Pricing dynamic guarantees under stochastic interest rates

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Abstract

Dynamic guarantees in equity-indexed annuities provide a floor level of protection over the investment period. Earlier pricing models of dynamic guarantees assume constant interest rates, which may not be too desirable since the contracts are usually of long maturity. This article develops an analytic approach that prices dynamic guarantees under stochastic interest rate environment. The valuation procedure involves the derivation of the density function of the joint process of fund value and interest rate subject to a reflecting barrier on the fund value.

1 Introduction

Equity-indexed annuities with embedded option features are popular innovative products introduced by US life insurance companies in the 1990s. The dynamic floor protection feature embedded in some of these equity-indexed insurance products provides the investor with a floor protection on the fund value over the investment period. Gerber and Pafumi (2000) first derive the price formula of a dynamic protection fund when the underlying fund process is assumed to follow the geometric Brownian process. Later, Gerber and Shiu (2003) provide a comprehensive treatise of linking the pricing model of dynamic guarantees with the lookback option model. They derive the Laplace-Stieltjes transform of the expected excess of the running maximum of a Brownian process above a positive constant over a finite time interval. Imai and Boyle (2001) extend the pricing of the dynamic protection feature under the constant elasticity of variance process of the underlying fund value.
They also propose a numerical algorithm for pricing dynamic guarantees under discrete monitoring of the fund value process. Chu and Kwok (2004) analyze the withdrawal right embedded in dynamic protection funds.

In all of the above pricing models of dynamic guarantees, the interest rate is assumed to be constant. However, since equity-indexed annuities are usually of long maturity, the assumption of constant interest rate may not be quite adequate. Lin and Tan (2003) propose numerical procedure for pricing equity-indexed annuities under stochastic interest rate. Bernard et al. (2005) construct valuation model for finding the market value of life insurance contracts under stochastic interest rates and default risk. Following the approach used in Collin-Dufresne and Goldstein’s (2001) risky debt model under stochastic interest rate, they compute the first passage time density to the default (absorbing) barrier of the log-normal asset value process. We would like to comment that Collin-Dufresne and Goldstein’s technique cannot be applied to price dynamic guarantees under stochastic interest rate. This is because the protection floor level in dynamic guarantees is a reflecting barrier of the fund value process while the defaulting threshold in a risky debt model is an absorbing barrier. In the commenting note on Gerber-Shiu’s paper (2003), Deelstra (2003) remarks that it is not feasible to derive a closed form expression for the Laplace-Stieltjes transform of the expected excess of the running maximum of a Brownian process above a positive constant over a finite time interval when the force of interest is stochastic.

Relating to the study of the dynamics of restricted stochastic processes, Ricciardi and Sacerdote (1987) derive an integral equation that governs the density function of a restricted one-dimensional Ornstein-Uhlenbeck process with a reflecting boundary. Similar technique is also used by Chvosta et al. (2005) to study the kinetics and energetics of a reflected diffusion process with time dependent and space-homogeneous force. In this article, we propose a valuation methodology for pricing guarantees under stochastic interest rate environment. In the solution procedure, we derive the integral equation that governs the density function of the joint process of fund value and interest rate subject to a reflecting barrier on the fund value. Our work thus provides an analytic approach for this apparently daunting pricing problem.

The paper is organized as follows. In Section 2, we present the model formulation of dynamic guarantees under stochastic interest rate environment. The value function is expressed as a double integral involving the density function of the joint process of fund value and interest rate subject to a reflecting barrier on the fund value. In Section 3, we present the partial dif-
ferential equation formulation of the density function of the restricted joint process and derive the relevant governing integral equation for the determination of the density function. The last section summarizes the results and concludes.

2 Model formulation

Let $F_t$ denote the value of the underlying fund at time $t$ and $r_t$ denote the riskless short rate process. Under the risk neutral measure $Q$, $F_t$ follows the Geometric Brownian motion and $r_t$ follows the Vasicek dynamics:

$$\frac{dF_t}{F_t} = (r_t - q) \, dt + \sigma_F \sqrt{1 - \rho^2} \, dZ_1(t) + \sigma_F \rho \, dZ_2(t)$$  \hspace{1cm} (1a)$$

$$dr_t = k(\theta - r_t) \, dt + \sigma_r \, dZ_2(t)$$  \hspace{1cm} (1b)$$

where $q$ is the constant dividend yield of the fund process, $Z_1(t)$ and $Z_2(t)$ are independent Brownian motions under $Q$, $\sigma_F$ and $\sigma_r$ are constant volatility parameters, $\rho$ is the constant correlation coefficient between $r_t$ and $\ln F_t$, $k$ and $\theta$ are the respective constant reversion rate and long-term mean in the drift coefficient of the Vasicek dynamics of $r_t$. Let $K$ denote the constant protection level of the dynamic guarantee, that is, whenever the fund value drops to $K$, sufficient money will be supplied so that the upgraded fund value does not fall below $K$. In this case, $K$ is the reflecting barrier of the modified (upgraded) fund value process (Gerber and Pafumi, 2000). Accordingly, the value of the dynamic fund protection at time zero is given by (Imai and Boyle, 2001)

$$V(0) = E_Q \left[ e^{-\int_0^T r_u \, du} F_T \max \left\{ 1, \frac{K}{\min_{0 \leq u \leq T} F_u} \right\} \right] - E_Q \left[ e^{-\int_0^T r_u \, du} F_T \right],$$  \hspace{1cm} (2)$$

where $E_Q$ represents the expectation under the risk neutral measure $Q$. We write $Y_t = \ln \frac{F_t}{K}$, then under the $T$-forward measure $Q_T$ where the discount
bond price $P(0, T)$ is used as the numeraire, we obtain

\[
dY_t = \left[ r_t - q - \frac{\sigma_F^2}{2} - \rho \sigma_F \sigma_r B(T - t) \right] dt \\
+ \sigma_F \sqrt{1 - \rho^2} dZ_1^T(t) + \sigma_F \rho dZ_2^T(t),
\]

\[
dr_t = k \left[ \theta - r_t - \frac{\sigma_r^2}{k} B(T - t) \right] dt + \sigma_r dZ_2^T(t),
\]

where $Z_1^T(t)$ and $Z_2^T(t)$ are independent Brownian motions under $Q_T$ and

\[
B(T - t) = \frac{1}{k} [1 - e^{-k(T-t)}].
\]

Under the interest rate dynamics as specified by Eq. (1b), the discount bond price $P(0, T)$ is found to be

\[
P(0, T) = \exp \left( \theta - \frac{\sigma_r^2}{2k^2} \right) \left[ B(T) - T - \frac{\sigma_r^2}{4k} B(T)^2 - B(T) r_0 \right].
\]

Let $\tilde{F}_t$ be the modified fund value as defined by

\[
\tilde{F}_t = F_t \max \left\{ 1, \frac{K}{\min_{0 \leq u \leq t} F_u} \right\}.
\]

Imai and Boyle (2001) show that

\[
\tilde{F}_t = K e^{\tilde{Y}_t},
\]

where $\tilde{Y}_t$ is obtained from $Y_t$ by restricting its state space to $[0, \infty)$ and placing a reflecting barrier at 0. Under the $T$-forward measure $Q_T$, we obtain

\[
V(0) = KP(0, T) E_{Q_T}[e^{\tilde{Y}_T}] - F_0 e^{-qT}.
\]

Let $\tilde{f}(y, r, T|y_0, r_0, 0)$ denote the transition density of the restricted joint process $\{(\tilde{Y}_t, r_t) : t \geq 0\}$, where $y_0 = \ln \frac{F_0}{K}$. It is seen that

\[
E_{Q_T}[e^{\tilde{Y}_T}] = \int_0^\infty \int_{-\infty}^\infty e^y \tilde{f}(y, r, T|y_0, r_0, 0) dr dy.
\]
Once the density function \( \tilde{f} \) is known, the value of the modified fund \( V(0) \) can be obtained by evaluating the double integral in Eq. (9).

In the next section, we will show that the density functions of the restricted and unrestricted joint processes are related by

\[
\tilde{f}(y, r, T|y_0, r_0, 0) = f(y, r, T|y_0, r_0, 0) \nonumber \]

\[
\frac{\partial}{\partial y} \int_{-\infty}^{\infty} \int_{0}^{t} f(y, r, T|0, \tilde{r}, \tilde{t}) M(\tilde{r}, \tilde{t}; y_0, r_0, 0) \, d\tilde{t} \, d\tilde{r}, \quad (10)
\]

where \( f(\cdot|\cdot) \) is the density function of the unrestricted joint process \( \{(Y_t, r_t) : t \geq 0\} \). The function \( M(\tilde{r}, \tilde{t}; y_0, r_0) \) satisfies the following integral equation:

\[
\left[ r - q - \rho \sigma_F \sigma_r B(T - t) \right] f(0, r, T|y_0, r_0, 0) - \frac{\sigma^2_F}{2} \frac{\partial f}{\partial y}(0, r, T|y_0, r_0, 0) - \frac{\sigma F}{2} \frac{\partial f}{\partial r}(0, r, T|y_0, r_0, 0) \nonumber \]

\[
= \int_{-\infty}^{\infty} \int_{0}^{t} \frac{\partial}{\partial y} \left\{ \left[ r - q - \frac{\sigma^2_F}{2} - \rho \sigma_F \sigma_r B(T - t) \right] f(0, r, T|0, \tilde{r}, \tilde{t}) \right. \nonumber \]

\[
\left. - \frac{\sigma^2_F}{2} \frac{\partial f}{\partial y}(0, r, T|0, \tilde{r}, \tilde{t}) - \frac{\rho \sigma F \sigma_r}{2} \frac{\partial f}{\partial r}(y, r, T|0, \tilde{r}, \tilde{t}) \right\} M(\tilde{r}, \tilde{t}; y_0, r_0) \, d\tilde{t} \, d\tilde{r}. \quad (11)
\]

The analytic expression for the density function \( f(y, r, T|y_0, r_0, 0) \) of the unrestricted joint process can be found in Bernard’s paper (2005). Also, numerical procedure of solving \( M \) from the above integral equation can be constructed by following a similar procedure as presented in their paper.

3 Transition density function of restricted process subject to reflecting boundary

In this section, we show how to derive the density function of the restricted joint process with a reflecting boundary. First, for \( t \geq t' \geq 0 \), let \( p(x_1, x_2, t; x'_1, x'_2, t') \) denote the transition density function of the unrestricted diffusion process \( \{(X_{1,t}, X_{2,t}) : t \geq 0\} \) in the unrestricted domain \( \mathbb{R}^2 \). The density
function is governed by the following forward Fokker-Planck equation:

\[
\frac{\partial p}{\partial t}(x_1, x_2, t; x_1', x_2', t') = -\frac{\partial}{\partial x_1} \left[ \mu_1(x_2, t)p(x_1, x_2, t; x_1', x_2', t') \right] - \frac{\partial}{\partial x_2} \left[ \mu_2(x_2, t)p(x_1, x_2, t; x_1', x_2', t') \right]
+ \frac{D_{11}}{2} \frac{\partial^2 p}{\partial x_1^2}(x_1, x_2, t; x_1', x_2', t') + \frac{D_{12}}{2} \frac{\partial^2 p}{\partial x_1 \partial x_2}(x_1, x_2, t; x_1', x_2', t')
+ \frac{D_{22}}{2} \frac{\partial^2 p}{\partial x_2^2}(x_1, x_2, t; x_1', x_2', t')
\] (12a)

with initial condition:

\[
p(x_1, x_2, t'; x_1', x_2', t') = \delta(x_1 - x_1')\delta(x_2 - x_2').
\] (12b)

Suppose we define the components of the probability current vector by

\[
J_1(x_1, x_2, t; x_1', x_2', t') = \mu_1(x_2, t)p(x_1, x_2, t; x_1', x_2', t')
- \frac{D_{11}}{2} \frac{\partial p}{\partial x_1}(x_1, x_2, t; x_1', x_2', t') - \frac{D_{12}}{2} \frac{\partial p}{\partial x_2}(x_1, x_2, t; x_1', x_2', t')
\] (13a)

\[
J_2(x_1, x_2, t; x_1', x_2', t') = \mu_2(x_2, t)p(x_1, x_2, t; x_1', x_2', t')
- \frac{D_{12}}{2} \frac{\partial p}{\partial x_1}(x_1, x_2, t; x_1', x_2', t') - \frac{D_{22}}{2} \frac{\partial p}{\partial x_2}(x_1, x_2, t; x_1', x_2', t')
\] (13b)

then the forward Fokker-Planck equation (12a) can be expressed as

\[
\frac{\partial p}{\partial t}(x_1, x_2, t; x_1', x_2', t') = -\frac{\partial J_1}{\partial x_1}(x_1, x_2, t; x_1', x_2', t') - \frac{\partial J_2}{\partial x_2}(x_1, x_2, t; x_1', x_2', t').
\] (14)

Let \( \tilde{p}(x_1, x_2, t; x_1', x_2', t') \) be the transition density function of the restricted diffusion process \( \{(X_{1,t}, X_{2,t}) : t \geq 0\} \) in the half-plane \([b, \infty) \times \mathbb{R}\) with the reflecting barrier at \(x_1 = b\). It is known that \( \tilde{p} \) satisfies the same forward Fokker-Planck equation and initial condition as specified in Eqs. (12a,b). In addition, \( \tilde{p} \) has to observe the following reflecting boundary condition:

\[
\tilde{J}_1(b, x_2, t; x_1', x_2', t') = \mu_1(x_2, t)\tilde{p}(b, x_2, t; x_1', x_2', t')
- \frac{D_{11}}{2} \frac{\partial \tilde{p}}{\partial x_1}(b, x_2, t; x_1', x_2', t')
- \frac{D_{12}}{2} \frac{\partial \tilde{p}}{\partial x_2}(b, x_2, t; x_1', x_2', t') = 0.
\] (15)
Suppose we define the function

\[
R(x_1, x_2, t; x'_1, x'_2, t') = p(x_1, x_2, t; x'_1, x'_2, t') \quad - \frac{\partial}{\partial x_1} \int_{-\infty}^{\infty} \int_{t_0}^{t} p(x_1, x_2, t; b, \tilde{x}_2, t') M(\tilde{x}_2, \tilde{t}; x'_1, x'_2) \, d\tilde{t} \, d\tilde{x}_2,
\]

(16)

it can be shown by direct differentiation that \( R \) satisfies the forward Fokker-Planck equation and the initial condition. In order to observe the reflecting boundary condition (15), the function \( M(\tilde{x}_2, \tilde{t}; x'_1, x'_2) \) has to satisfy the following integral equation:

\[
J_1(b, x_2, t; x'_1, x'_2, 0) = \int_{-\infty}^{\infty} \int_{t_0}^{t} \frac{\partial J_1}{\partial x_1}(b, x_2, t; b, \tilde{x}_2, \tilde{t}) M(\tilde{x}_2, \tilde{t}; x'_1, x'_2) \, d\tilde{t} \, d\tilde{x}_2.
\]

(17)

Hence, \( R \) is the solution to the transition density function of the restricted joint process with reflecting boundary at \( x_1 = b \). The analytic expression for \( \tilde{f} \) and integral equation for \( M \) in the last section [see Eqs. (10, 11)] can then be obtained by referencing to the result in Eqs. (16, 17).

4 Conclusion

Since the life of an equity-indexed annuity contract is usually of long maturity, the assumption of constant interest rate may not be too desirable. In this paper, we have proposed an analytic approach for pricing dynamic guarantees under stochastic interest rate environment. The difficulties of the analytic procedure stem from the presence of the reflecting barrier arising from the nature of the protection floor. We manage to obtain an integral representation of the price function of the dynamic guarantee in terms of the density function of a two-dimensional restricted joint process of fund value and interest rate subject to a reflecting barrier on the fund value.

References


