Singular Stochastic Control Models for Optimal Dynamic Withdrawal Policies in Variable Annuities

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Agenda

- Product nature of the Guaranteed Minimum Withdrawal Benefit (GMWB) in variable annuities
- Construction of a continuous singular stochastic control model
  - withdrawal rate as the stochastic control variable
- Analysis of optimal dynamic withdrawal policies
  - asymptotic behavior of the separating boundaries
  - solution to the pricing model under various asymptotic limits
- Conclusions

Product nature of GMWB

- Variable annuities — deferred annuities that are fund-linked.
- The single lump sum paid by the policyholder at initiation is invested in a portfolio of funds chosen by the policyholder — equity participation.
- The policyholder is allowed to withdraw funds on an annual or semi-annual basis until the entire principal is returned. The GMWB promises to return the entire annuitization amount.
- The benefit is funded by charging proportional fee on the policy fund value at the rate η.
- In 2004, 69% of all variable annuity contracts sold in the US include the GMWB option.

Numerical example

- Let the initial fund value be $\$100,000$ and the withdrawal rate be 7% per annum. Suppose the investment account earns ten percent in the first two years but earns returns of minus sixty percent in each of the next three years.

Year	Rate earned during the year	Fund before withdrawals	Amount withdrawn	Fund after withdrawals	Guaranteed withdrawals remaining balance
1	10%	$110,000$	$7,000$	$103,000$	$93,000$
2	10%	$113,300$	$7,000$	$106,300$	$86,000$
3	-60%	$42,520$	$7,000$	$35,520$	$79,000$
4	-60%	$14,208$	$7,000$	$7,208$	$72,000$
5	-60%	$2,883$	$7,000$	0	$65,000$

- At the end of year five before any withdrawal the value of the fund, $\$2,883$, is not enough to cover the annual withdrawal payment of $\$7,000$.

The guarantee kicks in:

The value of the fund is set to be zero and the policyholder's ten remaining withdrawal payments are financed under the writer's guarantee. The policyholder's income stream of annual withdrawals is protected irrespective of the market performance.

- If the market does well, then there will be funds left at policy's maturity. The remaining balance in the fund account is paid to the policyholder.
- If performance is bad, the investment account balance will have shrunk to zero before the principal is repaid and will remain there.
- Benefit can be seen as a guaranteed stream of G per annum plus a call option on the terminal account value W_T. The strike price of the call is zero.

References

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Continuous singular stochastic control model under dynamic withdrawal

- A_t is the account balance of the guarantee, A_t is a non-negative and non-increasing $\{\mathcal{F}_t\}_{t\geq 0}$-adaptive process.
- At initiation, $A_0 = w_0$; the withdrawal guarantee becomes insignificant when $A_t = 0$.
- As withdrawal continues, A_t decreases over the life of the policy until it hits the zero value.

The dynamics of the value of the policy fund account W_t under a risk neutral measure follows

$$dW_t = (r - \eta)W_t \, dt + \sigma W_t \, dB_t + dA_t, \quad t < \tau,$$

$$A_t = A_0 - \int_0^t \gamma_s \, ds, \quad 0 \leq \gamma_s \leq \lambda,$$

η is the proportional fee charged in the policy fund value, γ_s is the withdrawal rate process and λ is some upper bound.

Proportional Penalty Charge

Penalty charges are incurred when the withdrawal rate γ exceeds the contractual withdrawal rate G. Supposing a proportional penalty charge k is applied on the portion of γ above G, then the net amount received by the policyholder is $G + (1-k)(\gamma - G)$ when $\gamma > G$.

Let $f(\gamma)$ denote the rate of cash flow received by the policyholder as resulted from the continuous withdrawal process, we then have

$$f(\gamma) = \begin{cases} \gamma & \text{if } 0 \leq \gamma \leq G \\ G + (1-k)(\gamma - G) & \text{if } \gamma > G \end{cases} .$$

The policyholder receives the continuous withdrawal cash flow $f(\gamma_u)\ du$ over $(u, u + du)$ throughout the life of the policy and the remaining balance of the investment account at maturity.

Rational behavior of policyholder

The policyholder strikes the balance between

- time value of cash flows
- proportional penalty charge
- optionality of the terminal payoff

The no-arbitrage value $\overline{V}$ of the variable annuity with GMWB is given by

$$\overline{V}(W, A, t) = \max_{\gamma} \mathsf{E}_t \left[e^{-r(T-t)} \max(W_T, (1-k)A_T) + \int_t^T e^{-r(u-t)} f(\gamma_u)\, du \right].$$

Here, γ is the *control variable* for the withdrawal rate that is chosen to maximize the expected value of the discounted cash flows.

- The first term gives the optionality of remaining terminal fund value W_T or remaining guarantee amount net of penalty $(1-k)A_T$.
- The second term represents the discounted cash flow stream.

Hamilton-Jacobi-Bellman (HJB) equation

The dynamic withdrawal rate γ is the stochastic control variable. The governing equation for $\overline{V}$ is found to be

$$\frac{\partial \overline{V}}{\partial t} + \mathcal{L}\overline{V} + \max_{\gamma} h(\gamma) = 0$$

where

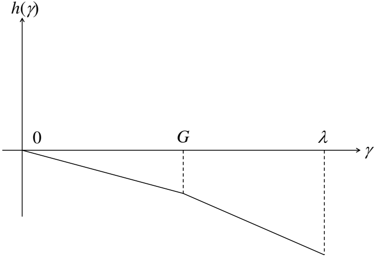
$$\mathcal{L}\overline{V} = \frac{\sigma^2}{2} W^2 \frac{\partial^2 \overline{V}}{\partial W^2} + (r-\eta) W \frac{\partial \overline{V}}{\partial W} - r\overline{V}$$

$$\begin{aligned} h(\gamma) &= f(\gamma) - \gamma \frac{\partial \overline{V}}{\partial W} - \gamma \frac{\partial \overline{V}}{\partial A} \\ &= \begin{cases} \gamma\left(1 - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}\right) & \text{if } 0 \le \gamma < G \\ kG + \gamma\left(1 - k - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}\right) & \text{if } \gamma \ge G \end{cases} . \end{aligned}$$

Write $\beta = 1 - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}$, then

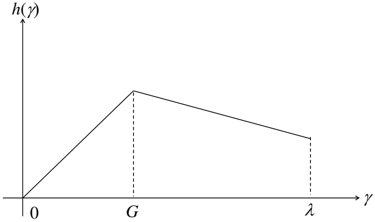
$$h(\gamma) = \begin{cases} \beta\gamma & \text{if } 0 < \gamma < G \\ \beta\gamma - k(\gamma - G) & \text{if } \gamma \geq G \end{cases} = \begin{cases} \beta\gamma & \text{if } 0 \leq \gamma \leq G \\ (\beta - k)\gamma + kG & \text{if } \gamma > G \end{cases}$$

(i) $\beta \leq 0$

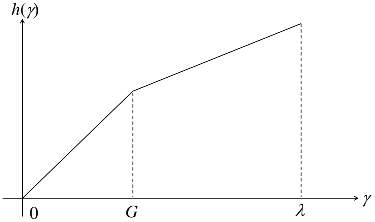


Maximum value of $h(\gamma)$ is achieved at $\gamma = 0$ (zero withdrawal). This occurs when $\frac{\partial \overline{V}}{\partial W} + \frac{\partial \overline{V}}{\partial A} \geq 1$.

(ii) $0 < \beta < k \Longleftrightarrow 1 - k < \frac{\partial \overline{V}}{\partial W} + \frac{\partial \overline{V}}{\partial A} < 1$, it is optimal to withdraw at G.



(iii) $\beta \geq k \Longleftrightarrow \frac{\partial \overline{V}}{\partial W} + \frac{\partial \overline{V}}{\partial A} \leq 1 - k$, it is optimal to withdraw at the maximum rate λ.



Penalty approximation approach

The function $h(\gamma)$ is piecewise linear so its maximum value is achieved at either $\gamma = 0, \gamma = G$ or $\gamma = \lambda$.

Recall $0 \leq \gamma \leq \lambda$. Note that

$$\max_{\gamma} h(\gamma) = \begin{cases} kG + \lambda\left(1 - k - \dfrac{\partial \overline{V}}{\partial W} - \dfrac{\partial \overline{V}}{\partial A}\right) & \text{if } \dfrac{\partial \overline{V}}{\partial W} + \dfrac{\partial \overline{V}}{\partial A} \leq 1 - k \\ G\left(1 - \dfrac{\partial \overline{V}}{\partial W} - \dfrac{\partial \overline{V}}{\partial A}\right) & \text{if } 1 - k < \dfrac{\partial \overline{V}}{\partial W} + \dfrac{\partial \overline{V}}{\partial A} < 1 \\ 0 & \text{if } \dfrac{\partial \overline{V}}{\partial W} + \dfrac{\partial \overline{V}}{\partial A} \geq 1 \end{cases} .$$

We obtain the following equation for $\overline{V}$:

$$\frac{\partial \overline{V}}{\partial t} + \mathcal{L}\overline{V} + \min\left[\max\left(1 - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}, 0\right), k\right] G$$
$$+ \lambda \max\left(1 - k - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}, 0\right) = 0. \qquad \text{(A)}$$

The set of variational inequalities are given by

$$\frac{\partial \overline{V}}{\partial t} + \mathcal{L}\overline{V} \leq 0 \qquad (i)$$

$$\frac{\partial \overline{V}}{\partial t} + \mathcal{L}\overline{V} + G\left(1 - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}\right) \leq 0 \qquad (ii)$$

$$\frac{\partial \overline{V}}{\partial t} + \mathcal{L}\overline{V} + kG + \lambda\left(1 - k - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}\right) \leq 0 \qquad (iii)$$

and equality holds in at least one of the above three cases.

Continuation region with zero withdrawal

Suppose $\frac{\partial \overline{V}}{\partial W} + \frac{\partial \overline{V}}{\partial A} \geq 1, \max_\gamma h(\gamma)$ is achieved by taking $\gamma = 0$.

We have equality for (i), and strict inequalities for (ii) and (iii). That is,

$$\begin{aligned}
&\frac{\partial \overline{V}}{\partial t} + \mathcal{L}\overline{V} = 0 \\
&\frac{\partial \overline{V}}{\partial t} + \mathcal{L}\overline{V} + G\left(1 - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}\right) < 0 \\
&\frac{\partial \overline{V}}{\partial t} + \mathcal{L}\overline{V} + kG + \lambda\left(1 - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}\right) < 0.
\end{aligned}$$

This corresponds to the continuation region with no withdrawal.

Withdrawal at the contractual rate G

Similarly, when $1-k < \frac{\partial \overline{V}}{\partial w} + \frac{\partial \overline{V}}{\partial A} < 1$, we have equality for (ii) and strict inequalities for (i) and (iii). This corresponds to the region with withdrawal at rate G.

Withdrawal of a finite amount

When $\frac{\partial \overline{V}}{\partial W} + \frac{\partial \overline{V}}{\partial A} \leq 1 - k$, it is optimal to choose λ as the withdrawal rate. We have strict equality for (iii). Suppose we take $\lambda \to \infty$, then

$$\frac{\partial \overline{V}}{\partial W} + \frac{\partial \overline{V}}{\partial A} = 1 - k$$

in order to satisfy the strict equality in (iii).

This scenario corresponds to an immediate withdrawal of a finite amount. The net cash received is $1 - k$ times the withdrawal amount since proportional penalty charge k is imposed.

Linear complementarity formulation of the singular stochastic control model

To obtain $V(W, A, t)$ from $\overline{V}(W, A, t)$, we allow the upper bound λ on γ to be infinite. Conversely, Eq. (A) is visualized as the corresponding penalty approximation

Taking the limit $\lambda \to \infty$, we obtain the following linear complementarity formulation of the value function $V(W, A, t)$:

$$\min\left[-\frac{\partial V}{\partial t} - \mathcal{L}V - \max\left(1 - \frac{\partial V}{\partial W} - \frac{\partial V}{\partial A}, 0\right) G, \frac{\partial V}{\partial W} + \frac{\partial V}{\partial A} - (1-k)\right] = 0,$$

$$W > 0, \quad 0 < A < w_0, \quad t > 0.$$

In summary, the linear complimentarity formulation can be expressed as follows:

1. When $\frac{\partial V}{\partial W} + \frac{\partial V}{\partial A} > 1$, which corresponds to zero withdrawal, we have

$$-\frac{\partial V}{\partial t} - (r-\eta)W\frac{\partial V}{\partial W} - \frac{\sigma^2}{2}W^2\frac{\partial^2 V}{\partial W^2} + rV = 0.$$

2. When $1 \geq \frac{\partial V}{\partial W} + \frac{\partial V}{\partial A} > 1-k$, which corresponds to optimal continuous withdrawal at the rate G, we have

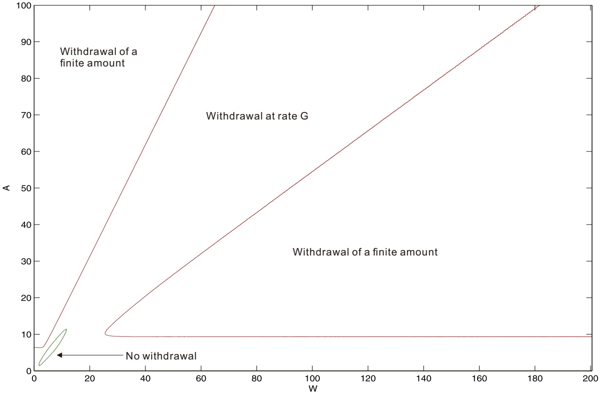
$$-\frac{\partial V}{\partial t} - (r-\eta)W\frac{\partial V}{\partial W} - \frac{\sigma^2}{2}W^2\frac{\partial^2 V}{\partial W^2} + rV - G\left(1 - \frac{\partial V}{\partial W} - \frac{\partial V}{\partial A}\right) = 0.$$

3. In the region that corresponds to optimal withdrawal at the infinite rate (withdrawal of a finite amount), we have

$$\frac{\partial V}{\partial W} + \frac{\partial V}{\partial A} = 1-k.$$

A glance at the optimal withdrawal policies

A typical plot of the separating boundaries that signifies various withdrawal strategies of the GMWB in the (W,A)-plane.



Key features of the separating regions

- Oblique asymptotes that separate "$\gamma = \infty$" and "$\gamma = G$" regions.

- Horizontal asymptote: at large value of W, the optimal withdrawal policy is changed from "$\gamma = \infty$" to "$\gamma = G$" when A falls below some threshold value A^{**}.

- An island of "$\gamma = 0$" region.

Summary of the withdrawal strategies

- "$\gamma = \infty$" region - capture the time value of cash but faces with proportional penalty charge.

- "$\gamma = G$" region - strike the balance between penalty charge and time value of cash.

- "$\gamma = 0$" - take advantage of the optionality in the terminal payoff: $\max(W_T, (1-k)A_T)$.

We consider various limiting cases.

1. Dimension reduction of the pricing model under $G = 0$.
2. Perpetuality of the policy life, $T \to \infty$.
3. Infinitely large value of the policy fund value W_t (far-field condition).
4. At time close to expiry, $t \to T^-$.
5. Limiting small value of guarantee account value A_t.

Simplified pricing model under penalty charge that is applied on any withdrawal, $G = 0$

Homogeneity property of the value function

With $G = 0$, the value function $V(W, A, t)$ becomes homogeneous in A and W. The dimension of the pricing model can be reduced to one by normalizing $V(W, A, t)$ by A and defining the similarity variable $Y = W/A$.

Let $P(Y, t) = V(W, A, t)/A$, the linear complementarity formulation can be expressed in terms of $P(Y, t)$ as

$$\min(-\frac{\partial P}{\partial t} - \frac{\sigma^2}{2} Y^2 \frac{\partial^2 P}{\partial Y^2} - (r - \eta) Y \frac{\partial P}{\partial Y} + rP, (1 - Y) \frac{\partial P}{\partial Y} + P - (1 - k)) = 0,$$

terminal condition: $P(Y, T) = \max(Y, 1 - k)$;
boundary conditions: (i) $\frac{\partial P}{\partial Y}(\infty, t) = e^{-\eta(T-t)}$, (ii) $P(0, t) = 1 - k$.

Optimal dynamic withdrawal policies under $G = 0$

- Either $\gamma = 0$ or $\gamma = \infty$

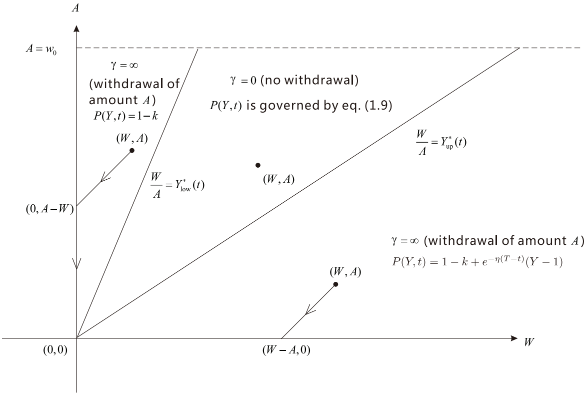
By using convexity property of $P(Y, t)$, we can show that once it is optimal to withdraw under $G = 0$, then the whole guarantee account will be withdrawn to complete depletion immediately.

Recall that $\gamma = \infty$ if and only if

$$H(Y,t) = (Y-1)\frac{\partial P(Y,t)}{\partial Y} - P(Y,t) + (1-k) = 0.$$

When a finite amount δ_0 is withdrawn, Y becomes $\widetilde{Y} = \dfrac{W - \delta_0}{A - \delta_0}$.

To complete the proof, it suffices to show that $H(\widetilde{Y}, t) = 0$.



The separation of the solution domain under $G = 0$ into withdrawal regions ($\gamma = \infty$) and continuation region ($\gamma = 0$) is illustrated.

The separating boundaries are a pair of straight lines $\frac{W}{A} = Y^*_{\text{low}}(t)$, $Y^*_{\text{low}}(t) < 1$ and $\frac{W}{A} = Y^*_{\text{up}}(t)$, $Y^*_{\text{up}}(t) > 1$.

When (W, A) falls within either one of the withdrawal regions, the whole guarantee amount A is depleted immediately (see the two arrows shown in the two regions where $\gamma = \infty$).

Determination of $P(Y,t)$ in the continuation region

In the continuation (no withdrawal) region $\mathcal{D}_0$, $P(Y,t)$ is governed by

$$\frac{\partial P}{\partial t} + \frac{\sigma^2}{2} Y^2 \frac{\partial^2 P}{\partial Y^2} + (r-\eta) Y \frac{\partial P}{\partial Y} - rP = 0, \quad Y^*_{\mathsf{low}}(t) < Y < Y^*_{\mathsf{up}}(t), \quad 0 < t < T.$$

1. Value matching conditions:

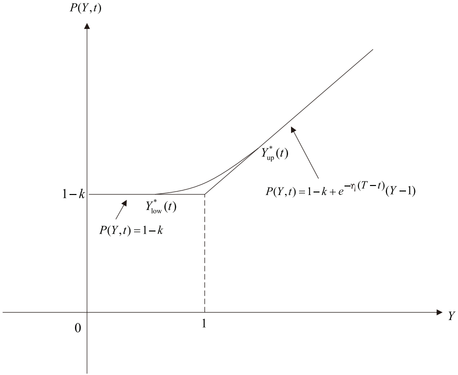
$$P\left(Y^*_{\mathsf{low}}(t), t\right) = 1 - k \quad \text{and} \quad P\left(Y^*_{\mathsf{up}}(t), t\right) = 1 - k + e^{-\eta(T-t)}\left[Y^*_{\mathsf{up}}(t) - 1\right].$$

2. Smooth pasting conditions:

$$\frac{\partial P}{\partial Y}\left(Y^*_{\mathsf{low}}(t), t\right) = 0 \quad \text{and} \quad \frac{\partial P}{\partial Y}\left(Y^*_{\mathsf{up}}(t), t\right) = e^{-\eta(T-t)}.$$

The corresponding obstacle constraint is given by

$$P(Y,t) \geq 1 - k + \max\left(e^{-\eta(T-t)}(Y-1), 0\right), \qquad t < T.$$



The plot of $P(Y,t)$ against Y and the obstacle function: $1-k+\max(e^{-\eta(T-t)}(Y-1),0)$. In the continuation (no withdrawal) region: $Y^*_{\mathsf{low}}(t) < Y < Y^*_{\mathsf{up}}(t)$, $P(Y,t)$ is governed by eq. (1.9). In the two separate withdrawal regions: $Y \leq Y^*_{\mathsf{low}}(t)$ and $Y \geq Y^*_{\mathsf{up}}(t)$, $P(Y,t)$ assumes the same value as that of the obstacle function.

Value function $P(Y,t)$

The value function can be expressed as

$$P(Y,t) = (1-k)e^{-r(T-t)} + c(Y,t;1-k) + M(Y,t),$$

where $M(Y,t)$ represents the withdrawal premium and $c(Y,t;1-k)$ is the time-t price of the European call option with strike $1-k$.

Let $\tau^* = -\frac{\ln(1-k)}{\eta}$. One can show that $Y^*_{\mathsf{up}}(t)$ is not defined for $t \geq T - \tau^*$ and $Y^*_{\mathsf{low}}(t)$ is defined for all t.

The withdrawal premium is given by

$$\begin{aligned}
M(Y,t) = {} & (1-k)r\int_t^{T-\widehat{\tau}^*} e^{-r(u-t)} N(d_{12}(Y,u-t;Y^*_{\mathsf{up}}(u)))\ du \\
& -(r-\eta)\int_t^{T-\widehat{\tau}^*} e^{-r(u-t)} e^{-\eta(T-u)} N(d_{12}(Y,u-t;Y^*_{\mathsf{up}}(u)))\ du \\
& +(1-k)r\int_t^{T} e^{-r(u-t)} N(-d_{22}(Y,u-t;Y^*_{\mathsf{low}}(u)))\ du,
\end{aligned}$$

where $\widehat{\tau}^* = \min(\tau^*, T-t)$,

$$d_{12}\left(Y,u-t;Y^*_{\mathsf{up}}(u)\right) = \frac{\ln\frac{Y}{Y^*_{\mathsf{up}}(u)} + \left(r-\eta-\frac{\sigma^2}{2}\right)(u-t)}{\sigma\sqrt{u-t}},$$

$$d_{22}\left(Y,u-t;Y^*_{\mathsf{low}}(u)\right) = \frac{\ln\frac{Y}{Y^*_{\mathsf{low}}(u)} + \left(r-\eta-\frac{\sigma^2}{2}\right)(u-t)}{\sigma\sqrt{u-t}}.$$

Parameter	Value
Interest rate r	0.05
Maximum no penalty withdrawal rate G	0/year
Volatility σ	0.3
Insurance fee η	0.0312856
Initial lump-sum premium w_0	100
Initial guarantee account balance A_0	100
Initial personal annuity account balance W_0	100

The GMWB contract parameter values used in the numerical calculation of the free boundaries.

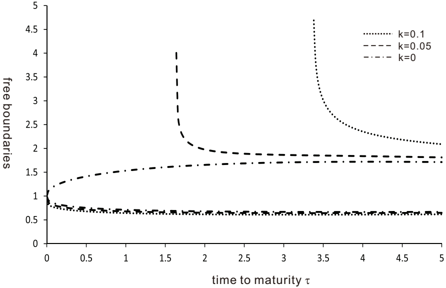
Recursive integration scheme

The numerical values of $Y^*_{\text{low}}(\tau)$ and $Y^*_{\text{up}}(\tau)$ at varying values of τ and $k = 0.05$ are shown.

		$\tau = 5$				$\tau = 10$			
		Recursive scheme			Huang-Forsyth	Recursive scheme			Huang-Forsyth
	n	40	80	120		40	80	120	
$k =$	$Y^*_{\text{up}}(\tau)$	1.80919	1.81057	1.81101	1.80998	1.62899	1.62868	1.62937	1.62172
0.05	$Y^*_{\text{low}}(\tau)$	0.64776	0.64781	0.64782	0.65014	0.68767	0.68765	0.68764	0.69027

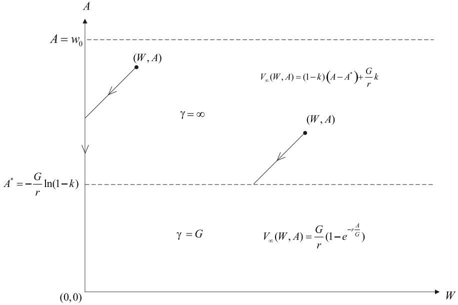
Here, n is the total number of sub-intervals used in the recursive integration scheme. We observe good agreement with the numerical results reported in Huang and Forsyth (2012).

Plot of the withdrawal boundaries $Y^*_{\mathsf{up}}(\tau)$ and $Y^*_{\mathsf{low}}(\tau)$ against time to maturity τ under $G = 0$ with varying values of k.



When $k > 0$, $Y^*_{\mathsf{up}}(\tau)$ is not defined for $\tau \le \tau^*$, where $\tau^* = -\frac{\ln(1-k)}{\eta}$. The threshold value τ^* for $k = 0.1$ and $k = 0.05$ are 3.3677 and 1.6395, respectively.
When $k = 0$, $Y^*_{\mathsf{low}}(0) = 1 - k$ and $Y^*_{\mathsf{up}}(\tau)$ is defined for all values of τ. We also observe that $Y^*_{\mathsf{low}}(\tau)$ is not sensitive to change in value of k.

Perpetuality - closed form solution can be found



The separation of the solution domain into the infinite withdrawal region ($\gamma = \infty$) and the region of withdrawal at the contractual rate ($\gamma = G$). The separating boundary is the horizontal line $A = A^* = -\frac{G}{r}\ln(1-k)$.

When (W, A) falls within the infinite withdrawal region, the amount $A - A^*$ is withdrawn immediately, so A drops to A^* immediately.

Far field boundary conditions at infinitely large policy fund value

The optimal choice of zero withdrawal should be ruled out as $W \to \infty$ since optionality of terminal payoff has very low value.

The value function of the far field, $W \to \infty$, is determined by finding δ such that

$$V(W,A,t) = \sup_{0\leq\delta\leq A} \left\{ (1-k)\,\delta + \int_t^{T^*} Ge^{-ru}\,du + e^{-r(T-t)}E_t\left[W_T\right] \right\},$$

where

$$T^* = \min\left(T, t + \frac{A-\delta}{G}\right).$$

Let A^{**} denote the solution for the equation

$$1 - k - e^{-\eta(T-t)} - e^{-r\frac{A}{G}} \left\{1 - e^{-\eta\left[(T-t)-\frac{A}{G}\right]}\right\} = 0$$

1. If $1 - k - e^{-\eta(T-t)} > 0$ and $A \leq A^{**}$, then

$$V(W, A, \tau) \approx \frac{G}{r}\left(1 - e^{-r\frac{A}{G}}\right) + e^{-\eta(T-t)}W - \frac{G}{r-\eta}e^{-\eta(T-t)}\left(1 - e^{-(r-\eta)\frac{A}{G}}\right).$$

   The optimal withdrawal policy is to withdraw in the rate of G.

2. If $1 - k - e^{-\eta(T-t)} > 0$ and $A > A^{**}$, then

$$V(W, A, t) \approx \left[1 - k - e^{-\eta(T-t)}\right](A - A^{**}) + \frac{G}{r}\left(1 - e^{-r\frac{A^{**}}{G}}\right) + e^{-\eta(T-t)}W - \frac{Ge^{-\eta(T-t)}}{r-\eta}\left[1 - e^{-(r-\eta)\frac{A^{**}}{G}}\right].$$

   The optimal withdrawal policy is to withdraw the finite amount $A - A^{**}$ immediately, then followed by withdrawal at the rate G.

3. If $e^{-\eta(T-t)} \geq 1-k$

$$V(W,A,t) \approx e^{-\eta(T-t)}W + \frac{G}{r}\left[1-e^{-r\min\left(\frac{A}{G},T-t\right)}\right] - \frac{Ge^{-\eta(T-t)}}{r-\eta}\left[1-e^{-(r-\eta)\min\left(\frac{A}{G},T-t\right)}\right].$$

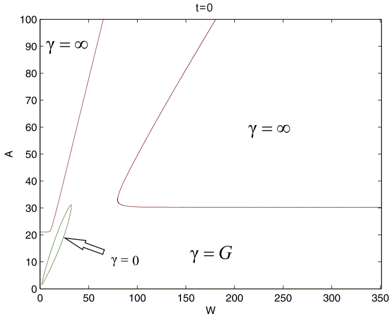
The optimal withdrawal policy is to withdraw at the rate of G.

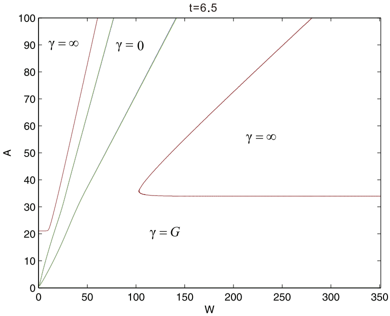
Summary

- When the optionality value is ignored, the remaining factors for the policyholder to weigh are the penalty charge and insurance fee.
- When the penalty charge rate k is larger than the insurance fee incurred in the remaining period $T-t$, as quantified by $1-e^{-\eta(T-t)}$, the rational holder will choose to bear the insurance fee rather than suffer the larger penalty charge.

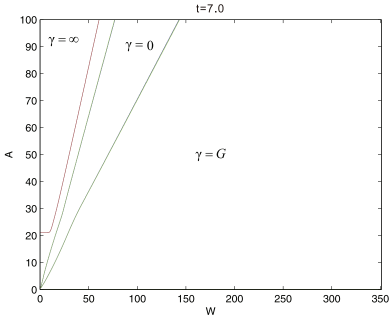
	Huang-Forsyth	asymptotic formulas	percentage difference
$A = 20,\ W = 80$	63.18349	63.184184	0.00110%
$A = 20,\ W = 100$	77.810965	77.811287	0.00041%
$A = 30,\ W = 80$	65.035297	65.030709	-0.00705%
$A = 30,\ W = 100$	79.657330	79.657813	0.00061%
$A = 40,\ W = 80$	66.763615	66.717224	-0.06949%
$A = 40,\ W = 100$	81.345396	81.344328	-0.00131%
$A = 50,\ W = 80$	68.821701	68.403672	-0.60741%
$A = 50,\ W = 100$	83.038705	83.030776	-0.00955%

Comparison of the numerical value for the policy value obtained from Huang-Forsyth's (2012) numerical calculations and asymptotic formulas at large value of W. Very good agreement between the two sets of numerical values is observed even at moderate values of W.





The plots of the optimal withdrawal regions with penalty parameter $k = 0.1$ at varying values of the calendar time t. The horizontal asymptote: $A = A^{**}$ exists when the calendar time is sufficiently far from expiry.



The horizontal asymptote: $A = A^{**}$ disappears when time is sufficiently close to expiry.

There is a narrow strip of "$\gamma = G$" region that lies between "$\gamma = 0$" region and "$\gamma = \infty$" region.

At time close to expiry

At time close to expiry, $t \to T^-$, the value of optionality associated with the terminal payoff almost vanishes. The optimal strategy of zero withdrawal is almost ruled out (except under the unlikely event of $(1-k)A \approx W$).

To show the claim, we consider the value function at time close to expiry $V(W, A, T^-)$. By continuity of the value function, we have

$$V(W, A, T^-) = \begin{cases} (1-k)A & \text{if } (1-k)A > W \\ W & \text{if } (1-k)A < W \end{cases}.$$

For either payoff of $(1-k)A$ or W, we observe that the gradient constraint: $\frac{\partial V}{\partial W} + \frac{\partial V}{\partial A} > 1$ is violated. Hence, the region of zero withdrawal ($\gamma = 0$) almost vanishes as $t \to T^-$, except in an asymptotically narrow strip along the separating boundary line $(1-k)A = W$.

1. $W > (1-k)A$

- Given that $t \to T^-$, the terminal payoff is almost surely to be W_T.
- As $\gamma = 0$ is ruled out when $t \to T^-$, the choice of either $\gamma = G$ or $\gamma = \infty$ depends on the relative magnitude of various depreciation factors; namely, $e^{-\eta(T-t)}$ due to insurance fee η and $1-k$ due to penalty charge.
- When $T-t$ is small so that $e^{-\eta(T-t)}$ is almost surely smaller than $1-k$. As a result, it is optimal to choose $\gamma = G$.
- The asymptotic value function is given by

$$
\begin{aligned}
V(W, A, t) &\approx \int_t^T Ge^{-ru}\, du + e^{-r(T-t)} E_t\left[W_T\right] \\
&= \frac{G}{r}\left[1 - e^{-r(T-t)}\right] + e^{-\eta(T-t)}\left\{W - \frac{G}{r-\eta}\left[1 - e^{-(r-\eta)(T-t)}\right]\right\}, \quad t \to T^-.
\end{aligned}
$$

2. $W < (1-k)A$

- The terminal payoff is almost surely to be $(1-k)A$.
- In order to minimize loss of time value of the cash amount received, the optimal strategy is to withdraw the finite amount $A - G(T-t)$ immediately, followed by continuous withdrawal at the rate G in the remaining time until maturity date T.
- The asymptotic value function is given by

$$\begin{aligned} V(W,A,t) &\approx \int_t^T Ge^{-ru}\,du + (1-k)\left[A - G\left(T-t\right)\right] \\ &= \frac{G}{r}\left[1 - e^{-r(T-t)}\right] + (1-k)\left[A - G(T-t)\right], \quad t \to T^-. \end{aligned}$$

Asymptotic analysis when $A \to 0$

The value function at $A \to 0$ (low level of guarantee account) tends asymptotically to that at $k = 0$ (zero penalty charge).

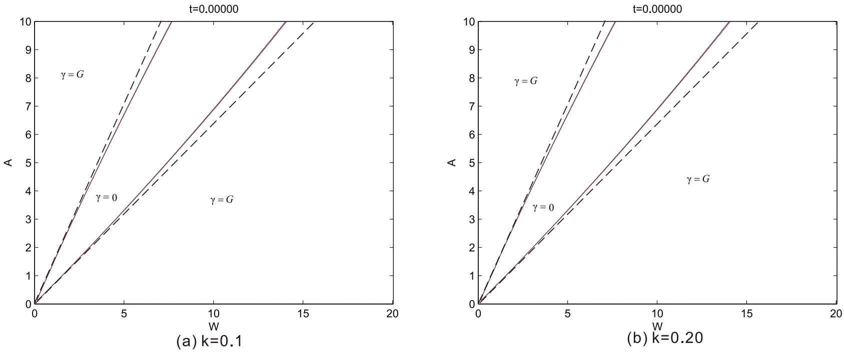
- When $k = 0$, $\gamma = G$ is ruled out.
- When $A \to 0$, $\gamma = G$ and $\gamma = \infty$ are almost indifferent since withdrawal of a very small amount at continuous withdrawal rate G over a short time interval is almost identical to an immediate withdrawal of a finite amount at $\gamma = \infty$.
- For both cases of $k = 0$ and $A \to 0$, the value of optionality at maturity has a similar impact on the decision of zero withdrawal.

Outline of the theoretical proof

We consider the value function with $k > 0$ and adopting sub-optimal withdrawal policies of the optimal withdrawal policies of those of the zero penalty ($k = 0$) counterpart.

- The value function under $k > 0$ is bounded above by the value function under $k = 0$ and the value function reduces to a lower value when sub-optimal withdrawal policies are adopted.

- It suffices to show that the value function under $k > 0$ and adoption of sub-optimal withdrawal policies tends to that under $k = 0$ as $A \to 0$.



The optimal withdrawal strategy with penalty $k = 0.1$ and 0.20 at $t = 0$. The dashed lines are the optimal boundaries when setting $k = 0$.

Conclusions

- Complete solution is available for $G = 0$
  - Homogeneity property of the value function
  - Integral equations for the determination of the optimal withdrawal boundaries
- Analytic analysis of various limiting cases for $G > 0$
  - Perpetuality of policy life
  - Far field boundary condition at infinitely large policy fund value
  - Time close to expiry
  - Small value of guarantee account

- When the underlying fund value is large, it is optimal to withdraw an immediate amount provided that the guarantee account value is sufficiently high and the current time is sufficiently far from expiry.
- When the underlying fund value is sufficiently small, it is always optimal to withdraw an immediate amount provided that the guarantee account value is not too low.
- When the ratio of the underlying fund value to the guarantee account value falls within certain range, it may become optimal to adopt the policy of zero withdrawal.