_a}})^+, G(T_a)A_{T_a})]} \{p_{T_a} f^{I}_{T_a}(\gamma_{T_a}; A_{T_a}, G(T_a))$$

$$+ e^{-r}E[V^{(I)}(W, A, v, T_a + 1; G(T_a))|(W_{T_a+1}, A_{T_a+1}) = h^I_{T_a}(W_{T_a}, A_{T_a}, \gamma_{T_a}, G(T_a), v_{T_a} = v)]\}.$$
guaranteed withdrawal rate is set to be \( G(T_a) \). As a remark, if \( V_C^{(A)}(T_a) \geq V_C^{(I)}(T_a) \), then it is optimal for the policyholder to choose to remain in the accumulation phase on the event date \( T_a \). Otherwise, it is optimal to initiate the income phase at year \( T_a \) and the control variable \( \tau_I \) is set to be \( T_a \) accordingly. Besides the optimal timing of activation of the income phase, the policyholder can also determine the optimal withdrawal strategy \( \gamma_{T_a} \) for the value function \( V^{(A)}(W, A, v, T_a) \) for fixed \( W \) and \( A \) at year \( T_a \) using the Fourier transform method, details of which are discussed in the following sections.

For an earlier event date, \( 1 \leq i \leq T_a - 1 \), we have

\[
V^{(A)}(W, A, v, i) = p_{i-1}q_{i-1}W_i + \max\{V_C^{(A)}(i), V_C^{(I)}(i)\},
\]

where

\[
V_C^{(A)}(i) = \sup_{\gamma_i \in [-B A_i, (W_i - \eta_{i} A_i)^+]} \left\{ p_i f_i^A(\gamma_i; A_i) + e^{-r} E[V^{(A)}(W, A, v, i + 1)| (W_{i+}, A_{i+}) = h_i^A(W_{i}, A_{i}, \gamma_{i}), v_{i+} = v] \right\},
\]

\[
V_C^{(I)}(i) = \sup_{\gamma_i \in [0, \max((W_i - \eta_{i} A_i)^+, G(i))]} \left\{ p_i f_i^I(\gamma_i; A_i, G(i)) + e^{-r} E[V^{(I)}(W, A, v, i + 1; G(i))| (W_{i+}, A_{i+}) = h_i^I(W_{i}, A_{i}, \gamma_{i}; G(i)), v_{i+} = v_i] \right\}.
\]

Here, \( V_C^{(A)}(i) \) corresponds to the case that the policyholder chooses not to activate the income phase at year \( i \). Since the policyholder is entitled to choose to stay in the accumulation phase or activate the income phase in the next year \( i + 1 \), we evaluate the conditional expectation of \( V^{(A)}(W, A, v, i + 1) \). Since \( V_C^{(I)}(i) \) corresponds to the case that the policyholder chooses to activate the income phase in year \( i \), so we consider the evaluation of the conditional expectation of \( V^{(I)}(W, A, v, i + 1; G(i)) \). Since the GLWB contract on the initiation date, where \( i = 0 \), is in the accumulation phase and there is no withdrawal event at the beginning of the contract, we have

\[
V(W, A, v, 0) = e^{-r} E[V^{(A)}(W, A, v, 1)].
\]  

### 3 Bang-bang analysis

The design of the numerical algorithm would be much simplified if the choices of the optimal withdrawal amount \( \gamma_i \) are limited to a finite number of discrete values. When an additional purchase is not allowed, though it has been commonly assumed in the actuarial mathematics literature without proof that the optimal withdrawal policies are either zero withdrawal, withdrawal at the contractual amount or complete surrender, it is instructive to perform a rigorous bang-bang analysis of the strategy space of the optimal withdrawal policies. The technical analysis relies on the convexity and monotonicity properties of the value function. As part of the technical procedure, it is necessary to require the two-dimensional Markov process \( \{(W_t, v_t)\}_t \) to observe the following mathematical properties:

**Property 1 (Convexity preservation)** For any convex terminal payoff function \( \Phi(W_T) \), the corresponding European price function as defined by

\[
\phi(w, v) = e^{-r(T-t)} E[\Phi(W_T)| W_t = w, v_t = v], \quad t \leq T,
\]

is also convex with respect to \( w \).
Property 2 (Scaling) For any positive $K$, the two stochastic processes $\{(W_t, v_t)\}_t$ and $\{\left(\frac{W_t}{K}, v_t\right)\}_t$ have the same distribution law given that their initial values are the same with each other almost surely.

Ekström and Tysk (2007) analyze the property of convexity preservation for option prices in models with jump and show that many popular jump diffusion models, such as Merton’s jump diffusion model and Kou’s model, satisfy Property 1. Hobson (2010) gives a sufficient condition for convexity preservation in stochastic volatility models and the corresponding condition required for the Heston model also satisfies Property 1. Moreover, it is obvious that all the above mentioned models satisfy Property 2. The class of two-dimensional Markov processes that observe both Properties 1 and 2 indeed include the most popular models of asset price processes.

By virtue of Property 2, together with the invariant forms of the bonus rate $b$, guaranteed withdrawal $G$, cap multiplier for additional purchase $B$ and penalty charge $k$, the value functions $V^{(I)}$ and $V^{(A)}$ satisfy the following scaling properties for any positive scalar $K$:

$$V^{(I)}(KW, KA, v, t; G_0) = KV^{(I)}(W, A, v, t; G_0)$$  (3.1a)

$$V^{(A)}(KW, KA, v, t) = KV^{(A)}(W, A, v, t).$$  (3.1b)

By virtue of the above scaling properties, we can achieve reduction in dimensionality of the pricing model by one when we calculate the conditional expectations in the dynamic programming procedure. The scaling properties are also crucial in establishing the bang-bang control analysis.

Theorem 3 characterizes the bang-bang control strategies for GLWB$^{(I)}$ and GLWB$^{(A)}$.

**Theorem 3** Assume that $\{(W_t, v_t)\}_t$ satisfies both Properties 1 and 2, GLWB$^{(I)}$ and GLWB$^{(A)}$ observe the following optimal withdrawal strategy, respectively.

1. On any withdrawal date $i$, the optimal withdrawal strategy $\gamma_i$ for GLWB$^{(I)}$ with a positive guaranteed rate $G_0$ is limited to (i) $\gamma_i = 0$; (ii) $\gamma_i = G_0A_i$; or (iii) $\gamma_i = W_i - \eta_b A_i$.

2. On any withdrawal date $i$, the optimal strategy on this withdrawal date for GLWB$^{(A)}$ is either

   (2a) to initiate the income phase on this withdrawal date if $V^{(I)}_C(i) > V^{(A)}_C(i)$ and the subsequent optimal withdrawal strategy $\gamma_i$ is limited to (i) $\gamma_i = 0$; (ii) $\gamma_i = G(i)A_i$; or (iii) $\gamma_i = W_i - \eta_b A_i$;

   (2b) or to remain in the accumulation phase on this withdrawal date if $V^{(I)}_C(i) \leq V^{(A)}_C(i)$ and the optimal withdrawal strategy $\gamma_i$ is limited to (i) $\gamma_i = -BA_i$; (ii) $\gamma_i = 0$; or (iii) $\gamma_i = W_i - \eta_b A_i$.

In summary, when the policy is already in the income phase, the withdrawal policies are limited to zero withdrawal, withdrawal at the contractual rate or complete surrender. When the policy is in the accumulation phase, the policyholder may choose to enter into the income phase or stay in the accumulation phase. The subsequent optimal policies while staying in the accumulation phase are limited to maximum allowable purchase, zero withdrawal or complete surrender. The proof of Theorem 3 is presented Appendix A. Interestingly, one may deduce the following set of dominated strategies as stated in Corollary 4.

**Corollary 4** When positive bonus rate and penalty charge rate are applied on the withdrawal date $i$ and the income phase has not been activated before date $i$, the strategy of staying in the
Fourier transform algorithms

In this section, we construct the efficient Fourier transform algorithms for pricing the GLWB product with policy fund value under the Heston model and complex path dependent features arising from the ratchet and bonus events, dynamic control of withdrawals and additional purchases, together with optimality in the time of initiation of the income phase. By the recursive backward induction procedure, the computation starts with the discounted expectation of the value function in the income phase. We then proceed backward in time to compute the value function in the accumulation phase. Thanks to the bang-bang analysis, it suffices to consider the choice set of \( \gamma_i \) at year \( i \) to be \( \{-B A_i, 0, (W_i - \eta A_i)^+\} \) and \( \{0, (W_i - \eta A_i)^+, G(i) A_i\} \) in the accumulation phase and income phase, respectively. Assuming annualized event dates and adopting the dynamic programming procedure, the value function \( V^{(A)}(W, A, v, i) \) on the withdrawal date \( i \) in the accumulation phase can be expressed as

\[
V^{(A)}(W, A, v, i) = p_{i-1} q_{i-1} W_i + \max \{V^{(A)}_C(i), V^{(I)}_C(i)\},
\]

where

\[
V^{(A)}_C(i) = \sup_{\gamma_i \in \{-B A_i, 0, (W_i - \eta A_i)^+\}} \{p_i f^A_i(\gamma_i; A_i)
+ e^{-r} E[V^{(A)}(W, A, v, i + 1)|(W_{i+}, A_{i+}) = h^A_i(W_i, A_i, \gamma_i), v_{i+} = v]\},
\]

\[
V^{(I)}_C(i) = \sup_{\gamma_i \in \{0, (W_i - \eta A_i)^+, G(i) A_i\}} \{p_i f^I_i(\gamma_i; A_i, G(i))
+ e^{-r} E[V^{(I)}(W, A, v, i + 1; G(i))|(W_{i+}, A_{i+}) = h^I_i(W_i, A_i, \gamma_i, G(i)), v_{i+} = v_i]\}.
\]

We observe continuity of the value function across the two phases at \( T_a + 1 \), where

\[
V^{(A)}(W, A, v, T_a + 1) = V^{(I)}(W, A, v, T_a + 1; G(T_a + 1)).
\]

Since the benefit base \( A_t \) remains unchanged between consecutive event dates, we achieve dimensionality reduction by defining the normalized policy fund value \( \hat{W}_t \) and normalized value functions \( \hat{V}^{(I)}(\hat{W}, v, t; G(i)) \) and \( \hat{V}^{(A)}(\hat{W}, v, t) \) as follows:

\[
\hat{W}_t = \frac{W_t}{A_t},
\]

\[
\hat{V}^{(I)}(\hat{W}, v, t; G(i)) = \frac{V^{(I)}(W, A, v, t; G(i))}{A_t},
\]

\[
\hat{V}^{(A)}(\hat{W}, v, t) = \frac{V^{(A)}(W, A, v, t)}{A_t}.
\]
By considering the possible choices of withdrawal strategies and the jump conditions on \( \tilde{W}_i \) and \( A_i \) across the event date \( i \) under the respective withdrawal strategy, the normalized value function in the accumulation phase \( \tilde{V}^{(A)}(\bar{W}, v, i) \) can be written as

\[
\tilde{V}^{(A)}(\bar{W}, v, i) = p_i q_{i-1} \tilde{W}_i + \max \left\{ p_i \left[ G(i) + (1 - \kappa_i)(\tilde{W}_i - \eta_b - G(i)) \right], \right.
\]

\[
p_i G(i) + e^{-r} \max \left\{ (1, \left[ \tilde{W}_i - \eta_b - G(i) \right]_1 \right\}, \right.
\]

\[
E \left[ \tilde{V}^{(I)}(\bar{W}, v, i + 1; G(i)) \right| \tilde{W}_{i+} = \max \left\{ (\tilde{W}_i - \eta_b - G(i))^{+}, \max \left\{ (1, \left[ \tilde{W}_i - \eta_b - G(i) \right]_1 \right\}, v_i \right. \right]
\]

\[
- p_i B + e^{-r} \max \left\{ (1 + b_i + B, [(\tilde{W}_i - \eta_b)^+ + B]_1 \right\}
\]

\[
E \left[ \tilde{V}^{(A)}(\bar{W}, v, i + 1) \right| \tilde{W}_{i+} = \max \left\{ (\tilde{W}_i - \eta_b + B) \right\}
\]

\[
e^{-r} \max \left\{ (1 + b_i, (\tilde{W}_i - \eta_b)_1 \right\} E \left[ \tilde{V}^{(A)}(\bar{W}, v, i + 1) \right| \tilde{W}_{i+} = \max \left\{ (\tilde{W}_i - \eta_b + B) \right\}, v_i \right] \right\}. \quad (4.2)
\]

For notational convenience, we define the following functions

\[
\phi_i^{(1)}(x) = \frac{(x - \eta_b - G(i))^{+}}{\max \left\{ (1, \left[ x - \eta_b - G(i) \right]_1 \right\}}, \right.
\]

\[
\phi_i^{(2)}(x) = \frac{(x - \eta_b)^+ + B}{\max \left\{ (1 + b_i + B, [(x - \eta_b)^+ + B]_1 \right\}}, \right.
\]

\[
\phi_i^{(3)}(x) = \frac{(x - \eta_b)^+}{\max \left\{ (1 + b_i, (x - \eta_b)_1 \right\}}, \right.
\]

\[
\psi_i^{(1)}(x) = \max \left\{ (1, \left[ x - \eta_b - G(i) \right]_1 \right\}, \right.
\]

\[
\psi_i^{(2)}(x) = \max \left\{ (1 + b_i + B, [(x - \eta_b)^+ + B]_1 \right\}, \right.
\]

\[
\psi_i^{(3)}(x) = \max \left\{ (1 + b_i, (x - \eta_b)_1 \right\}. \right.
\]

Here, \( \{ \phi_i^{(j)} \}_{j=1,2,3} \) relate \( \tilde{W}_i \) and \( \tilde{W}_{i+} \) across the withdrawal date \( i \) under the three respective withdrawal strategies while \( \{ \psi_i^{(j)} \}_{j=1,2,3} \) give the multiplier for the benefit base arising from the corresponding jump condition. In terms of these functions, the normalized value function in the accumulation phase \( \tilde{V}^{(A)}(\bar{W}, v, i) \) can be rewritten into a more concise representation:

\[
\tilde{V}^{(A)}(\bar{W}, v, i) = p_i q_{i-1} \tilde{W}_i + \max \left\{ p_i \left[ G(i) + (1 - \kappa_i)(\tilde{W}_i - \eta_b - G(i)) \right], \right.
\]

\[
p_i G(i) + e^{-r} \psi_i^{(1)}(\tilde{W}_i) E \left[ \tilde{V}^{(I)}(\bar{W}, v, (i + 1); G(i)) \right| \tilde{W}_{i+} = \phi_i^{(1)}(\tilde{W}_i), v_i \right], \right.
\]

\[
- p_i B + e^{-r} \psi_i^{(2)}(\tilde{W}_i) E \left[ \tilde{V}^{(A)}(\bar{W}, v, i + 1) \right| \tilde{W}_{i+} = \phi_i^{(2)}(\tilde{W}_i), v_i \right], \right.
\]

\[
e^{-r} \psi_i^{(3)}(\tilde{W}_i) E \left[ \tilde{V}^{(A)}(\bar{W}, v, i + 1) \right| \tilde{W}_i = \phi_i^{(3)}(\tilde{W}_i), v_i \right], \quad i = 1, 2, \ldots, T_a. \quad (4.3)
\]

Recall that \( \{ G_{n_1}, \ldots, G_{n_K} \} \) denote all possible outcomes for the contractual withdrawal rate \( G_0 \). We introduce another set of functions that serve to capture the corresponding jump conditions...
on $\tilde{W}_i$ and $A_i$ across the withdrawal date $i$ in the income phase as follows:

$$\phi_i^{(4)}(x) = \frac{(x - \eta_b)^+}{\max(1, (x - \eta_b)1_{i \in T_i})},$$

$$\phi_i^{(5)}(x; G_{nk}) = \frac{(x - \eta_b - G_{nk})^+}{\max(1, (x - \eta_b - G_{nk})1_{i \in T_i})},$$

$$\psi_i^{(4)}(x) = \max(1, (x - \eta_b)1_{i \in T_i}),$$

$$\psi_i^{(5)}(x; G_{nk}) = \max(1, (x - \eta_b - G_{nk})1_{i \in T_i}).$$

In terms of these functions, the normalized value function in the income phase $\tilde{V}^{(I)}(\tilde{W}, v, i; G_{nk})$ can be expressed as follows

$$\tilde{V}^{(I)}(\tilde{W}, v, i; G_{nk}) = p_{i-1}q_{i-1}\tilde{W}_i + \max\left\{ p_i \left[ G_{nk} + (1 - \kappa_i)(\tilde{W}_i - \eta_b - G_{nk}) \right], e^{-r}\psi_i^{(4)}(\tilde{W}_i)E\left[ \tilde{V}^{(I)}(\tilde{W}, v, i + 1; G_{nk}) \left| \tilde{W}_{i+1} = \phi_i^{(4)}(\tilde{W}_i), v_i \right. \right], p_iG_{nk} + e^{-r}\psi_i^{(5)}(\tilde{W}_i)E\left[ \tilde{V}^{(I)}(\tilde{W}, v, i + 1; G_{nk}) \left| \tilde{W}_{i+1} = \phi_i^{(5)}(\tilde{W}_i), v_i \right. \right]\right\}, \quad i = 1, 2, \ldots, T - 1.$$  \hfill (4.4)

As a result, we can also evaluate $\tilde{V}^{(A)}(\tilde{W}, v, i)$ in an alternative way as follows:

$$\tilde{V}^{(A)}(\tilde{W}, v, i) = p_{i-1}q_{i-1}\tilde{W}_i + \max\left\{ -p_{i-1}q_{i-1}\tilde{W}_i + \tilde{V}^{(I)}(\tilde{W}, v, i; G(i)) \right.$$  

$$- p_iB + e^{-r}\psi_i^{(2)}(\tilde{W}_i)E\left[ \tilde{V}^{(A)}(\tilde{W}, v, i + 1) \left| \tilde{W}_{i+1} = \phi_i^{(2)}(\tilde{W}_i), v_i \right. \right],$$

$$e^{-r}\psi_i^{(3)}(\tilde{W}_i)E\left[ \tilde{V}^{(A)}(\tilde{W}, v, i + 1) \left| \tilde{W}_i = \phi_i^{(3)}(\tilde{W}_i), v_i \right. \right]\right\}, \quad i = 1, 2, \ldots, T_a. \hfill (4.5)$$

We formulate the backward induction calculations combined with the dynamic programming procedure for the normalized value functions in the income and accumulation phases as follows:

1. The backward induction procedure is initiated by observing the following terminal condition corresponding to the respective scheduled withdrawal rate $G_{nk}$:

$$\tilde{V}^{(I)}(\tilde{W}, v, T; G_{nk}) = p_{T-1}\tilde{W}_T,$$

where $k = 1, \cdots, K$.

2. Time-stepping calculations between the consecutive event dates

First, we evaluate $\tilde{V}^{(I)}(\tilde{W}, v, i; G_{nk})$ recursively by eq. (4.4) for $T_a + 1 \leq i \leq T - 1$ and $1 \leq k \leq K$. Next starting $i$ from $T_a$ to 1, we calculate $\tilde{V}^{(I)}(\tilde{W}, v, i; G_{nk})$ and $\tilde{V}^{(A)}(\tilde{W}, v, i)$ according to eqs. (4.4) and (4.3), respectively.

3. The fair value of the normalized value function at initiation is obtained by setting

$$\tilde{V}^{(A)}(\tilde{W}, v, 0) = e^{-r}E\left[ \tilde{V}^{(A)}(\tilde{W}, v, 1) \left| \tilde{W}_0 = \frac{W_0}{A_0}, v_0 \right. \right].$$
Remark
The above backward induction works for the general class of two-dimensional càdlàg Markov processes. Though our proposed Fourier transform algorithm is constructed under the Heston model, the formulation can be applicable for the 3/2 stochastic volatility model, Merton’s jump diffusion model and the double-exponential jump diffusion model with some slight modifications.

In the numerical valuation of the conditional expectation of the normalized value functions, two technical challenges remain. Firstly, the Fourier transforms of the normalized value functions may be not well defined since the normalized value functions do not tend to zero at the two ends of the domain of definition. Secondly, how does one perform effective conditional expectation calculation of the normalized value functions in the variance domain? We show how to circumvent these difficulties.

Normalized value functions at low policy fund value
Recall that in the above backward induction in calculating
\[ E[\mathcal{V}^{(A)}(\tilde{W}, v, i + 1) | \tilde{W}_{i +}, v_i], \]
we have to consider the two separate cases: (1) \( \tilde{W}_{i +} > 0 \) (2) \( \tilde{W}_{i +} = 0 \). For the first case, we present the Fourier transform algorithm to calculate the two-dimensional expectation in our later discussion. For the second case, since
\[ E[\mathcal{V}^{(A)}(\tilde{W}, v, i + 1) | \tilde{W}_{i +} = 0, v_i] = E[\mathcal{V}^{(A)}(0, v, i + 1) | v_i], \]
so one has to calculate the solution for the normalized value functions at zero policy fund value \( \mathcal{V}^{(A)}(0, v, i + 1) \).

In fact, the normalized value functions do not decay to zero when \( W_i \) approaches to 0 and \( \infty \). As a result, any choice of the damping factor cannot guarantee the existence of the Fourier transform of the damped normalized value functions. Recall that \( \eta_b \) is the percentage of the benefit base charged on the policy fund value as the annual rider fee. Fortunately, one can always find the solutions for the normalized value functions when \( \tilde{W}_i \leq \eta_b \), which plays an important role in constructing the new functions based on the normalized value functions such that the Fourier transform of these damped new functions are well defined. Therefore, let us first show how to derive the solutions for the normalized value functions when \( \tilde{W}_i \leq \eta_b \).

For any \( i \in \{1, 2, \ldots, T\} \), we restrict our attention to the special case that \( \tilde{W}_i \leq \eta_b \), where \( \mathcal{V}^{(i)}(\tilde{W}, v, i; G_{nk}) \) has the closed form representations for \( 1 \leq k \leq K \). In the income phase, since \( \tilde{W}_{i +} = 0 \), we have \( \tilde{W}_t = 0 \) for any \( t > i \) due to no additional purchase. As a result, complete surrender is excluded on all subsequent withdrawal dates. In fact, it is always optimal to withdraw starting from the withdrawal date \( i \). This leads to
\[
\mathcal{V}^{(i)}(\tilde{W}, v, i; G_{nk}) = p_{i-1} q_{i-1} \tilde{W}_i + \sum_{j=i}^{T-1} p_j G_{nk} e^{-r(j-i)}, \quad \text{if} \quad \tilde{W}_i \leq \eta_b. \tag{4.6}
\]
Here, the second term is the sum of the discounted expected withdrawal amounts.

Alternatively, the above closed form solution can also be obtained directly from eq. (4.4). We take \( q_{T-1} = 1 \) so that
\[
\mathcal{V}^{(i)}(\tilde{W}, v, T; G_{nk}) = p_{T-1} \tilde{W}_T.
\]
For notational convenience, we write
\[
g^{(i)}(i; G_{nk}) = \sum_{j=i}^{T-1} p_j G_{nk} e^{-r(j-i)},
\]
so that
\[
\tilde{V}(t)(\tilde{W}, v, i; G_{n_k}) = p_{i-1}q_{i-1}\tilde{W}_i + g^{(I)}(i; G_{n_k}), \quad \text{if } \tilde{W}_i \leq \eta_b. \tag{4.7}
\]
In addition,
\[
E[\tilde{V}(t)(\tilde{W}, v, i + 1; G_{n_k})|\tilde{W}_{i+} = 0, v_i] = g^{(I)}(i + 1; G_{n_k}). \tag{4.8}
\]
According to eq. (4.7), the closed form solution for \(\tilde{V}(t)(\tilde{W}, v, i; G_{n_k})\) shows no dependence on the current variance in the above special case. Unfortunately, \(\tilde{V}(A)(\tilde{W}, v, i)\) does not retain this nice property except at \(T_a + 1\). In fact, we can define
\[
\tilde{V}(A)(\tilde{W}, v, i) = p_{i-1}q_{i-1}\tilde{W}_i + g^{(A)}(v, i), \quad \text{if } \tilde{W}_i \leq \eta_b. \tag{4.9}
\]
Provided that \(\tilde{W}_i \leq \eta_b\) for \(i = 1, \ldots, T_a\), based on eqs. (4.5) (4.7) and (4.9), we can derive
\[
g^{(A)}(v, i) = \max \left\{ g^{(I)}(i; G(i)), \right. \\
- p_i B + e^{-r}[(1 + b_i) + B]E \left[ \tilde{V}(A)(\tilde{W}, v, i + 1)|\tilde{W}_{i+} = \frac{B}{(1 + b_i) + B}; v_i \right], \tag{4.10}
\]
\[
e^{-r(1 + b_i)}E \left[ g^{(A)}(v_{i+1}, i + 1)|v_i \right], \quad \text{if } \tilde{W}_i \leq \eta_b.
\]
Here, \(g^{(A)}(v, i)\) does not admit an analytical representation except when \(B = 0\). Fortunately, starting with the terminal condition \(g^{(A)}(v, T_a + 1) = g^{(I)}(T_a + 1; G(T_a + 1))\), \(g^{(A)}(v, i)\) can be calculated using the Fourier transform method to be shown later.

It is desirable to construct a new set of modified normalized value functions with the property that they are equal to zero once \(\tilde{W}_i \leq \eta_b\). This would guarantee that the generalized Fourier transforms of these two modified functions with respect to \(\log \tilde{W}_i\) are well defined by adopting some proper damping factors. We define the two modified normalized value functions by
\[
U^{(A)}(\tilde{W}, v, i) = \tilde{V}(A)(\tilde{W}, v, i) - (p_{i-1}q_{i-1}\tilde{W}_i + g^{(A)}(v, i)),
\]
\[
U^{(I)}(\tilde{W}, v, i; G_{n_k}) = \tilde{V}(I)(\tilde{W}, v, i; G_{n_k}) - (p_{i-1}q_{i-1}\tilde{W}_i + g^{(I)}(i; G_{n_k})). \tag{4.11}
\]

### 4.1 Expectation calculations of the normalized value functions

Now we are ready to calculate the conditional expectations of the normalized value functions, which is a key step in the backward induction. In order to compute the conditional expectation \(E[\tilde{V}(A)(\tilde{W}, v, i + 1)|\tilde{W}_{i+}, v_i]\), provided that \(\tilde{W}_{i+} > 0\), one has to evaluate the two-dimensional expectation integral \(E[U^{(A)}(\tilde{W}, v, i + 1)|\tilde{W}_{i+}, v_i]\). The latter can be calculated relatively easily since the generalized Fourier transforms of the modified normalized value functions are guaranteed to exist. We would like to apply the Fourier transform method in the log normalized policy fund value dimension and a quadrature rule in the variance dimension since the transition density of the variance \(v_t\) has an analytic form. However, the Feller condition is difficult to satisfy in practice, and the density of variance grows extremely fast in the left tail when the Feller condition fails. To resolve this difficulty, Fang and Oosterlee (2011) propose to transform the density function from the variance domain to the log-variance domain. Interested readers may refer to Fang and Oosterlee (2011), Zeng and Kwok (2014) for more details.

More specifically, based on the dynamics for the policy fund value process \(W_t\), an application of eq. (4.11) gives
\[
E[\tilde{V}(A)(\tilde{W}, v, i + 1)|\tilde{W}_{i+}, v_i] = E[U^{(A)}(\tilde{W}, v, i + 1)|\tilde{W}_{i+}, v_i] + p_i q_i e^{-r\eta_b} \tilde{W}_{i+} + E[g^{(A)}(v_{i+1}, i + 1)|v_i]. \tag{4.12}
\]
We define the log-variance \( \gamma_t = \ln v_t \). By the tower property and conditional on the log-variance process at time \( i + 1 \), we have
\[
E[U^{(A)}(\tilde{W}, v, i + 1)|\tilde{W}_i, v_i] = E \left[ E \left[ U^{(A)}(\tilde{W}, e^\gamma, i + 1)|\tilde{W}_i, \gamma_{i+1}, \gamma_i \right]|\tilde{W}_i, \gamma_i \right].
\]
Now we apply an appropriate \( J \)-point quadrature integration rule (say, the Gauss-Legendre quadrature rule) to effect numerical evaluation of the outer expectation integral, which involves integration over the density function \( p_\gamma(\gamma_{i+1}|\gamma_i) \). By performing discretization along the dimension of \( \gamma_{i+1} \) at the discrete nodes \( \zeta_j, j = 1, 2, \ldots, J \), we obtain
\[
E[U^{(A)}(\tilde{W}, v, i)|\tilde{W}_i, v_i] \approx \sum_{j=1}^J w_j p_\gamma(\zeta_j|\gamma_i) E \left[ U^{(A)}(\tilde{W}, e^\zeta, i + 1)|\tilde{W}_i, \gamma_{i+1} = \zeta_j, \gamma_i \right], \quad (4.13)
\]
where \( w_j \) is the weight at the quadrature node \( \zeta_j, j = 1, 2, \ldots, J \).

To perform the inner expectation calculation, we adopt the Fourier transform method that is commonly used in option pricing (Lord et al., 2008; Kwok et al., 2012). Let \( X_t = \log \tilde{W}_t \), the generalized Fourier transform of \( U^{(A)}(\tilde{W}, e^\zeta, i + 1) \) with respect to \( X_{i+1} \) is defined by
\[
\hat{U}^{(A)}(\beta, e^\omega, i + 1) = \int_{-\infty}^{\infty} e^{(\alpha+i\beta)X_{i+1}} U^{(A)}(\tilde{W}, e^\omega, i + 1) dX_{i+1} = \int_{\log \eta_b}^{\infty} e^{(\alpha+i\beta)X_{i+1}} U^{(A)}(\tilde{W}, e^\omega, i + 1) dX_{i+1}.
\] (4.14)
Here, the parameter \( \alpha \) is a damping factor, which should be properly chosen to insure the existence of the generalized Fourier transform of \( U^{(A)}(\tilde{W}, e^\zeta, i + 1) \). With reference to the conditional moment generating function \( \Psi(\omega, \gamma_t, \gamma_s) = E[e^{\omega(X_t-X_s)}|\gamma_t, X_s, \gamma_s] \), the renowned Parseval relation leads to the following inverse Fourier transform representation:
\[
E \left[ U^{(A)}(\tilde{W}, e^\zeta, i + 1)|\tilde{W}_i, \gamma_{i+1} = \zeta_j, \gamma_i \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(\alpha+i\beta)\log \tilde{W}_i} \hat{U}^{(A)}(\beta, e^\zeta, i + 1) \Psi (-\alpha-i\beta, \zeta_j, \gamma_i) d\beta. \quad (4.15)
\]
Combining eqs. (4.12), (4.13) and (4.15), and interchanging the order of integration and summation, we obtain
\[
E[\hat{V}^{(A)}(\tilde{W}, v, i + 1)|\tilde{W}_i, v_i] = p_\eta q_l e^{r-\eta_l} \tilde{W}_i + \sum_{j=1}^J g(e^{\omega_j}, i + 1) p_\gamma(\zeta_j|\gamma_i) w_j + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(\alpha+i\beta)\log \tilde{W}_i} \sum_{j=1}^J \hat{U}^{(A)}(\beta, e^{\omega_j}, i + 1) \tilde{V} (-\alpha-i\beta, \zeta_j, \gamma_i) w_j d\beta, \quad i = 0, 1, \ldots, T_\alpha. \quad (4.16)
\]
Here, \( \tilde{V}(\omega, \gamma_t, \gamma_s) = \Psi(\omega, \gamma_t, \gamma_s) p_\gamma(\gamma_t|\gamma_s) \) and \( \Psi(\omega, \gamma_t, \gamma_s) \) admits a closed form representation (Fang and Oosterlee, 2011; Zeng and Kwok, 2014). Similarly, for \( i = 1, \ldots, T-1 \), we have
\[
E[\hat{V}^{(I)}(\tilde{W}, v, i + 1; G_{n_k})|\tilde{W}_i, v_i] = p_\eta q_l e^{r-\eta_l} \tilde{W}_i + g^{(I)}(i + 1; G_{n_k}) + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(\alpha+i\beta)\log \tilde{W}_i} \sum_{j=1}^J \hat{U}^{(I)}(\beta, e^{\omega_j}, i + 1; G_{n_k}) \tilde{V} (-\alpha-i\beta, \zeta_j, \gamma_i) w_j d\beta. \quad (4.17)
\]
The expectation corresponding to $i = T - 1$ admits a simple closed form representation

$$E[\tilde{V}^{(i)}(\tilde{W}, v, T; G_{n_k}) | \tilde{W}_{T-1}^+, v_{T-1}] = p_{T-1} e^{\gamma_{\nu} - \eta_{\nu}} \tilde{W}_{T-1}^+. \quad (4.18)$$

Alternatively, this conditional expectation can be derived easily based on eq. (4.17) by observing that $U^{(i)}(\tilde{W}, v, T; G_{n_k})$ is a zero function.

**Implementation procedures**

We would like to present the details of the implementation procedure of the Fourier transform algorithm for computing the value functions. Substituting eq. (4.17) into eq. (4.4) and taking advantage of the relation (4.11) between $U^{(i)}(\tilde{W}, v, i; G_{n_k})$ and $\tilde{V}^{(i)}(\tilde{W}, v, i; G_{n_k})$, we manage to construct the Fourier transform algorithm that calculates $U^{(i)}(\tilde{W}, v, i; G_{n_k})$. Note that $U^{(A)}(\tilde{W}, v, i)$ can be evaluated in a similar manner. The detailed steps in constructing these recursive equations and the terminal condition are provided in Appendix B. The initial value function $V^{(A)}(W_0, A_0, e^{\delta_{\nu}}, 0)$ can be recovered at the last time step. The implementation procedures of the Fourier transform algorithm for calculating the value functions by applying the Fourier transform method to the log fund value process and the quadrature rule to the log-variance dimension are summarized as below.

Let the infinite Fourier domain for $\beta$ be truncated to the finite truncation domain $[-Mh, Mh]$, and consider $X_i = l\Delta$, $l = 1, 2, \ldots, L$. Here, $M$ and $L$ are referred as the truncation levels.

**Step 1: Preparation at $T - 1$**

Calculate the generalized Fourier transform using the following formula

$$\tilde{U}^{(i)}( mh, e^{\delta_{\nu}}, T - 1; G_{n_k}) = \sum_{l=1}^{L} e^{(a + ih)l\Delta} \max \{ p_{T-1}(1 - \kappa_{T-1})(e^{l\Delta} - \eta_{\nu} - G_{n_k}), p_{T-1} e^{-\gamma_{\nu}(e^{l\Delta} - \eta_{\nu} - G_{n_k})} \} \Delta, \quad (4.19)$$

where $m = -M, \ldots, M$, $j = 1, \ldots, J$ and $k = 1, \ldots, K$. Here, $\alpha$ is the damping factor and $\alpha < -1$ is required to guarantee the existence of the generalized Fourier transform.

**Step 2: Backward induction from $T - 2$ to $T_a + 1$**

$$U^{(i)}( e^{l\Delta}; e^{\delta_{\nu}}, i; G_{n_k})$$

$$= -g^{(i)}(i; G_{n_k}) + \max \{ p_i G_{n_k} + p_l (1 - \kappa_{i})(e^{l\Delta} - \eta_{\nu} - G_{n_k}),$$

$$e^{-r_{\nu} \phi_{i}^{(4)}(e^{l\Delta})} \left[ p_i q_i e^{-\gamma_{\nu} \phi_{i}^{(4)}(e^{l\Delta})} + g^{(i)}(i + 1; G_{n_k}) + \frac{1}{2\pi} \sum_{m=-M}^{M} e^{-(a + mh)\log_{\phi_{i}^{(4)}}(e^{l\Delta})} \right] \right]$$

$$\sum_{J} \sum_{j=1}^{J} \sum_{k=1}^{K} \tilde{U}^{(i)}( mh, e^{\delta_{\nu}}, i + 1; G_{n_k}) \Psi ( -\alpha - imh, \zeta_{j}, \zeta_{p} ) w_{j}h ,$$

$$p_i G_{n_k} + e^{-r_{\nu} \phi_{i}^{(5)}(e^{l\Delta}; G_{n_k})} \left[ p_i q_i e^{-\gamma_{\nu} \phi_{i}^{(5)}(e^{l\Delta}; G_{n_k})} + g^{(i)}(i + 1; G_{n_k}) + \frac{1}{2\pi} \sum_{m=-M}^{M} e^{-(a + mh)\log_{\phi_{i}^{(5)}}(e^{l\Delta}; G_{n_k})} \right]$$

$$\sum_{J} \sum_{j=1}^{J} \sum_{k=1}^{K} \tilde{U}^{(i)}( mh, e^{\delta_{\nu}}, i + 1; G_{n_k}) \Psi ( -\alpha - imh, \zeta_{j}, \zeta_{p} ) w_{j}h \} \right \}, \quad (4.20a)$$
where \( l = 1, \ldots, L, \ p = 1, \ldots, J \) and \( k = 1, \ldots, K \). The corresponding generalized Fourier transform is calculated according to the following formula:

\[
\hat{U}^{(I)} (mh, e^{\xi_p}, i; G_{nk}) = \sum_{l=1}^{L} e^{(\alpha + imh)\Delta} U^{(I)} (e^{l\Delta}, e^{\xi_p}, i; G_{nk}) \Delta. \tag{4.20b}
\]

Repeat Step 2 for \( i = T - 2, \ldots, T_a + 1 \).

**Step 3: Preparation at \( T_a + 1 \)**

The mandated activation to the income phase at \( T_a + 1 \) leads to

\[
\hat{U}^{(A)} (mh, e^{\xi_p}, T_a + 1) = \sum_{l=1}^{L} e^{(\alpha + imh)\Delta} U^{(I)} (e^{l\Delta}, e^{\xi_p}, T_a + 1; G(T_a + 1)) \Delta,
\]

\[
g^{(A)} (e^{\xi_p}, T_a + 1) = g^{(I)} (T_a + 1; G(T_a + 1)). \tag{4.21}
\]

**Step 4: Backward induction from \( T_a \) to 1**

1. Calculate the function \( g^{(A)} (e^{\xi_p}, i) \) based on the following formula:

\[
g^{(A)} (e^{\xi_p}, i) = \max \left\{ g^{(I)} (i; G(i)),
- p_i B + e^{-r} ((1 + b_i) + B) \left[ \frac{B p_i q_i e^{r - \eta_p}}{(1 + b_i) + B} + \sum_{j=1}^{J} g^{(A)} (e^{\xi_j}, i + 1) p_j (\zeta_j | \zeta_p) w_j
+ 1_{(B > 0)} \frac{h}{2\pi} \sum_{m=-M}^{M} e^{-(\alpha + imh) \log (1 + b_i) + B} \sum_{j=1}^{J} \hat{U}^{(A)} (mh, e^{\xi_j}, i + 1) \tilde{\Psi} (-\alpha - imh, \zeta_j, \zeta_p) w_j \right],
\right. \]

\[
e^{-r} (1 + b_i) \sum_{j=1}^{J} g (e^{\xi_j}, i + 1) p_j (\zeta_j | \zeta_p) w_j \right\} \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r...
(3) Calculate $U^{(A)}(e^{i\Delta}, e^{\psi}, i)$ and $\hat{U}^{(A)}(m h, e^{\psi}, i)$ based on the following formulas:

$$U^{(A)}(e^{i\Delta}, e^{\psi}, i)$$

$$= - g^{(A)}(e^{\psi}, i) + \max \left\{ U^{(1)}(e^{i\Delta}, e^{\psi}, i; G(i)) + g^{(1)}(i; G(i)),
- p_i \beta + e^{-\psi^2(t)}(e^{i\Delta}) \left[ p_i q_i e^{-\eta p} \phi_i^2(e^{i\Delta}) + \sum_{j=1}^J g(e^{\psi}, i + 1) p_r(\zeta_j | \zeta_p) w_j 
+ \frac{1}{2\pi} \sum_{m=-M}^M e^{-(\alpha + imh) \log \phi^2(i\Delta)} \sum_{j=1}^J \hat{U}^{(A)}(m h, e^{\psi_j}, i + 1) \tilde{\Psi} (-\alpha - imh, \zeta_j, \zeta_p) w_j h \right],
\right.$$  

$$e^{-\psi^2(t)}(e^{i\Delta}) \left[ p_i q_i e^{-\eta p} \phi_i^2(e^{i\Delta}) + \sum_{j=1}^J g(e^{\psi}, i + 1) p_r(\zeta_j | \zeta_p) w_j 
+ \frac{1}{2\pi} \sum_{m=-M}^M e^{-(\alpha + imh) \log \phi^2(i\Delta)} \sum_{j=1}^J \hat{U}^{(A)}(m h, e^{\psi_j}, i + 1) \tilde{\Psi} (-\alpha - imh, \zeta_j, \zeta_p) w_j h \right]\right\},$$

and

$$\hat{U}^{(A)}(m h, e^{\psi}, i) = \sum_{l=1}^L e^{(\alpha + imh)\Delta} U^{(A)}(e^{i\Delta}, e^{\psi}, i) \Delta,$$

where $l = 1, \cdots, L$, $p = 1, \cdots, J$ and $m = -M, \cdots, M$.

Repeat Step 3 for $i = T_a, \cdots, 1$.

Step 4: Inversion of the Fourier transform at the final step to recover the value function

$$V^{(A)}(W_0, A_0, e^{\psi}, 0)$$

$$= e^{-\psi} \left[ p_0 q_0 e^{-\eta p} W_0 + \sum_{j=1}^J g(e^{\psi_j}, 1) p_r(\zeta_j | \zeta_p) w_j A_0 \right] + \frac{A_0}{2\pi} \sum_{m=-M}^M e^{-(\alpha + imh) \log W_0/A_0}$$

$$\times \sum_{j=1}^J \hat{U}^{(A)}(m h, e^{\psi_j}, 1) \tilde{\Psi} (-\alpha - imh, \zeta_j, \zeta_p) w_j h \right],$$

where $p = 1, \cdots, J$.

Remarks

1. Since the benefit base $A_t$ is substituted into the pricing formulation only in the final step of the algorithm through the initial benefit base $A_0$, the Fourier transform algorithm can be used to find the value function at varying values of $A_0$ simultaneously with minimal additional computational cost.

2. Huang and Kwok (2016) present the regression-based Monte Carlo simulation algorithms for pricing and hedging of the GLWB in variable annuities without considering the optimal initiation, age-dependent scheduled withdrawal rates and additional purchases. The regression-based Monte Carlo method would fail to price the GLWB products to the level of complexities under the general framework considered in this paper. On the contrary,
Table 1: Parameter values of the Heston model.

<p>| | | | | | |</p>
<table>
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</tr>
</thead>
<tbody>
<tr>
<td>r</td>
<td>κ</td>
<td>ε</td>
<td>θ</td>
<td>ν₀</td>
<td>ρ</td>
</tr>
<tr>
<td>0.04</td>
<td>1.15</td>
<td>0.39</td>
<td>0.0348</td>
<td>0.0348</td>
<td>-0.64</td>
</tr>
</tbody>
</table>

our newly derived Fourier transform algorithm can be employed for their GLWB formulation by making some slight modification. The corresponding modified version is much easier compared to our Fourier transform algorithm in this paper since we do not need to calculate the two normalized value functions at each time step in the modified version.

5 Numerical results

In this section, we first demonstrate the high level of accuracy and efficiency of the Fourier transform algorithm for pricing GLWB under the Heston model through some carefully designed test cases. The numerical performance of the Fourier transform algorithm is compared with that of a modified regression-based Monte Carlo algorithm based on an extension version of Huang and Kwok (2016). Secondly, we show the performance of our Fourier transform algorithm under a general framework and perform the sensitivity analysis of the GLWB price with respect to varying model parameters and contractual features. Furthermore, we explore the characterization of the optimal withdrawal strategy regions for GLWB in the $W$-$v$ plane. Finally, we investigate the optimal initiation regions with respect to the calendar time and the optimal initiation regions with respect to the initial age from the perspective of diverse policyholders. In particular, we examine the impact of model parameters and contractual features on these two kinds of optimal initiation regions.

5.1 Numerical accuracy and efficiency

Firstly, we compare numerical accuracy and computational efficiency of the Fourier transform algorithm with the regression-based Monte Carlo algorithm in pricing GLWB with the optimal initiation feature. In our sample calculations, we consider the simplified scenario where $B = 0$, $T_a = \infty$ and $G(t) = 0.05$ for any $t$; that is, additional purchases is not allowed and there is no restriction for the policyholder to activate the income phase by a given age.

We assume that the policy fund value dynamics is governed by the Heston model as follows:

\[
\begin{align*}
\frac{dW_t}{dt} &= (r - \eta)p dt + \sqrt{v_t}(\rho dB^1_t + \sqrt{1 - \rho^2} dB^2_t), \\
\frac{dv_t}{dt} &= \kappa(\theta - v_t) dt + \epsilon dB^1_t,
\end{align*}
\]

where $B^1_t$ and $B^2_t$ are two independent Brownian motions under the risk neutral measure $Q$. Tables 1 and 2 list the parameter values in the Heston model and the relevant contractual features in the GLWB product, which are regarded as the “Base” case in our sample calculations. We adopt the calibrated values of the Heston model parameters obtained by Bakshi et al. (1997) based on minimizing the sum of squared pricing errors between the market prices of S&P 500 options and the model-determined prices.

In Table 3, we list the prices of the GLWB product with the optimal initiation feature under four different scenarios: “Base”, “No bonus”, “No surrender” and “No ratchet” using the Fourier transform algorithm and the regression-based Monte Carlo algorithm. For the first three scenarios, good agreement of numerical results obtained from the two numerical methods is observed. This confirms high accuracy of the Fourier transform algorithm. When the ratchet
feature is not included, the regression-based Monte Carlo algorithm fails to provide reasonably stable numerical results even with a large number of simulation paths since the simulated normalized policy fund value may become too erratic. However, the Fourier transform algorithm does not have such numerical difficulties. The CPU times (seconds) required in the computations are also listed. The Fourier transform algorithm converges quite rapidly with respect to the truncation parameters. By comparing the CPU times, we observe that the Fourier transform method is more computationally efficient than the regression-based Monte Carlo method.

### 5.2 Pricing behaviors of the GLWB

Next, we present the performance of the Fourier transform algorithm for pricing GLWB under the generalized cases of inclusion of all contractual features, where the regression-based Monte Carlo method may fail to give reasonably stable numerical results. In our calculations, we set $T_a = 20$ and consider different choices on the contractual withdrawal rate $G(t)$ and cap multiplier of the benefit base $B$ for additional purchases (see Tables 4 and 5).

In Table 5, we present the numerical results of the GLWB prices obtained from the Fourier transform algorithm, and the CPU times required are also reported. As expected, increasing the additional purchase parameter $B$ or the contractual withdrawal rate would lead to a higher GLWB price. In addition, though the Cox-Ingersoll-Ross model parameters fail to satisfy the Feller condition, our Fourier transform algorithm remains to converge rapidly for the logvariance dimension even under such scenario. Finally, though the time dependent feature of the contractual withdrawal rate adds one additional dimension to our pricing problem, the CPU times do not increase substantially. This is because the computational time of our Fourier transform is mainly attributed to the calculations of the kernel function $\bar{\Psi} (-\alpha - i m h, \zeta_j, \zeta_p)$, which involves the valuation of the modified Bessel function. Interested readers may refer to Zeng and Kwok (2014) for more details.
Table 3: Comparison of the numerical results for GLWB obtained from the Fourier transform algorithm with the regression-based Monte Carlo algorithm under $B = 0$, $T_a = \infty$ and $G(t) = 0.05$ for any $t$. The CPU times (seconds) required in the computations are listed. Here, SD denotes the standard deviation for the regression-based Monte Carlo algorithm, $M$, $L$ and $J$ denote the truncation parameter for the Fourier transform of the normalized policy fund value, the normalized policy fund value and discretization parameter for the log-variance, respectively, and $J = 2^5$. The regression-based Monte Carlo method fails to give reasonable numerical results when there is no ratchet feature in the contract.

<table>
<thead>
<tr>
<th>Case</th>
<th>Monte Carlo</th>
<th>$L = 2M = 2^d$</th>
<th>Fourier Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Price (SD)</td>
<td>$d$</td>
<td>Price</td>
</tr>
<tr>
<td>Base</td>
<td>100.22 (0.0202)</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>No bonus</td>
<td>100.13 (0.0211)</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>No surrender</td>
<td>98.52 (0.0232)</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>No ratchet</td>
<td>— —</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Values of the contractual withdrawal rate $G(t)$ of the GLWB product.

<table>
<thead>
<tr>
<th>Withdrawal rate</th>
<th>$G(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Withdraw 1</td>
<td>$0 \leq t \leq T$ : 5%</td>
</tr>
<tr>
<td>Withdraw 2</td>
<td>$0 \leq t \leq 5$ : 5%, $6 \leq t \leq 10$ : 5.5%, $11 \leq t \leq T$ : 6%</td>
</tr>
<tr>
<td>Withdraw 3</td>
<td>$0 \leq t \leq 15$ : 5%+0.1%$t$, $16 \leq t \leq T$ : 6.5%</td>
</tr>
</tbody>
</table>

Table 5: Numerical results for the GLWB prices obtained from the Fourier transform algorithm with respect to varying contractual withdrawal rates and upper bounds on additional purchases. The CPU times (seconds) required in the Fourier transform calculations are also listed. The truncation level parameters are set to be $L = 2M = 2^8$.

<table>
<thead>
<tr>
<th>B</th>
<th>Withdrawal rate</th>
<th>$J = 2^5$</th>
<th>$J = 2^6$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Price</td>
<td>CPU</td>
<td>Price</td>
</tr>
<tr>
<td>0.3</td>
<td>1</td>
<td>100.23669</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>103.60563</td>
<td>3.1</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>105.63209</td>
<td>9.2</td>
</tr>
<tr>
<td>0.5</td>
<td>1</td>
<td>100.26647</td>
<td>1.9</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>104.82287</td>
<td>3.0</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>108.52587</td>
<td>9.3</td>
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</tbody>
</table>
Next, we conduct sensitivity analysis of the GLWB price function with respect to contractual features and model parameters. Without loss of generality, we set $T_a = 20$, $B = 0.3$ and the contractual withdrawal rate to be “Withdraw 2”.

**Cycle of ratchet events and penalty charge**

The cycle of ratchet events refers to the number of years lapsed between successive ratchet event dates. In Figure 1, we plot the GLWB price against the cycle of ratchet events under two penalty charge schemes: Penalty 1 and Penalty 2. Here, “Penalty 1” refers to the penalty charge setting taken from Table 2, while “Penalty 2” is obtained by adding 3% to the penalty charge fee $k(t)$ when $t \leq 25$. The GLWB price is seen to decrease with longer cycle of ratchet events and higher penalty charge. The plots also reveal that the ratchet provision and penalty charge scheme may have strong impact on the GLWB price.

**Correlation coefficient and volatility of variance**

Figure 2 examines the impact of the correlation coefficient $\rho$ and volatility of variance $\epsilon$ on the price of GLWB. The GLWB price decreases with an increasing value of $\epsilon$. The GLWB price is seen to be an increasing function of the correlation coefficient $\rho$.

**Optimal withdrawal strategy regions in the $\tilde{W}$-$v$ plane**

Now we would like to explore the characterization of the optimal withdrawal strategy regions for GLWB using the Fourier transform algorithm. In particular, we examine the impact of contractual withdrawal rate and additional purchases on the optimal withdrawal strategy regions. At a fixed time, the separation of the optimal withdrawal strategy regions is characterized by the normalized policy fund value $\tilde{W}$ and variance $v$. Figure 3 illustrates the separation of optimal withdrawal strategy regions in the $\tilde{W}$-$v$ plane on the first withdrawal date under four different scenarios. As revealed from Figure 3, when the contractual withdrawal rate is an increasing function of time, GLWB has a smaller withdrawal region. This is because the policyholder would choose not to withdraw prematurely and prefer to wait until a higher contractual withdrawal rate at a later time. At the same time, the region of optimal additional purchase increases under higher contractual withdrawal rate. In addition, a larger value of $B$ would enlarge the region of additional purchase. As a final remark, the region of optimal additional purchases increases as the variance $v$ increases while the optimal withdrawal region decreases with higher variance.

5.3 **Optimal initiation policies**

**Optimal initiation boundary against the calendar time $t$**

By fixing the variance, for each $t$, we consider the optimal value of $\tilde{W}$ at which the value function in the accumulation phase equals that in the income phase. Similar to the optimal exercise boundary for an American option, we plot the optimal initiation boundary as a function of time $t$. We examine the effect of the variance, bonus rate and contractual withdrawal rate on the optimal initiation boundary in the $\tilde{W}$-$t$ plane. We let the colored region denote the optimal initiation region in which the income phase should be initiated. In Figures 4a-4c, we assume a constant contractual withdrawal rate and plot the optimal initiation region in the $\tilde{W}$-$t$ plane for different values of the variance and bonus rate. The optimal initiation region decreases as the variance or the bonus rate increases. Note that it is optimal to stay in the accumulation phase only in the areas with high normalized policy fund value $\tilde{W}$ and earlier withdrawal dates. Especially when the bonus rate is low, as observed from Figure 4c, it is always optimal to initiate the income phase immediately. This is because the incentive for the policyholder to
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Notation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial age</td>
<td>$x_0$</td>
<td>$50 \leq x_0 \leq 120$</td>
</tr>
<tr>
<td>Penalty for excess withdrawal</td>
<td>$x_0 k_t$</td>
<td>$0 \leq x_0 + t \leq 66 : 6%$, $66 &lt; x_0 + t \leq 67 : 5%$, $67 &lt; x_0 + t \leq 68 : 4%$, $68 &lt; x_0 + t \leq 69 : 3%$, $69 \leq x_0 + t \leq 70 : 2%$, $70 &lt; x_0 + t \leq 90 : 1%$, $90 &lt; x_0 + t \leq 122 : 0%$</td>
</tr>
<tr>
<td>Expiry time</td>
<td>$T_{x_0}$</td>
<td>$122 - x_0$ (years)</td>
</tr>
<tr>
<td>Survival probability</td>
<td>$x_0 p_t$</td>
<td>$e^{-e^{-\frac{x_0 + t - x_0}{w} - 0.75}} + e^{-\frac{x_0 - x_0}{w} - 0.75}$</td>
</tr>
</tbody>
</table>

Table 6: Contract parameter values of the GLWB product with diverse policyholders.

choose zero withdrawal or additional purchase is low when the bonus is very small.

We investigate the effect of the contractual withdrawal rate on the optimal initiation region. We assume the same parameter values as in Figure 4a, except that the contractual withdrawal rate increases on some specified dates (triggering dates). In our test, the contractual withdrawal rate rises from 5% to 5.5% on the triggering date $t = 6$ and it increases to 6% at $t = 11$. Figure 4d reveals that when the time goes beyond the latest of the triggering dates, the plot agrees with that of Figure 4a. Otherwise, the increase in the contractual withdrawal rate provides a strong incentive for the policyholder to delay initiation of the income phase. Especially when the calendar time is approaching a triggering date of changes of the contractual withdrawal rate, the optimal initiation boundary is zero; so initiation becomes non-optimal at any level of normalized policy fund value.

**Optimal initiation regions in the $\tilde{W} - x_0$ plane**

We would like to study the optimal initiation of a GLWB from the perspective of diverse policyholders. We let $x_{p_{x_0}}$, $k_{x_0}$ and $G_{x_0}(t)$ denote the survival probability, the penalty charge rate and contractual withdrawal rate at time $t$ for a policyholder with an initial age $x_0$, respectively. The model and contract parameter values are taken from Tables 1 and 2, except those listed in Table 6.

Similar to Huang et al. (2014), we can determine the optimal initiation region with respect to the initial age $x_0$. Compared to Huang et al. (2014), here we consider a discrete set of event dates and allow for stochastic volatility, dynamic withdrawal, additional purchase and mandated initiation time. Also, we construct the Fourier transform algorithm to determine the optimal initiation region rather than using the finite difference method. On the first withdrawal date ($t = 1$), for a fixed variance, we plot the optimal initiation region with respect to the initial age $x_0$. The effects of the investment, the penalty charge rate and the contractual withdrawal rate on the optimal initiation are revealed in Figures 5a-5d. It is optimal for young policyholders to accumulate regardless of the level of $\tilde{W}$ when more additional purchase is allowed. The additional purchase parameter $B$ has a pronounced impact on young policyholders. Secondly, setting the penalty charge rate for excess withdrawal to be 100% is equivalent to ruling out the complete surrender feature. The optimal initiation region becomes larger when compared with the scenario where the complete surrender is allowed and the penalty charge rate is a decreasing function of age. In fact, the lowering of the penalty rate motivates the policyholder to delay the initiation until a smaller penalty charge rate kicks in. Finally, the effect of the contractual withdrawal rate on this optimal initiation region is similar to that on the optimal initiation region in the $\tilde{W} - t$ plane. We consider the contractual withdrawal rate as an increasing function of age. In our test calculations, we assume that the contractual withdrawal rate rises from 5% to 5.5% at age 71 (triggering age) and jumps from 5.5% to 6% at age 76. Figure 5d
shows that an increase in the contractual withdrawal rate motivates the policyholders who are younger than the last triggering age to delay initiation. This effect becomes more profound for policyholders at an age immediately before any triggering age since the corresponding optimal initiation boundary is zero.

6 Conclusion

We present the comprehensive pricing model for the GLWB product with the accumulation phase and income phase, additional purchases, age-dependent scheduled withdrawal rate, bonus and ratchet provisions under the Heston stochastic volatility process for the policy fund value. The pricing model includes the optimal stopping rule of initiation and dynamic withdrawal as the stochastic control process. Through a rigorous bang-bang analysis, we show that the strategy space of the optimal policies is limited to four choices, thus simplifying the construction of the Fourier transform algorithm for pricing GLWB products with complex path dependent features. The success of the bang-bang analysis relies on the convexity and monotonicity properties of the price function. The results are applicable to many common processes for the policy fund value. In the design of the Fourier transform algorithm, the dimension of the pricing model is reduced by one since the benefit base remains constant between two consecutive event dates. Then we apply the Fourier transform in the log normalized policy fund value dimension and a quadrature rule in the log-variance dimension. The Fourier transform algorithm is seen to be efficient, accurate and reliable even under the level of complexities of path dependence in our comprehensive pricing model of the GLWB while the regression-based Monte Carlo simulation algorithm may fail to give a reliable numerical solution. The CPU time required for numerical evaluation of the price function to achieve 4 significant figures accuracy using the Fourier transform algorithm is typically within a few seconds.

We perform sensitivity analysis of the GLWB price function with respect to various contractual features and model parameters. We also examine the impact of the contractual withdrawal rate and upper bound of additional purchases on the optimal initiation policies. The optimal initiation policies are seen to depend sensibly on the age-dependent contractual withdrawal rate. Policyholders would wait for a more favorable contractual withdrawal rate for optimal entry into the income phase.

Appendix A - Proof of Theorem 3

The proof requires two results in convex analysis, stated as Property A.1 and Property A.2 below:

**Property A.1** Let \( \mathcal{A} \) be a convex set, and let \( \mathcal{B} \) and \( \mathcal{C} \) be vector spaces over \( \mathbb{R} \). If \( g : \mathcal{A} \rightarrow \mathcal{B} \) is convex, \( h : \mathcal{B} \rightarrow \mathcal{C} \) is convex and monotonic increasing, then \( h \circ g \) is convex on set \( \mathcal{A} \).

**Property A.2** Suppose we have a function \( f : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R} \) that satisfies (i) \( f(\cdot, y) \) is convex for any fixed \( y \in \mathbb{R}^+ \); (ii) for any positive constant \( K \), \( f(Kx, Ky) = Kf(x, y) \), then \( f(\cdot, \cdot) \) is convex.

The proof of Property A.1 can be found in Boyd and Vandenberghe (2004). To show Property A.2, it suffices to show

\[
f(\hat{x}, \hat{y}) \leq \theta f(x_1, y_1) + (1 - \theta)f(x_2, y_2), \quad \forall \theta \in (0, 1),
\]

(A.1)
where
\[
\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \theta \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1-\theta) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.
\]

By virtue of the homogeneous property in Property A.2, eq. (A.1) is equivalent to
\[
f \left( \frac{x}{y} , 1 \right) \leq \hat{\theta} f \left( \frac{x_1}{y_1} , 1 \right) + (1-\hat{\theta}) f \left( \frac{x_2}{y_2} , 1 \right),
\]
with \( \hat{\theta} = \frac{\theta y_1}{y} \in (0,1) \). Indeed, by observing \( \frac{x}{y} = \theta \frac{x_1}{y_1} + (1-\hat{\theta}) \frac{x_2}{y_2} \) and convexity of \( f(\cdot, 1) \), we establish eq. (A.2).

Next we perform the bang-bang analysis for GLWB\(^{(k)}\) in the income phase, then extend the analysis in a similar manner to GLWB\(^{(A)}\) in the accumulation phase. The proof of the bang-bang control policy for GLWB\(^{(k)}\) consists of the following steps:

1. Suppose \( V^{(k)}(\cdot, \cdot, v, i+1; G_0) \) is convex, the optimal strategy on the withdrawal date \( i \) for GLWB\(^{(k)}\) is limited to a finite number of choices (see the details in Claim A.3) and \( V^{(k)}(\cdot, \cdot, v, i; G_0) \) is also convex (see the details in Claim A.4).

2. Since the terminal payoff \( V^{(k)}(\cdot, \cdot, v, T; G_0) \) is convex, Claims A.3 and A.4 can be applied inductively to show the bang bang control for GLWB\(^{(k)}\) on any withdrawal date (see the details in Claim A.5).

First, we list the statements of Claims A.3, A.4 and A.5 below. The detailed proofs of these claims are presented later.

**Claim A.3** If \( V^{(k)}(\cdot, \cdot, v, i+1; G_0) \) is convex, the optimal strategy \( \gamma_i \) on the withdrawal date \( i \) for GLWB\(^{(k)}\) is limited to (i) \( \gamma_i = 0 \); (ii) \( \gamma_i = G_0 A_i \); (iii) \( \gamma_i = W_i - \eta_b A_i \).

**Claim A.4** If \( V^{(k)}(\cdot, \cdot, v, i+1; G_0) \) is convex, then \( V^{(k)}(\cdot, \cdot, v, i; G_0) \) is also convex.

**Claim A.5** On any withdrawal date \( 1 \leq i \leq T - 1 \), the optimal strategy \( \gamma_i \) for GLWB\(^{(k)}\) with any positive \( G_0 \) is limited to (i) \( \gamma_i = 0 \); (ii) \( \gamma_i = G_0 A_i \); (iii) \( \gamma_i = W_i - \eta_b A_i \). Moreover \( V^{(k)}(\cdot, \cdot, i; G_0) \) is convex.

To establish Claim A.3, we define \( C_{i,v,G_0}^{(k)}(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) and \( C_{i,v}^{(A)}(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) as follows:

\[
C_{i,v,G_0}^{(k)}(x, y) = E[V^{(k)}(W, A, v, i+1; G_0)|W_i+ = x, A_i+ = y, v_i+ = v], \quad i = 1, 2, \ldots, T - 1,
\]

and

\[
C_{i,v}^{(A)}(x, y) = E[V^{(A)}(W, A, v, i+1)|W_i+ = x, A_i+ = y, v_i+ = v], \quad i = 1, 2, \ldots, T_a.
\]

On any withdrawal date, \( 1 \leq i \leq T - 1 \), we deduce from eq. (2.7) that

\[
V^{(k)}(W, A, v, i; G_0) = p_{i-1} q_{i-1} W_i + \sup_{\gamma_i \in [0, \max(W_i-\eta_b A_i; G_0 A_i)]} \{ p_i f_i^{(k)}(\gamma_i; A_i, G_0) \}
\]

\[
+ e^{-r} C_{i,v,G_0}^{(k)} h_i^{(k)}(W_i, A_i, \gamma_i; G_0) \}
\]

(A.3)

If \( V^{(k)}(\cdot, \cdot, v, i+1; G_0) \) is convex, then \( C_{i,v,G_0}^{(k)}(\cdot, \cdot) \) is convex due to Property 1, eq. (3.1a) and Property A.2. Notice that \( h_i^{(k)}(W_i, A_i, \cdot; G_0) \) is convex on \([0, G_0 A_i]\) and \([G_0 A_i, W_i-\eta_b A_i]\) (the later
interval becomes empty when \( W_i - \eta_b A_i < G_0 A_i \). Together with monotonicity of \( C_{i,v,G_0}^{(I)}(x,y) \) on both \( x \) and \( y \), \( C_{i,v,G_0}^{(I)} \circ h_i^* (W_i, A_i, \cdot; G_0) \) is convex on \([0, G_0 A_i]\) and \([G_0 A_i, W_i - \eta_b A_i]\) by Property A.1. As a result, we can easily show that \( p_i f_i^I(\cdot; A_i, G_0) + e^{-r} C_{i,v,G_0}^{(I)} \circ h_i^I (W_i, A_i, \cdot; G_0) \) is convex on \([0, G_0 A_i]\) and \([G_0 A_i, W_i - \eta_b A_i]\).

Since the supremum of a convex function on a closed bounded convex set must occur at one of the extreme points of the set (Corollary 32.3.2 in Rockafellar, 1997), the optimal strategy \( \gamma_i \) shown in (A.3) is limited to (i) \( \gamma_i = 0 \), (ii) \( \gamma_i = G_0 A_i \) and (iii) \( \gamma_i = W_i - \eta_b A_i \). Hence, Claim A.3 is proved.

To show Claim A.4, it suffices to show that \( V^{(I)}(\cdot, A, v, i; G_0) \) is convex due to eq. (3.1a) and Property A.2. We define
\[
\hat{W} = \theta W_1 + (1 - \theta) W_2, \quad \forall \theta \in (0, 1),
\]
and let \( \hat{\gamma}^* \) be the optimal strategy for \( V^{(I)}(\hat{W}, A, v, i; G_0) \). Also, we let \( \gamma_1 \) and \( \gamma_2 \) be the candidate strategy (not necessary to be optimal) for \( V^{(I)}(W_1, A, v, i; G_0) \) and \( V^{(I)}(W_2, A, v, i; G_0) \), respectively. We observe that \( \hat{\gamma}^* \) is limited to the following three choices according to Claim A.3: (i) \( \hat{\gamma}^* = 0 \); (ii) \( \hat{\gamma}^* = G_0 A_i \); (iii) \( \hat{\gamma}^* = \hat{W} - \eta_b A_i \).

For the case of \( \hat{\gamma}^* = \hat{W} - \eta_b A_i \), we set \( \gamma_1 = \max(W_1 - \eta_b A_i, G_0 A_i) \) and \( \gamma_2 = \max(W_2 - \eta_b A_i, G_0 A_i) \). We have
\[
V^{(I)}(\hat{W}, A, v, i; G_0) = p_{i-1} q_{i-1} \hat{W} + p_i f_i^I(\hat{\gamma}^*; A_i, G_0) \leq \theta V^{(I)}(W_1, A, v, i; G_0) + (1 - \theta) V^{(I)}(W_2, A, v, i; G_0).
\]
The above inequalities hold since \( \gamma_1 \) and \( \gamma_2 \) are the admissible strategies for \( V^{(I)}(W_1, A, v, i; G_0) \) and \( V^{(I)}(W_2, A, v, i; G_0) \), respectively.

For the case of \( \hat{\gamma}^* = G_0 A_i \), we set \( \gamma_1 = \gamma_2 = G_0 A_i \). Since \( h_i^I(\cdot, A_i, G_0 A_i; G_0) \) is convex by virtue of eq. (2.5b), so \( C_{i,v,G_0}^{(I)} \circ h_i^I(\cdot, A_i, G_0 A_i; G_0) \) is also convex. Since \( \hat{\gamma}^* \) is the adopted optimal strategy, we have
\[
V^{(I)}(\hat{W}, A, v, i; G_0) = p_{i-1} q_{i-1} \hat{W} + p_i f_i^I(G_0 A_i; A_i, G_0) + e^{-r} E[V^{(I)}(W, A, v, i + 1; G_0)|(W_{i+1}, A_{i+1}) = h_i^I(\hat{W}, A_i, G_0 A_i; G_0), v_{i+1} = v] \leq \theta V^{(I)}(W_1, A, v, i; G_0) + (1 - \theta) V^{(I)}(W_2, A, v, i; G_0).
\]
For the last case where \( \hat{\gamma}^* = 0 \), we can establish in a similar manner that
\[
V^{(I)}(\hat{W}, A, v, i; G_0) \leq \theta V^{(I)}(W_1, A, v, i; G_0) + (1 - \theta) V^{(I)}(W_2, A, v, i; G_0).
\]
Hence, Claim A.4 is proved.

By virtue of Claims A.3 and A.4, and observing the terminal payoff $V(I)(\cdot, \cdot, v, T; G_0)$ to be convex, we deduce Claim A.5.

Next, we prove the bang-bang control for GLWB(A). Suppose the income phase has not been initiated before $T_a$, then $V(A)(W, A, v, T_a)$ in eq. (2.8) can be written as

$$V(A)(W, A, v, T_a) = p_{T_a-1}q_{T_a-1}W_{T_a} + \max\{V_C(A)(T_a), V_C(I)(T_a)\},\quad (A.4)$$

where

$$V_C(A)(T_a) = \sup_{\gamma_{T_a} \in \{-BA_{T_a}, (W_{T_a} - \eta A_{T_a})^+\}}\left\{p_{T_a}f_T^A(\gamma_{T_a}; A_{T_a}) + e^{-r}C_{T_a, v, G(T_a+1)}(W_{T_a}, A_{T_a}, \gamma_{T_a})\right\},$$

$$V_C(I)(T_a) = \sup_{\gamma_{T_a} \in \{0, \max((W_{T_a} - \eta A_{T_a})^+, G(T_a)A_{T_a})\}}\left\{p_{T_a}f_T^I(\gamma_{T_a}; A_{T_a}, G(T_a)) + e^{-r}C_{T_a, v, G(T_a)}(W_{T_a}, A_{T_a}, \gamma_{T_a}; G(T_a))\right\}.$$

To show the calculation of $V_C(A)(T_a)$, we observe that $f_T^A(\cdot; A_{T_a})$ and $h_T^A(W_{T_a}, A_{T_a}, \cdot)$ are both convex on $[-BA_{T_a}, 0]$ and $[0, (W_{T_a} - \eta A_{T_a})^+]$ by eqs. (2.4) and (2.5a). Since $V(I)(\cdot, \cdot, v, T_a + 1; G(T_a + 1))$ is convex by Claim A.5, we can similarly show that $C_{T_a, v, G(T_a+1)}(W_{T_a}, A_{T_a}, \cdot)$ is convex on $[-BA_{T_a}, 0]$ and $[0, (W_{T_a} - \eta A_{T_a})^+]$. By Corollary 32.3.2 in Rockafellar (1997), the optimal strategy $\gamma_{T_a}$ for $V_C(A)(T_a)$ is limited to: (i) $\gamma_{T_a} = -BA_{T_a}$; (ii) $\gamma_{T_a} = 0$: (iii) $\gamma_{T_a} = (W_{T_a} - \eta A_{T_a})^+$. Also, one can show that $V_C(A)(T_a)$ is convex with respect to $(W_{T_a}, A_{T_a})$ using a similar argument of proving Claim A.4.

Similarly for $V_C(I)(T_a)$, the optimal strategy $\gamma_{T_a}$ for $V_C(I)(T_a)$ is limited to: (i) $\gamma_{T_a} = 0$; (ii) $\gamma_{T_a} = G(T_a)A_{T_a}$; (iii) $\gamma_{T_a} = W_{T_a} - \eta A_{T_a}$. Moreover, $V_C(I)(T_a)$ is convex with respect to $(W_{T_a}, A_{T_a})$.

We define the two mappings $F_T^A : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $F_T^I : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as follows:

$$F_T^A(x, y) = V_C(A)(T_a) \text{ with } W_{T_a} = x \text{ and } A_{T_a} = y$$

and

$$F_T^I(x, y) = V_C(I)(T_a) \text{ with } W_{T_a} = x \text{ and } A_{T_a} = y$$

Also, we define the mapping $F_T : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by

$$F_T(x, y) := \left(\begin{array}{c} F_T^A(x, y) \\ F_T^I(x, y) \end{array}\right),$$

where $F_T^A$ and $F_T^I$ are seen to be convex. Then eq. (A.4) can be rewritten as

$$V(A)(W, A, v, T_a) = p_{T_a-1}q_{T_a-1}W_{T_a} + \max \circ F_T(W_{T_a}, A_{T_a}),$$

where the operator “max” is defined by $\max(x, y) = x\mathbb{1}_{\{x \geq y\}} + y\mathbb{1}_{\{x < y\}}$.

By applying Property A.1, $V(A)(\cdot, \cdot, v, T_a)$ is convex due to convexity of the two operators: max and $F_T$. The above argument can be applied inductively to obtain Claim A.6 as stated below.

**Claim A.6** On any withdrawal date $i$, the optimal strategy on this date for GLWB(A) is either to

1. initiate the income phase on date $i$ if $V_C(I)(i) > V_C(A)(i)$ and the optimal strategy $\gamma_i$ is limited to (i) $\gamma_i = 0$; (ii) $\gamma_i = G(i)A_i$; (iii) $\gamma_i = W_i - \eta A_i$; or to

2. remain in the accumulation phase on date $i$ if $V_C(I)(i) \leq V_C(A)(i)$ and the optimal strategy $\gamma_i$ is limited to (i) $\gamma_i = -BA_i$; (ii) $\gamma_i = 0$; (iii) $\gamma_i = W_i - \eta A_i$.

Combining Claims A.5 and A.6, we can establish parts (1), (2a) and (2b) in Theorem 3.
Appendix B - Backward induction for calculating the modified normalized value functions and terminal condition

We show how to derive a backward induction for calculating $U^{(f)}(\tilde{W}, v, i; G_{n_k})$ and $U^{(A)}(\tilde{W}, v, i)$ and derive the terminal condition for $\tilde{U}^{(f)}(\tilde{W}, v, T - 1; G_{n_k})$. Recall that when $\tilde{W}_i \leq \eta_b$, $U^{(f)}(\tilde{W}, v, i, G_{n_k})$ equals to zero. Therefore we can restrict our attention to the condition that $\tilde{W}_i > \eta_b$. Based on eqs. (4.4), (4.17), and (4.8), we can express the normalized value functions in terms of the Fourier transform integrals. When we compute $E[\tilde{V}^{(f)}(\tilde{W}, v, i + 1; G_{n_k})|\tilde{W}_i = \phi_i^{(5)}(\tilde{W}_i), v_i]$ in eq. (4.4), it is necessary to distinguish the two cases by comparing $\phi_i^{(5)}(\tilde{W}_i; G_{n_k})$ with zero. An unified formula for $V^{(f)}(\tilde{W}, v, i; G_{n_k})$ can be expressed as follows

\[
\tilde{V}^{(f)}(\tilde{W}, v, i; G_{n_k}) = p_{i-1}q_{i-1}\tilde{W}_i + \max \left\{ \frac{p_i}{G_{n_k}} + (1 - \kappa_i)(\tilde{W}_i - \eta_b - G_{n_k}) \right\},
\]

\[
e^{-r\psi^{(4)}(\tilde{W}_i)} \left[ p_i q_i e^{-r}\phi_i^{(4)}(\tilde{W}_i) + g^{(4)}(i + 1; G_{n_k}) \right. \left. + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(\alpha + i\beta)\log\phi_i^{(4)}(\tilde{W}_i)} \sum_{j=1}^{J} \tilde{U}^{(f)}(\beta, e^{\gamma_j}, i + 1) \Psi (-\alpha - i\beta, \gamma_j) w_j d\beta \right],
\]

\[
p_i q_i G + e^{-r}\phi_i^{(5)}(\tilde{W}_i; G_{n_k}) \left[ p_i q_i e^{-r}\phi_i^{(5)}(\tilde{W}_i; G_{n_k}) + g^{(5)}(i + 1; G_{n_k}) + \mathbb{1}\{\phi_i^{(5)}(\tilde{W}_i; G_{n_k}) > 0\} \right. \left. \sum_{j=1}^{J} \tilde{U}^{(f)}(\beta, e^{\gamma_j}, i + 1) \Psi (-\alpha - i\beta, \gamma_j) w_j d\beta \right] \}. \]

The last term is included conditional on $\phi_i^{(5)}(\tilde{W}_i; G_{n_k}) > 0$. Using the relation (4.11) between $U^{(f)}(\tilde{W}, v, i; G_{n_k})$ and $\tilde{V}^{(f)}(\tilde{W}, v, i; G_{n_k})$, and performing the computation on a set of nodes, we can obtain a recursive equation (4.20a) for $U^{(f)}(\tilde{W}, v, i; G_{n_k})$. Likewise, $\tilde{V}^{(A)}(\tilde{W}, v, i)$ can be derived easily by combining eqs. (4.5) and (4.16) when $\tilde{W}_i > \eta_b$. Therefore, an application of eq. (4.11) gives the result for $U^{(A)}(\tilde{W}, v, i)$ in eq. (4.23a).

At time $T - 1$, based on eqs. (4.4) and (4.18), one can easily obtain the closed form representation for the normalized value function as follows

\[
\tilde{V}^{(f)}(\tilde{W}, v, T - 1; G_{n_k}) = p_{T-2} q_{T-2} \tilde{W}_{T-1} + \max \left\{ p_{T-1} G_{n_k} + p_{T-1} (1 - \kappa_{T-1})(\tilde{W}_{T-1} - \eta_b - G_{n_k}), p_{T-1} G_{n_k} + p_{T-1} e^{-\eta_b}(\tilde{W}_{T-1} - \eta_b - G_{n_k})^+ \right\}.
\]

We then have

\[
U^{(f)}(\tilde{W}, v, T - 1; G_{n_k}) = \tilde{V}^{(f)}(\tilde{W}, v, T - 1; G_{n_k}) - \left( p_{T-2} q_{T-2} \tilde{W}_{T-1} + g^{(f)}(T - 1; G_{n_k}) \right) = \max \left\{ p_{T-1} (1 - \kappa_{T-1})(\tilde{W}_{T-1} - \eta_b - G_{n_k}), p_{T-1} e^{-\eta_b}(\tilde{W}_{T-1} - \eta_b - G_{n_k})^+ \right\},
\]

leading to eq. (4.19) for $\tilde{U}^{(f)}(m_h e^{\gamma_j}, T - 1; G_{n_k})$. 

29
References


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Figure 1: Plot of the price of GLWB against the cycle of ratchet events under two penalty schemes: Penalty 1 and Penalty 2.

Figure 2: Plot of the price of GLWB against volatility of variance $\epsilon$ under varying values of correlation coefficient $\rho$. 
Figure 3: Plots of the optimal withdrawal boundaries for GLWB at time $t = 1$ under different values for the contractual withdrawal rate and upper bound of additional purchase. Here, “Withdraw 1” and “Withdraw 2” are depicted in Table 4, $T_a = 20$, and other model and contract parameter values are shown in Tables 1 and 2.
Figure 4: Plots of the optimal initiation regions in the $\tilde{W}$-$t$ plane under varying values of the contractual withdrawal rate $G(t)$, bonus rate $b_i$ and variance $v_t$. In the “Base case” shown in Figure 4a, we choose $b_i = 0.06$, $v_t = 0.04$, $G(t) = 0.05$. We modify one parameter from the “Base case” as labelled in each of Figures 4b-4d. Here, $B = 0.3$, $T_a = 20$, and other parameter values are shown in Tables 1 and 2.
Figure 5: Plots of the optimal initiation region in the $\tilde{W}-x_0$ plane on the first withdrawal date under varying values of the contractual withdrawal rate $G_{x_0}(t)$, additional purchases parameter $B$ and penalty charge rate $x_0k_t$. In the “Base case” shown in Figure 5(a), we choose $B = 0.3$, $G_{x_0}(t) = 0.05$ and the penalty scheme is taken from Table 6. We modify one parameter from the “Base case” as labelled in each of Figures 5b-5d. Here, $T_a = 20$, $v_t = 0.04$ and other parameter values are shown in Tables 1, 2 and 6.