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A Front–Fixing Finite Difference Method for the Valuation of American Options

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ABSTRACT

The difficulty for the accurate valuation of American type financial options lies on the unknown free boundaries associated with the early exercise feature. This article proposes a front–fixing transformation to transform the unknown free boundary into a known and fixed line in the transformed plane. An efficient finite difference method is then developed, which produces the optimal exercise boundary and multiple option values simultaneously. Numerical results reveal that the front–fixing finite difference method has accuracy comparable to that of the binomial method, and it becomes computationally competitive when multiple option positions are priced.

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I. INTRODUCTION

The valuation of American options has long been an intriguing problem. It is widely acknowledged that an analytical formula does not exist for the value of an American option where early exercise may be optimal. As a result, the valuation of American options routinely resorts to numerical or quasi–analytical methods. Since most traded options are American style, there is considerable interest in searching for new valuation techniques.

The numerical methods are typified by the finite difference method (Brennan and Schwartz 1977), and the binomial method (Cox et al. 1979). The binomial schemes are pedagogically appealing, easy to implement, and adapt easily to options with nonstandard features. Rigorous justification for convergence has also been established for these methods (Jaillet et al. 1990; Amin and Khanna 1994).

The quasi–analytical approach to the valuation problem includes the methods proposed by Geske and Johnson (1984), MacMillan (1986), and Barone–Adesi and Whaley (1987). These methods generate approximate solutions of an American option by either restricting the early exercise at discrete dates, or by solving some modified form of the Black–Scholes equation. Recent developments of the quasi–analytical approach include the analytical method of lines by Carr and Faguet (1994), the integral equation approach by Huang et al. (1996), and the capped option approximation by Broadie and Detemple (1996). Both the integral equation approach and capped option approach require some interactive procedure for determining the early exercise boundary. Also, the use of Richardson extrapolation is a critical component of the analytical method of lines and the integral equation approach. It may be reasonable to believe that these methods can be generalized to other types of options. However, for some exotic options (such as Asian options which do not have analytical formula even without the early exercise feature), the prospect of the quasi–analytical approach may not be promising.

Recently, Wilmott et al. (1993) developed a new framework to price exotic options, such as barrier, Asian, and lookback options. They model these exotic options by a linear complimentarily formulation of partial differential equations, which can then be solved effectively by the projected SOR method. The projection requires an embedded iteration at each time step.

In this article we introduce a known technique for solving free boundary problems to the field of option pricing. By the so–called front–fixing transformation (Landau 1950), we let the unknown boundary be included into the equation in exchange for a fixed boundary. The presence of fixed boundary facilitates effective discretization of the governing differential equation. We then propose a linearized difference scheme for the transformed equation. Our scheme does not require embedded iteration at each time step of evolution, like the projected SOR method. In addition to option values, the present method captures the whole optimal exercise boundary as part of the solution procedure. The procedure works well for any type of option as long as an appropriate front–fixing transformation exists, which we believe to be true at least for standard American options, barrier options, Asian options, and lookback options. In subsequent sections we will present the procedure and numerical test results with the prototype American put options.

The article is organized as follows. In Section II we introduce the front–fixing transformation which transforms an unknown front into a fixed boundary. In Section III, we propose a finite difference discretization of the transformed equation and outline the solution procedure. Numerical comparisons of the proposed algorithm with the binomial method are given in Section IV. Concluding remarks are presented in the last section.

II. THE FRONT–FIXING TRANSFORMATION

Let $P(S, \tau; X)$ denote the value of an American put option. Here, S is the price of the underlying asset, τ is the time to expiration, and X is the strike price. We assume that S follows the risk–neutral process:

$$dS = rSdt + \sigma Sdz \tag{1}$$

where r is the risk–free interest rate, σ is the volatility of the asset price, and dz is the standardized Weiner process. Both r and σ are assumed constant. It is well known that at any moment, there exists a critical asset price $B(\tau)$ such that it is optimal to exercise the put option when S is at or below $B(\tau)$. Hence, when $S \leq B(\tau)$, the put option takes the value:

$$P(S, \tau; X) = X - S \tag{2}$$

For asset price S above $B(\tau)$, $P(S, \tau)$ satisfies the celebrated Black–Scholes equation (Black and Scholes 1973):

$$\frac{\partial P}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} - rS\frac{\partial P}{\partial S} + rP = 0, \quad S \in (B(\tau), \infty) \tag{3}$$

augmented with the "smooth pasting" conditions at the exercise boundary $B(\tau)$:

$$P(B(\tau), \tau) = X - B(\tau), \quad \frac{\partial P}{\partial S}(B(\tau), \tau) = -1 \tag{4}$$

and the far field boundary condition:

$$\lim_{s \to \infty} P(S, \tau) = 0 \tag{5}$$

The terminal payoff of the option gives rise to the initial condition:

$$P(S, 0) = 0, \quad S \in (B(0), \infty) \quad \text{with } B(0) = X \tag{6}$$

Since $P(S, \tau)$ is linearly homogeneous in S and X, and S is linearly homogeneous in X, the equation and boundary conditions using the normalized functions $\tilde{P} = \frac{P}{X}$ and $\tilde{B}(\tau) = \frac{B(\tau)}{X}$ on the normalized variable $\tilde{S} = \frac{S}{X}$ are identical to Equations 3–6, except the strike price becomes 1. Assuming there is no confusion, we let P, B, and S stand for the normalized variables in subsequent discussions.

The difficulty for accurate valuation of the American put option lies on the unknown boundary $B(\tau)$. If we employ standard finite difference and finite element methods directly to Equations 3–6, we will encounter the difficulty of managing the computational mesh points or elements. It was first suggested by Landau (1950) that such difficulty can be removed by transforming the unknown and time varying boundary into a known and fixed line. The following transformation of state variable serves such a purpose:

$$y = \ln \frac{S}{B(\tau)} \tag{7}$$

The process for y now becomes:

$$dy = \left(r - \frac{\sigma^2}{2} - \frac{B'(\tau)}{B(\tau)} \right) dt + \sigma dz \tag{8}$$

By either forming a riskless portfolio or direct substitution, we can derive the equation and boundary conditions using the new variable y:

$$\frac{\partial P}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial y^2} - \left(r - \frac{\sigma^2}{2} \right) \frac{\partial P}{\partial y} + rP = \frac{B'(\tau)}{B(\tau)} \frac{\partial P}{\partial y} \tag{9}$$

$$P(y, 0) = 0 \quad y \in (0, \infty) \tag{10}$$

$$P(0, \tau) = 1 - B(\tau), \quad \frac{\partial P(0, \tau)}{\partial y} = -B(\tau), \quad P(\infty, \tau) = 0 \tag{11}$$

The presence of the term $\frac{B'(\tau)}{B(\tau)} \frac{\partial P}{\partial y}$ reveals the nonlinear nature of the valuation problem as exposed by the transformation. Note that transformation (7) is valid only if $B(\tau) > 0$ for all $\tau \geq 0$. This is indeed true as it has been shown (Samuelson 1979) that $B(\tau)$ is a monotonically decreasing function of τ with a nontrivial asymptotic limit:

$$B(\infty) = \frac{1}{1 + \gamma}, \quad \gamma = \frac{\sigma^2}{2r} \tag{12}$$

Unlike most other free boundary problems, there is no separate equation for $B(\tau)$ in the present case. At $y = 0$, Equation 9 becomes:

$$-\frac{\sigma^2}{2} \frac{\partial^2 P(0, \tau)}{\partial y^2} - \frac{\sigma^2}{2} B(\tau) + r = 0 \tag{13}$$

after some cancellations. Since the left boundary value $P(O, \tau)$ is an unknown, Equation 13 will be needed in the numerical procedure.

III. FINITE DIFFERENCE APPROXIMATION

The finite difference discretization of the previous governing equations amounts to the approximation of all derivatives by the appropriate difference quotients. For this purpose, we introduce a two-dimensional mesh of the size (h, k) in the first quadrant of the y–r plane. To present our finite difference scheme in a compact form, we define the following difference operators:

$$D_+ = \frac{E - I}{h}, \; D_- = \frac{I - E^{-1}}{h}, \; D_0 = \frac{E - E^{-1}}{2h} \tag{14}$$

where E is the spatial shifting operator such that for any discrete function P_j,

$$E^i P_j = P_{j+i} \tag{15}$$

In order to avoid nonlinearity and achieve a high order of accuracy, we adopt the following three-level discretization to Equation 9:

$$\frac{P_j^{n+1} - P_j^{n-1}}{2k} - \left\{\frac{\sigma^2}{2} D_+ D_- + \left(r - \frac{\sigma^2}{2}\right) D_0 - r\right\} \frac{(P_j^{n+1} + P_j^{n-1})}{2} = g^n D_0 P_j^n$$

$$j = 1, 2, \ldots, M \tag{16}$$

Here, P_j^n denotes the numerical approximation to $P(jh,nk)$, and:

$$g^n = \frac{B^{n+1} - B^{n-1}}{2kB^n} \tag{17}$$

which approximates $\frac{B'(nk)}{B(nk)}$. We choose M large enough so that we can comfortably set $P_{M+1}^n = 0$ for all n. The discretized version of Equation 13 is:

$$-\frac{\sigma^2}{2} D_+ D_- P_0^n - \frac{\sigma^2}{2} B^n + r = 0 \tag{18}$$

which involves a fictitious value P_{-1}^n. The discretization of the "smooth pasting condition" (Equation 11) by central differencing gives rise to:

$$P_0^n = 1 - B^n, \text{ and} \tag{19}$$

$$\frac{P_1^n - P_{-1}^n}{2h} = -B^n, \quad \text{for all } n \geq 1 \tag{20}$$

From Equations 18, 19, and 20, we can eliminate P_{-1}^n and obtain:

$$P_1^n = \alpha - \beta B^n, \quad n \geq 1 \tag{21}$$

where

$$\alpha = 1 + h^2\sigma^{-2}r, \quad \beta = [1 + (1 + h)^2]/2 \tag{22}$$

Note that the numerical discretization is not unique. We adopt the discretization in Equation 16 based on the following considerations. First, when $g^n = 0$, Equation 16 reduces to the Crank–Nicholson scheme used by Courtadon (1982) for the European call options. If we consider our finite difference scheme from the viewpoint of an approximate general jump process, then the underlying jump process has no biased variance. Second, the three–level discretization permits the explicit treatment of nonlinear term, without sacrificing the accuracy of the Crank–Nicholson discretization, which is known to be of order $O(k^2 + h^2)$.

We now explain how to advance from P_j^{n-1} and P_j^n to obtain P_j^{n+1}, $j = 0, 1, \ldots, M$. We first rewrite Equation 16 using matrix notations. Denote:

$$\alpha = \mu\sigma^2 + kr,\ b = \frac{\mu}{2}\left[\sigma^2 - h\left(r - \frac{\sigma^2}{2}\right)\right],\ c = \frac{\mu}{2}\left[\sigma^2 + h\left(r - \frac{\sigma^2}{2}\right)\right]$$

where $\mu = k/h^2$, and define matrix:

$$A = \begin{pmatrix} a & -c & 0 & \cdots & \cdots & 0 \\ -b & a & -c & 0 & \cdots & 0 \\ 0 & -b & a & -c & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & -b & a & -c \\ 0 & 0 & \cdots & 0 & -b & a \end{pmatrix}$$

In terms of A, Equation 16 can then be rewritten as:

$$(I + A)\ \mathbf{P}^{n+1} = (I - A)\mathbf{P}^{n-1} + bP_0^{n-1}\ \mathbf{e}_1 + bP_0^{n+1}\ \mathbf{e}_1\ \lambda g^n(2hD_0\mathbf{P}^n),$$
$$n > 1 \tag{23}$$

where $\lambda = k/h$ and:

$$\mathbf{P} = (P_1, P_2, \ldots, P_M)^T,$$
$$\mathbf{e}_1 = (1, 0, \ldots, 0)^T \tag{24}$$

The solution to Equation 23 can be expressed as:

$$\mathbf{P}^{n+1} = \mathbf{f}_1 + bP_0^{n+1}\ \mathbf{f}_2 + \lambda g^n\ \mathbf{f}_3 \tag{25}$$

where:

$$\mathbf{f}_1 = (I + A)^{-1}[(I - A)\mathbf{P}^{n-1} + bP_0^{n-1}\,\mathbf{e}_1],$$

$$\mathbf{f}_2 = (I + A)^{-1}\,\mathbf{e}_1 \tag{26}$$

$$\mathbf{f}_3 = (I + A)^{-1}(2hD_0\mathbf{P}^n)$$

Substituting P_1^{n+1} into Equation 21 and using Equation 17, we can solve for B^{n+1}:

$$B^{n+1} = \frac{\alpha - f_{1,1} - bf_{1,2} + \dfrac{\lambda f_{1,3} B^{n-1}}{2kB^n}}{\beta - bf_{1,2} + \dfrac{\lambda f_{1,3}}{2kB^n}} \tag{27}$$

The solutions for g^n and P_0^{n+1} then follow. The pseudo–code for the above scheme is:

[L, U] = LU–decompose I + A
$\mathbf{f}_2 = U^{-1} L^{-1} \mathbf{e}_1$
for n = 1, 2, . . . , N – 1 do
$\quad\mathbf{f}_1 \leftarrow U^{-1} L^{-1} (I - A)\mathbf{P}^{n-1}$
$\quad\mathbf{f}_3 \leftarrow U^{-1} L^{-1} (2hD_0\mathbf{P}^n)$
$\quad$Solve for B^{n+1}, g^n and P_0^{n+1}
$\quad\mathbf{P}^{n+1} \leftarrow \mathbf{f}_1 + bP_0^{n+1}\mathbf{f}_2 + \lambda g^n \mathbf{f}_3$

end

It takes 11M multiplications/divisions and 9M additions/subtractions to compute each $\mathbf{P}^n$.

Since Equation 16 is a three–level scheme, we need $\mathbf{P}^1$ in addition to $\mathbf{P}^0$ to initialize the computation. To maintain overall second order accuracy, we employ the following two–step predictor–corrector technique to obtain $\mathbf{P}^1$:

$$\left(I + \frac{A}{2}\right)\tilde{\mathbf{P}} = \left(I - \frac{A}{2}\right)\mathbf{P}^0 + \frac{b}{2}\tilde{P}_0 e_1 + k\tilde{g}D_0\mathbf{P}^0,$$

$$\left(I + \frac{A}{2}\right)\mathbf{P}^1 = \left(I - \frac{A}{2}\right)\mathbf{P}^0 + \frac{b}{2}P_0^1\mathbf{e}_1 + kg^1 D_0\left(\frac{\tilde{\mathbf{P}} + \mathbf{P}^0}{2}\right) \tag{28}$$

where:

$$\tilde{g} = \frac{\tilde{B} - B^0}{kB^0}, \quad g^1 = \frac{B^1 - B^0}{k\dfrac{\tilde{B} + B^0}{2}} \tag{29}$$

The code for $\mathbf{P}^n$ can be used to realize this predictor–corrector procedure with slight modification.

The specification of grid size (k, h) and the integer M is an important issue to be addressed. Following the convention of numerical schemes, we let k be one of the input parameters defined according to the number of time steps N, i.e, $k = T/N$. Regarding h, it is well known that the convergence of the finite difference solution for parabolic equation requires $k/h \to 0$ as $k \to 0$. From the viewpoint of approximating general jump process, it is desirable to have non-negative $1 - a$, b, and c, as they then can be interpreted as probabilities (multiplied by $1 - kr$, the time discount factor). The nonnegativity requirement leads to $h \geq \sigma\sqrt{k}$.

From experience, we recommend $h = 1.5\sigma\sqrt{k}$. This selection implies that our finite difference method is first order accurate in k. When penny accuracy is demanded, M should be chosen according to $P(Mh, T) < (100X)^{-1}$. Clearly, M is a function of all input parameters. At this point we cannot propose a general formula of M that guarantees penny accuracy in all situations. We have instead chosen M in a rather simple way. For $0 \leq T \leq O(1)$, we observe that the magnitude of the solution at high asset price ($y \gg 1$) depends on $\sigma\sqrt{T}$. We thus consider $Mh = c\sigma\sqrt{T}$, or $M = [c\sigma\sqrt{T}/h]$. Here c is a constant insensitive to the input parameters. When $T \leq 3$, we have chosen $c = 8$ uniformly. This selection is supported by our numerical results. For larger values of σ, T, or X, we may need larger c.

Given M chosen earlier, we can calculate the number of arithmetic operations needed for the entire iteration. The total numbers of multiplications/divisions and additions/subtractions are:

$$\text{No. of } \times/\div = 11MN = \left[\frac{22c}{3} N^{\frac{3}{2}}\right] \tag{30}$$

and

$$\text{No. of } +/- = 9MN = \left[6cN^{\frac{3}{2}}\right] \tag{31}$$

Note that the exponents over N are $\frac{3}{2}$. When the number of time steps doubles, the CPU time for the front–fixing method will increase by the factor of $(\sqrt{2})^3 = 2.8$. Meanwhile, the binomial method takes $N(N+1)$ multiplications and the same number of additions. When the number of time steps doubles, the CPU time for the binomial method will increase by the factor of 4. If the CPU time for one multiplication (division) significantly dominates the CPU time for one addition (subtraction), then the front–fixing method will take less CPU time than binomial method for *each run* when the number of time steps $N \geq \left(\frac{22c}{3}\right)^2$. Hence, if there are p option positions with the same maturity to be evaluated, we should consider the front–fixing method when the number of time steps $N \geq \left(\frac{22c}{3p}\right)^2$. If we take $p = c = 8$, for example, $N \geq 54$.

The asset price at which option value is desired in general does not fall on one of the grid points in the finite difference mesh. The finite difference method on the transformed equation produces option values at:

$$S_j = XB(T)e^{jh}, \quad j = 0, 1, \ldots, M \tag{32}$$

For option values at any designated asset prices other than these S_j, we adopt the cubic spline interpolation (Press et al. 1992). One can prove that interpolated option values over the interval $[B(T), B(T)e^{c\sigma\sqrt{T}}]$ will have the same accuracy as that of p_j^N. The delta values can be obtained by interpolating the centered difference of the option values at the grid points.

IV. NUMERICAL RESULTS

In this section we illustrate the performance of the front–fixing method with two test cases. The test cases cover medium–term and long–term options. For the same number of time steps, front–fixing method is tested against the standard binomial method. Throughout these test cases we take $h = \frac{3}{2}\sigma\sqrt{k}$ and $M = [8\sigma\sqrt{T}/h]$ for the front–fixing method. By varying the number of time steps, we tabulate the option values, deltas, root–mean–square–errors (RMSE), and CPU times of both the standard binomial method and the front–fixing method. In Examples 1 and 2, we generate the "exact" solutions needed for the computation of RMSE by the binomial method with 1,000 time steps. We would like to

Table 1A

Comparison of Speed and Accuracy for the Valuation of the American Put in Example 1

$r = 0.1$, $\sigma = 0.3$, $T = 1$, $X = 100$, $k = 0.01$

	Option Values			Delta		
Stock Price	**Binomial n = 1000**	**Binomial n = 100**	**F-F-F n = 100**	**Binomial n = 1000**	**Binomial n = 100**	**F-F-F n = 100**
77	23.0131	23.0000	23.0128	-0.9686	-0.9619	-0.9718
78	22.0615	22.0567	22.0621	-0.9318	-0.9342	-0.9353
79	21.1483	21.1442	21.1469	-0.8971	-0.8987	-0.9001
80	20.2687	20.2576	20.2662	-0.8632	-0.8634	-0.8661
RMSE		0.0092	0.0015		0.0036	0.0032
CPU(sec)		1.89	5.9500			
80	20.2689	20.2576	20.2662	-0.8631	-0.8634	-0.8661
85	16.3467	16.3412	16.3396	-0.7109	-0.7107	-0.7133
90	13.1228	13.1208	13.1124	-0.5829	-0.5832	-0.5848
95	10.4847	10.4798	10.4733	-0.4755	-0.4761	-0.4769
100	8.3348	8.3255	8.3277	-0.3856	-0.3860	-0.3866
105	6.6071	6.6108	6.5936	-0.3108	-0.3110	-0.3116
110	5.2091	5.2250	5.2004	-0.2491	-0.2493	-0.2497
115	4.0976	4.1034	4.0872	-0.1986	-0.1988	-0.1990
120	3.2059	3.1964	3.2023	-0.1575	-0.1574	-0.1578
RMSE		0.0086	0.0090		0.0003	0.0016
CPU(sec)		1.89	6.08			

emphasize here that the CPU times given in these examples are the CPU times for each run of either method.

Example 1. The first test case is the prototypical example used by Carr and Faguet (1994) with the following characteristics:

- Strike price X = \$100
- Risk–free interest rate r = 0.1
- Volatility $\sigma = 0.3$
- Time to expiration T = 1 (year)

Table 1A lists the option values and deltas obtained by the binomial and front–fixing methods for two sets of asset prices, where "F–F–F" stands for the front–fixing finite difference method. The asset prices in the first set are near the optimal exercise boundary B(T) = 76.25, and the asset prices in the second set lie within 20 percent range of the strike price. The RMSE indicates that the two methods have similar level of accuracy, and both are well within the truncation error O(k). When the asset price is near the optimal exercise boundary, the front–fixing method is slightly more accurate.

In Table 1B, we display the changes of RMSE and CPU time with respect to N. We define:

$$\text{Factor of RMSE decrease} = \frac{\text{RMSE}(N)}{\text{RMSE}(N/2)} \tag{33}$$

and

$$\text{Factor of CPU time increase} = \frac{\text{CPU}(N)}{\text{CPU}(N/2)} \tag{34}$$

Table 1B

RMSE and CPU Time versus Number of Time Steps

r = 0.1, σ = 0.3, T = 1, X = 100

	Binomial		F-F-F		Binomial		F-F-F	
Time Step N	RMSE	Factor of Decrease	RMSE	Factor of Decrease	CPU Time	Factor of Increase	CPU Time	Factor of Increase
16	5.1E-02		7.4E-02		6.7E-02		7.2E-01	
32	1.0E-02	0.20	3.4E-02	0.46	2.3E-01	3.50	1.2E+00	1.67
64	1.2E-02	1.22	1.5E-02	0.44	8.3E-01	3.57	2.7E+00	2.26
128	3.3E-03	0.27	6.9E-03	0.46	3.2E+00	3.90	6.8E+00	2.48
256	5.8E-03	1.79	3.8E-03	0.55	1.3E+01	3.91	1.8E+01	2.70
512	1.3E-03	0.22	2.8E-03	0.73	5.1E+01	4.03	5.0E+01	2.74

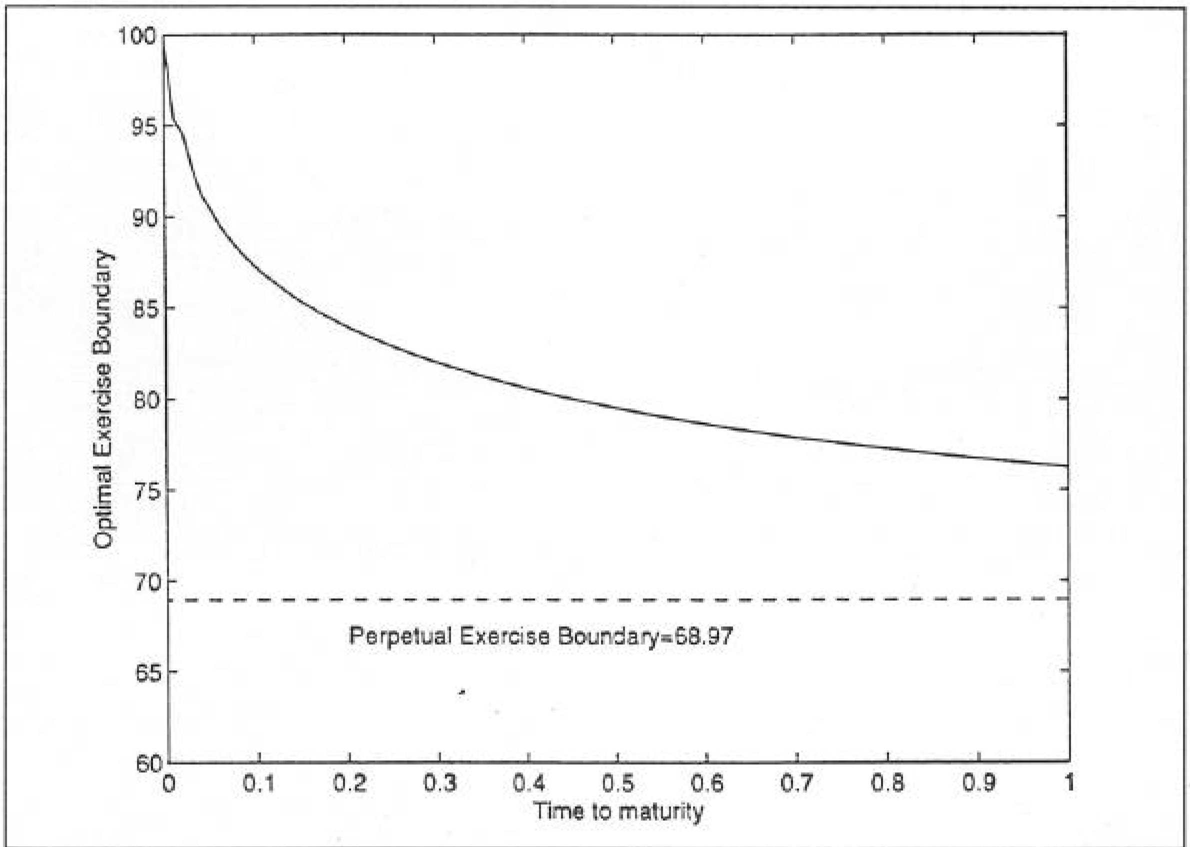
where RMSE(N) and CPU(N) denote the RMSE and CPU time of either method with N time steps. These two factors measure the order of the accuracy and the rate of increase of CPU times. It can be seen that when the number of time steps doubles, the RMSE of the front–fixing method decreases by a factor of around

and the CPU time increases by factors approaching $\sqrt{8}$. The factor of decrease for RSME confirms the first order temporal accuracy of the front–fixing method. Note that for N = 512, the run time of the front–fixing method becomes less than that of the binomial method. Figure 1 shows the early exercise boundary obtained by the front–fixing method for $0 \le \tau \le T$.

Example 2. The second example is a long–term option with the following characteristics:

- Strike price X = \$100
- Risk–free interest rate r = 0.06
- Volatility $\sigma = 0.4$
- Time to expiration T = 3 (years)

Figure 1
Optimal Exercise Boundary
r = 0.1, sigma = 0.3, T = 1, X = 100, N = 100



As shown in Table 2, the accuracy of option values by the front–fixing method is slightly better than the binomial method.

V. CONCLUSION

We have proposed and tested a new finite difference method for the numerical valuation of American options. The novelty of the proposed method is the front–fixing transformation. The new method has several advantages. First, it can evaluate option positions with the same maturity for essentially all possible asset prices simultaneously. It becomes increasingly more economical when the number of option positions increases. Second, it solves the optimal exercise boundary together with option prices without extra effort. Third, the accuracy of the method is comparable to that of the binomial method. Fourth and perhaps the most practical advantage is that the method is believed to be adaptive to other options as long as a front–fixing transformation exists. The types of options would include barrier and Asian options. However, the front–fixing transformation will not work for American options on multiple assets.

Table 2

Option Values and Deltas

$r = 0.1$, $\sigma = 0.3$, $T = 1$, $X = 100$, $k = 0.01$

	Option Values			Delta		
Stock Price	**Binomial n = 1000**	**Binomial n = 100**	**F-F-F n = 100**	**Binomial n = 1000**	**Binomial n = 100**	**F-F-F n = 100**
80	28.0708	28.0886	28.0607	-0.5040	-0.5039	-0.5060
85	25.6850	25.6895	25.6698	-0.4524	-0.4526	-0.4542
90	23.5395	23.5411	23.5203	-0.4073	-0.4073	-0.4088
95	21.6027	21.6320	21.5832	-0.3675	-0.3672	-0.3688
100	19.8513	19.8251	19.8335	-0.3323	-0.3328	-0.3334
105	18.2716	18.2967	18.2500	-0.3011	-0.3010	-0.3020
110	16.8385	16.8700	16.8141	-0.2732	-0.2731	-0.2741
115	15.5359	15.5039	15.5099	-0.2484	-0.2486	-0.2492
120	14.3500	14.3714	14.3233	-0.2262	-0.2261	-0.2269
RMSE		0.0235	0.0207		0.0002	0.0013
CPU(sec)		1.89	6.08			

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