Integral price formulas for lookback options

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We derive general integral representation of the price formulas for European options whose terminal payoff involves path dependent lookback variables. The intricacies in the derivaiton procedures using partial differential equation techniques stem from the degenerate nature of the pricing models, where the lookback state variables appear only in the auxiliary conditions but not in the governing differential equations. We also derive parity relation between the price functions of the floating strike and fixed strike lookback options.

Keywords: lookback options, integral price formulas, put-call parity

1. Background and model formulation

The mathematical formulation for the price function of an option whose terminal payoff involves path dependent lookback variables has been quite well explored in the literature. Let S denote the stock price variable and M denote the maximum price variable. Here, M represents the realized maximum of the stock price recorded from the initial time of the lookback period to the current time. Let t denote the calendar time variable, T be the maturity date of the lookback option and $\tau = T - t$ be the time to expiry. Under the Black-Scholes framework, the formulation for the price function $V(S, M, \tau)$ of the one-asset European lookback option model with terminal payoff $V_T(S, M)$ is given by (Goldman *et al.*, 1979)

$$
\begin{aligned}
&\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV, \quad 0 < S < M, \quad \tau > 0,\\
&\left.\frac{\partial V}{\partial M}\right|_{S=M} = 0, \quad \tau > 0,\\
&V(S, M, 0) = V_T(S, M),
\end{aligned} \tag{1.1}
$$

where r is the riskless interest rate and σ is the volatility of the stock price. For the sake of simplicity, we assume the stock to be zero dividend paying. The price function is essentially two-dimensional with state variables S and M. However, the differential equation exhibits the degenerate nature in the sense that it does not involve the lookback variable M. Rather, M only occurs in the Neumann boundary condition $\left.\frac{\partial V}{\partial M}\right|_{S=M} = 0$ and the terminal payoff function. The Neumann boundary condition signifies that if the current stock price equals the value of the current realized maximum then the option price is insensitive to M.

Lookback option prices are commonly evaluated using the probability approach. The solution procedure requires the determination of the density function of the joint processes of the stock price and its realized maximum (Conze and Viswanathan, 1991; He *et al.*, 1998).

In this paper, we demonstrate the use of the partial differential equation technique to derive general integral price formulas for lookback option models. First, we reformulate the pricing model (1.1) using the following new set of variables:

$$x = \ln \frac{M}{S}, \quad y = \ln M. \tag{1.2}$$

With the new set of variables, the lookback pricing model formulation can be rewritten as

$$\begin{aligned}
&\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} - \left(r - \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial x}, \quad x > 0, -\infty < y < \infty, \tau > 0,\\
&\left.\left(\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}\right)\right|_{x=0} = 0, \quad \tau > 0,\\
&V(x, y, 0) = V_T(e^{y-x}, e^y).
\end{aligned} \tag{1.3}$$

The triangular wedge shape of the original domain of definition $\mathcal{D} = \{(S, M) : 0 < S < M\}$ is now transformed into a new domain which is the semi-infinite two-dimensional plane $\widetilde{\mathcal{D}} = \{(x, y) : x > 0$ and $-\infty < y < \infty\}$. However, the boundary condition along $x = 0$ involves the function $\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}$.

In the next section, we first derive the integral price formulas for one-asset European options with general lookback payoff functions. We then generalize the derivation of the integral price formulas to two-asset lookback options. In Section 3, we deduce parity relation between the price functions of the floating strike and fixed strike lookback options. The paper ends with conclusive remarks in the last section.

2. Integral price formulas for European lookback options

In this section, we derive the integral price formula for the pricing model (1.3). The difficulties in the derivation procedure arise from the boundary condition along $x = 0$, which involves $\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}$.

We define the function

$$W(x, y, \tau) = \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}, \tag{2.1}$$

and in terms of $W(x, y, \tau)$, Eq. (1.3) can be rewritten as

$$\begin{aligned}
&\frac{\partial W}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 W}{\partial x^2} - \left(r + \frac{\sigma^2}{2}\right)\frac{\partial W}{\partial x}, \quad x > 0, -\infty < y < \infty, \tau > 0,\\
&W(0, y, \tau) = 0, \quad \tau > 0,\\
&W(x, y, 0) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) V_T(e^{y-x}, e^y).
\end{aligned} \tag{2.2}$$

The variable y appears only as a parameter in the above formulation. Hence, the solution of $W(x, y, \tau)$ is seen to be

$$W(x, y, \tau) = \int_0^\infty \overline{G}(\xi, \tau; x) W(\xi, y, 0)\, d\xi, \tag{2.3}$$

where the Green function $\overline{G}(\xi, \tau; x)$ corresponding to the semi-infinite domain $\widetilde{\mathcal{D}}$ is given by

$$\overline{G}(\xi, \tau; x) = [\psi(x - \xi, \tau) - \psi(x + \xi, \tau)] e^{\alpha(x-\xi)+\beta\tau} \tag{2.4a}$$

with

$$\alpha = \frac{r}{\sigma^2} - \frac{1}{2}, \quad \beta = -\frac{1}{2\sigma^2}\left(r - \frac{\sigma^2}{2}\right)^2, \psi(x,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{x^2}{2\sigma^2\tau}\right). \tag{2.4b}$$

Once $W(x,y,\tau)$ is known, we then solve for $V(x,y,\tau)$ using Eq. (2.1). First, we may rewrite Eq. (2.1) as

$$W(\xi, \xi + y - x, \tau) = \frac{d}{d\xi} V(\xi, \xi + y - x, \tau), \quad \text{for all } \xi > 0. \tag{2.5}$$

Upon integrating with respect to ξ from 0 to x, we obtain

$$V(x,y,\tau) = V(0, y - x, \tau) + \int_0^x W(\xi, \xi + y - x, \tau)\, d\xi. \tag{2.6}$$

The remaining step amounts to the determination of $V(0, y-x, \tau)$. Suppose we write $\phi(z,\tau) = V(0, -z, \tau)$, where $z = x - y$, it can be shown that ϕ satisfies

$$\begin{aligned}
&\frac{\partial \phi}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 \phi}{\partial z^2} - \left(r - \frac{\sigma^2}{2}\right)\frac{\partial \phi}{\partial z} + \frac{\sigma^2}{2}\frac{\partial W}{\partial x}(0, -z, \tau), \quad -\infty < z < \infty, \tau > 0, \\
&\phi(z, 0) = U(0, -z, 0) = V_T(e^{-z}, e^{-z}).
\end{aligned} \tag{2.7}$$

If we use $G(\eta, \tau; z)$ to denote the infinite domain Green function of the above problem

$$G(\eta, \tau; z) = e^{\alpha(z-\eta)+\beta\tau}\psi(z - \eta, \tau), \tag{2.8}$$

then the solution to $\phi(z,\tau)$ can be formally represented by

$$\begin{aligned}
\phi(z,\tau) = &\int_0^\tau \int_{-\infty}^\infty G(-\eta, \tau - u; z)\frac{\sigma^2}{2}\frac{\partial W}{\partial x}(0, \eta, u)\, d\eta du \\
&+ \int_{-\infty}^\infty G(-\eta, \tau; z) V_T(e^\eta, e^\eta)\, d\eta.
\end{aligned} \tag{2.9}$$

The integrand in the double integral still involves $\frac{\partial W}{\partial x}$. It would be more desirable to transform the double integral into the form that involves V_T only. By performing some analytic calculations (see Appendix A for the details), we obtain

$$\begin{aligned}
V(S, M, \tau) = &\int_0^\infty \overline{G}\left(\xi, \tau; \ln\frac{M}{S}\right) V_T(Me^{-\xi}, M)\, d\xi \\
&+ \int_0^\infty \int_{\ln M}^\infty \left[\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\overline{G}\left(\xi, \tau; \eta + \ln\frac{1}{S}\right)\right] V_T(e^{\eta-\xi}, e^\eta)\, d\eta d\xi.
\end{aligned} \tag{2.10}$$

It is relatively straightforward to show that the above solution satisfies the differential equation and auxiliary conditions as stated in Eq. (1.3).

Extension to two-asset lookback option models

The above derivation procedure can be extended in a straightforward manner to two-asset lookback option model with terminal payoff of the form $V_T(S_1, M_1, S_2)$, where S_1 and S_2 are the price of stock 1 and stock 2, respectively, and M_1 is the realized maximum of the price of stock 1. Let $\sigma_i, i = 1, 2$, denote the volatility of stock i, and ρ be the correlation coefficient between the joint lognormal price processes of stock 1 and stock 2. Let $V_2(S_1, M_1, S_2, \tau)$ denote the price function of the two-asset European lookback option model with terminal payoff $V_T(S_1, M_1, S_2)$. The formulation of $V_2(S_1, M_1, S_2, \tau)$ is given by

$$\begin{aligned}
\frac{\partial V_2}{\partial \tau} &= \frac{\sigma_1^2}{2} S_1^2 \frac{\partial^2 V_2}{\partial S_1^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V_2}{\partial S_1 \partial S_2} + \frac{\sigma_2^2}{2} S_2^2 \frac{\partial^2 V_2}{\partial S_2^2} \\
&\quad + rS_1 \frac{\partial V_2}{\partial S_1} + rS_2 \frac{\partial V_2}{\partial S_2} - rV_2, \quad 0 < S_1 < M_1, 0 < S_2 < \infty, \tau > 0, \\
\left.\frac{\partial V_2}{\partial M_1}\right|_{S_1 = M_1} &= 0, \quad \tau > 0, \\
V_2(S_1, M_1, S_2, 0) &= V_T(S_1, M_1, S_2).
\end{aligned} \tag{2.11}$$

Like the one-asset counterpart, we apply the following transformations of variables:

$$x_1 = \ln \frac{M_1}{S_1}, \quad y_1 = \ln M_1, \quad x_2 = \ln \frac{1}{S_2}. \tag{2.12}$$

The pricing formulation of the two-asset lookback option model can be expressed as

$$\begin{aligned}
\frac{\partial V_2}{\partial \tau} &= \frac{\sigma_1^2}{2} \frac{\partial^2 V_2}{\partial x_1^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 V_2}{\partial x_1 \partial x_2} + \frac{\sigma_2^2}{2} \frac{\partial^2 V_2}{\partial x_2^2} \\
&\quad - \left(r - \frac{\sigma_1^2}{2}\right) \frac{\partial V_2}{\partial x_1} - \left(r - \frac{\sigma_2^2}{2}\right) \frac{\partial V_2}{\partial x_2} - rV_2, \\
&\quad x_1 > 0, -\infty < y_1, x_2 < \infty, \tau > 0, \\
\left.\frac{\partial V_2}{\partial x_1} + \frac{\partial V_2}{\partial y_1}\right|_{x_1 = 0} &= 0, \quad \tau > 0, \\
V_2(x_1, y_1, x_2, 0) &= V_T(e^{y_1 - x_1}, e^{y_1}, e^{-x_2}).
\end{aligned} \tag{2.13}$$

We define

$$W_2(x_1, y_1, x_2, \tau) = \frac{\partial V_2}{\partial x_1} + \frac{\partial V_2}{\partial y_1} \tag{2.14}$$

and in terms of $W_2(x_1, y_1, x_2, \tau)$, Eq. (2.13) can be rewritten as

$$\begin{aligned}
\frac{\partial W_2}{\partial \tau} &= \frac{\sigma_1^2}{2} \frac{\partial^2 W_2}{\partial x_1^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 W_2}{\partial x_1 \partial x_2} + \frac{\sigma_2^2}{2} \frac{\partial^2 W_2}{\partial x_2^2} \\
&\quad - \left(r - \frac{\sigma_1^2}{2}\right) \frac{\partial W_2}{\partial x_1} - \left(r - \frac{\sigma_2^2}{2}\right) \frac{\partial W_2}{\partial x_2} - rW_2, \\
&\quad x_1 > 0, -\infty < y_1, x_2 < \infty, \tau > 0, \\
W_2(0, y_1, x_2, \tau) &= 0, \quad \tau > 0, \\
W_2(x_1, y_1, x_2, 0) &= \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}\right) V_T(e^{y_1 - x_1}, e^{y_1}, e^{-x_2}).
\end{aligned} \tag{2.15}$$

Let $\overline{G}_2(\xi_1, \xi_2, \tau; x_1, x_2)$ denote the Green function (semi-infinite domain) associated with the differential equation formulation in Eq. (2.15), then $W_2(x_1, y_1, x_2, \tau)$ can be formally represented by

$$W_2(x_1, y_1, x_2, \tau) = \int_{-\infty}^{\infty} \int_0^{\infty} \overline{G}_2(\xi_1, \xi_2, \tau; x_1, x_2) W_2(\xi_1, y_1, \xi_2, 0) \, d\xi_1 d\xi_2. \tag{2.16}$$

Once $W_2(x_1, y_1, x_2, \tau)$ has been obtained, we then solve Eq. (2.14) to obtain

$$V_2(x_1, y_1, x_2, \tau) = V_2(0, y_1 - x_1, x_2, \tau) + \int_0^{x_1} W_2(\xi_1, \xi_1 + y_1 - x_1, x_2, \tau)\, d\xi_1. \tag{2.17}$$

Let $\phi_2(z_1, x_2, \tau) = V_2(0, y_1 - x_1, x_2, \tau)$, where $z_1 = x_1 - y_1$, then ϕ_2 satisfies

$$\begin{aligned}
\frac{\partial \phi_2}{\partial \tau} =& \frac{\sigma_1^2}{2}\frac{\partial^2 \phi_2}{\partial z_1^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 \phi_2}{\partial z_1 \partial x_2} + \frac{\sigma_2^2}{2}\frac{\partial^2 \phi_2}{\partial x_2^2} \\
& - \left(r - \frac{\sigma_1^2}{2}\right)\frac{\partial \phi_2}{\partial z_1} - \left(r - \frac{\sigma_2^2}{2}\right)\frac{\partial \phi_2}{\partial x_2} - r\phi_2 + \frac{\sigma_1^2}{2}\frac{\partial W_2}{\partial z_1}(0, -z_1, x_2, \tau), \\
& \quad - \infty < z_1, x_2 < \infty, \tau > 0, \\
\phi_2(z_1, x_2, 0) =& \ V_T(e^{-z_1}, e^{-z_1}, e^{-x_2}).
\end{aligned} \tag{2.18}$$

Let $\overline{G}_2(\xi_1, \xi_1, \tau; z_1, x_2)$ denote the Green function (semi-infinite domain) associated with the differential equation formulation in Eq. (2.15). We can deduce the following integral formula for $V_2(S_1, M_1, S_2, \tau)$:

$$\begin{aligned}
V_2(S_1, M_1, S_2, \tau) =& \int_{-\infty}^{\infty}\int_0^{\infty} \overline{G}_2\left(\xi_1, \xi_2, \tau; \ln\frac{M_1}{S_1}, \ln\frac{1}{S_2}\right) V_T(M_1 e^{-\xi_1}, M_1, e^{-\xi_2})\, d\xi_1 d\xi_2 \\
&+ \int_{-\infty}^{\infty}\int_0^{\infty}\int_{\ln M_1}^{\infty} \left[\left(\frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \eta_1}\right)\overline{G}_2\left(\xi_1, \xi_2, \tau; \eta_1 + \ln\frac{1}{S_1}, \ln\frac{1}{S_2}\right)\right] \\
&\qquad V_T(e^{\eta_1 - \xi_1}, e^{\eta_1}, e^{-\zeta_2})\, d\eta_1 d\xi_1 d\xi_2.
\end{aligned} \tag{2.19}$$

3. Floating strike and fixed strike lookback options

The integral price formula (2.10) gives the value of a lookback option with general terminal payoff function $V_T(S, M)$. The two most common lookback options have payoff of the form: (i) floating strike payoff, $M - S$; (ii) fixed strike payoff, $\max(M - K, 0)$, where K is the fixed strike price. In this section, we first consider the valuation of lookback options with payoff of the form $Sf\left(\frac{M}{S}\right)$, which includes the floating strike payoff as a special example. We illustrate how to achieve dimension reduction of the pricing model under this special form of terminal payoff. After then, we deduce parity relation between the option values of floating strike and fixed strike lookback options.

By taking $V_T(S, M) = Sf\left(\frac{M}{S}\right)$ and applying the transformations of variables: $x = \ln\frac{M}{S}$ and $U(x, \tau) = \frac{V(S, M, \tau)}{S}$ to the pricing formulation (1.1), we obtain

$$\begin{aligned}
&\frac{\partial U}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 U}{\partial x^2} - \left(r + \frac{\sigma^2}{2}\right)\frac{\partial U}{\partial x}, \quad x > 0, \tau > 0, \\
&\left.\frac{\partial U}{\partial x}\right|_{x=0} = 0, \quad \tau > 0 \\
&U(x, 0) = f(e^x).
\end{aligned} \tag{3.1}$$

The new formulation involves only one space variable, so dimension reduction has been achieved. To resolve the difficulty of dealing with the Neumann boundary condition along $x = 0$, we extend the domain of definition from the semi-infinite domain to the full infinite domain. This is achieved by performing continuation of the initial condition to the domain $x < 0$ such that the price function can satisfy the Neumann boundary condition. Due to the presence of the drift term in the differential equation, the simple odd-even extension is not applicable. In Appendix C, we present the details of the construction of the continuation function. For example, for the floating strike payoff $M - S$, we have $U(x, 0) = e^x - 1, x > 0$. The continuation of the initial condition to the domain $x < 0$ is found to be (see Appendix B)

$$U(x, 0) = \frac{1 - e^{(2\widetilde{\alpha}-1)x}}{2\widetilde{\alpha} - 1}, \quad x < 0, \quad \text{where} \quad \widetilde{\alpha} = \frac{r}{\sigma^2} + \frac{1}{2}. \tag{3.2}$$

We obtain the integral price formula of lookback option with payoff $Sf\left(\frac{M}{S}\right)$ as follows (see Appendix B):

$$\begin{aligned} V(S, M, \tau) = S\left(\frac{M}{S}\right)^{\widetilde{\alpha}} e^{\widetilde{\beta}\tau} \int_1^\infty \Bigg[& \psi\left(\ln\frac{M}{S} + \ln\xi, \tau\right) + \psi\left(\ln\frac{M}{S} - \ln\xi, \tau\right) \\ & + 2\alpha \int_\xi^\infty \psi\left(\ln\frac{M}{S} + \ln\eta, \tau\right)\left(\frac{\eta}{\xi}\right)^{\widetilde{\alpha}-1} d\eta\Bigg] \frac{f(\xi)}{\xi^{\widetilde{\alpha}+2}}\, d\xi, \end{aligned} \tag{3.3}$$

where $\widetilde{\beta} = -\frac{1}{2\sigma^2}\left(r + \frac{\sigma^2}{2}\right)^2$ and $\psi(x, \tau)$ are defined in Eq. (2.4b). For the floating strike lookback option, we have $f(\xi) = \xi - 1$. The corresponding price function is found to be (assuming $r > 0$)

$$\begin{aligned} V_{f\ell}(S, M, \tau) = {} & Me^{-r\tau}\left[N(d_1) - \frac{\sigma^2}{2r}\left(\frac{M}{S}\right)^{2r/\sigma^2} N(-d_3)\right] \\ & - S\left[N(d_2) - \frac{\sigma^2}{2r} N(-d_2)\right], \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\frac{M}{S} - \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \\ d_2 &= d_1 - \sigma\sqrt{\tau}, \\ d_3 &= d_2 + \frac{2r}{\sigma}\sqrt{\tau}. \end{aligned} \tag{3.5}$$

Parity relation

The fixed strike lookback option has payoff of the form $(M - K)^+$, where $x^+ = \begin{cases} x & x > 0 \\ 0 & \text{otherwise} \end{cases}$. Unlike the floating strike counterpart, such payoff structure does not admit dimension reduction of the pricing model. Fortunately, there exists a parity relation between their price functions [see Wong and Kwok's paper (2003) for an alternative proof using the probability approach].

Let $V_{fix}(S, M, \tau)$ denote the price function of the fixed strike lookback option and write

$$\widetilde{V}(S, M, \tau) = V_{fix}(S, M, \tau) - V_{f\ell}(S, M, \tau) - S - Ke^{-r\tau}. \tag{3.6}$$

The governing equation for $\widetilde{V}$ is given by

$$\begin{aligned} & \frac{\partial\widetilde{V}}{\partial\tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2\widetilde{V}}{\partial S^2} + rS\frac{\partial\widetilde{V}}{\partial S} - r\widetilde{V}, \quad S > 0, \tau > 0, \\ & \left.\frac{\partial\widetilde{V}}{\partial M}\right|_{S=M} = 0, \quad \tau > 0 \\ & \widetilde{V}(S, M, 0) = (M - K)^+ - (M - S) - (S - K) \\ & \qquad\qquad\quad = \begin{cases} 0 & \text{if } M \geq K \\ K - M & \text{if } M < K \end{cases}. \end{aligned} \tag{3.7}$$

We claim that the solution to $\widetilde{V}(S,M,\tau)$ is given by

$$\widetilde{V}(S,M,\tau) = \begin{cases} 0 & \text{if } M \geq K \\ V_{f\ell}(S,K,\tau) - V_{f\ell}(S,M,\tau) & \text{if } M < K \end{cases}. \tag{3.8}$$

The solution observes continuity property at $M = K$; and the initial condition is satisfied since $V_{f\ell}(S,K,0) - V_{f\ell}(S,M,0) = (K-S) - (M-S) = K - M$. Also, $V_{f\ell}(S,K,\tau) - V_{f\ell}(S,M,\tau)$ satisfies the governing equation together with the Neumann condition [note that $V_{f\ell}(S,K,\tau)$ has no dependence on M]. Hence, by uniqueness of solution to problem (3.7), we obtain the following parity relation between V_{fix} and $V_{f\ell}$:

$$\begin{aligned} V_{fix}(S,M,\tau) &= \begin{cases} V_{f\ell}(S,M,\tau) + S - Ke^{-r\tau} & \text{if } M \geq K \\ V_{f\ell}(S,K,\tau) + S - Ke^{-r\tau} & \text{if } M < K \end{cases} \\ &= V_{f\ell}(S, \max(M,K), \tau) + S - Ke^{-r\tau}. \end{aligned} \tag{3.9}$$

4. Conclusion

The lookback option pricing models exhibit the interesting properties that the lookback variable does not appear explicitly in the governing equation, but only in the auxiliary conditions. The main contribution of this paper is the construction of analytic solution to pricing models with such degenerate feature. We demonstrate the use of partial differential equation techniques to obtain integral price formulas for European lookback option models. We also deduce parity relation between the price functions of floating strike and fixed strike lookback options.

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Appendix A – proof of Eq. (2.10)

Observe that $W(x,\eta,u)$ is governed by Eq. (2.2). We multiply each terms in the equation by $G(-\eta, \tau - u; z - x)$, then integrate from $x = 0$ to $x = \infty$ and $u = 0$ to $u = \tau - \epsilon$ (ϵ is a small positive constant) to obtain

$$\begin{aligned} 0 = &\int_0^{\tau-\epsilon} \int_0^\infty G(-\eta, \tau-u; z-x) \frac{\partial W}{\partial u}(x,\eta,u) \, dxdu \\ &- \frac{\sigma^2}{2} \int_0^{\tau-\epsilon} \int_0^\infty G(-\eta, \tau-u; z-x) \frac{\partial^2 W}{\partial x^2}(x,\eta,u) \, dxdu \\ &+ \left(r - \frac{\sigma^2}{2}\right) \int_0^{\tau-\epsilon} \int_0^\infty G(-\eta, \tau-u; z-x) \frac{\partial W}{\partial x}(x,\eta,u) \, dxdu. \end{aligned}$$

By performing parts integration and applying the homogeneous boundary condition $W(0,\eta,u)=0$, we obtain

$$\begin{aligned}0 = &\int_0^\infty [G(-\eta,\epsilon;z-x)W(x,\eta,\tau-\epsilon) - G(-\eta,\tau;z-x)]W(x,\eta,0)\ dx\\ &+\frac{\sigma^2}{2}\int_0^{\tau-\epsilon} G(-\eta,\tau-u;z-x)\frac{\partial W}{\partial x}(0,\eta,u)\ du\\ &+\int_0^{\tau-\epsilon}\int_0^\infty \left[\frac{\partial G}{\partial \tau}(-\eta,\tau-u;z-x) - \frac{\sigma^2}{2}\frac{\partial^2 G}{\partial z^2}(-\eta,\tau-u;z-x)\right.\\ &\qquad\left.+\left(r-\frac{\sigma^2}{2}\right)\frac{\partial G}{\partial z}(-\eta,\tau-u;z-x)\right]W(x,\eta,u)\ dxdu.\end{aligned}$$

Next, we take the limit $\epsilon\to 0^+$ and observe that

$$\begin{aligned}&\lim_{\epsilon\to 0^+} G(-\eta,\epsilon;z-x) = \delta(x-z-\eta)\\ &G(-\eta,\tau;z-x) = G(x,\tau;z+\eta),\end{aligned}$$

we then obtain

$$\begin{aligned}&\int_0^\tau G(-\eta,\tau-u;z)\frac{\sigma^2}{2}\frac{\partial W}{\partial x}(0,\eta,u)\ du\\ &= \int_0^\infty G(-\eta,\tau;z-\xi)W(\xi,\eta,0)\ d\xi - H(z+\eta)W(z+\eta,\eta,\tau)\\ &= \int_0^\infty \{[1-H(z+\eta)]G(\xi,\tau;z+\eta) + H(z+\eta)\widehat{G}(\xi,\tau;z+\eta)\}W(\xi,\eta,0)\ d\xi,\end{aligned}$$

where $H(x)$ is the Heaviside function. Next, we integrate each term in the above equation with respect to η over the interval $(-\infty,\infty)$ to give

$$\begin{aligned}&\int_0^\tau\int_{-\infty}^\infty G(-\eta,\tau-u;z)\frac{\sigma^2}{2}\frac{\partial W}{\partial x}(0,\eta,u)\ d\eta du\\ &= \int_0^\infty\int_{-\infty}^{-z} G(\xi,\tau;z+\eta)W(\xi,\eta,0)\ d\eta d\xi\\ &\quad+\int_0^\infty\int_{-z}^\infty \widehat{G}(\xi,\tau;z+\eta)W(\xi,\eta,0)\ d\eta d\xi,\end{aligned}$$

where

$$\widehat{G}(\xi,\tau;x) = e^{\alpha(x-\xi)+\beta\tau}\psi(\xi+x,\tau).$$

Substituting the above relations into Eq. (2.6), we have

$$\begin{aligned}V(x,y,\tau) = &\int_0^\infty\int_{-\infty}^{y} G(\xi,\tau;x+\eta-y)W(\xi,\eta,0)\ d\eta d\xi\\ &+\int_0^\infty\int_{y}^\infty \widehat{G}(\xi,\tau;x+\eta-y)W(\xi,\eta,0)\ d\eta d\xi\\ &+\int_{-\infty}^\infty G(0,\tau;x+\eta-y)W(0,\eta,0)\ d\eta.\end{aligned}$$

Lastly, by applying the following relations and performing parts integration

$$\begin{aligned}&W(\xi,\eta,0) = \left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)V_T(e^{\eta-\xi},e^\eta)\\ &\frac{\partial G}{\partial\xi}(\xi,\tau;x+\eta-y) = -\frac{\partial G}{\partial\eta}(\xi,\tau;x+\eta-y)\\ &\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)\widehat{G}(\xi,\tau;x+\eta-y) = \left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)\overline{G}(\xi,\tau;x+\eta-y),\end{aligned}$$

we obtain

$$
\begin{aligned}
V(x,y,\tau) = & \int_0^\infty \overline{G}(\xi,\tau;x) V_T(e^{y-\xi}, e^y)\ d\xi \\
& + \int_0^\infty \int_y^\infty \left[\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \overline{G}(\xi,\tau;x+\eta-y) \right] V_T(e^{\eta-\xi}, e^\eta)\ d\eta d\xi.
\end{aligned}
$$

Transforming back to the original variables S and M, we obtain the result in Eq. (2.10).

Appendix B – proof of Eq. (3.3)

Suppose we set $U(x,\tau) = \widetilde{U}(x,\tau) e^{\widetilde{\alpha}x + \widetilde{\beta}\tau}$, where $\widetilde{\alpha} = \frac{r}{\sigma^2} + \frac{1}{2}$ and $\widetilde{\beta} = -\frac{1}{2\sigma^2}\left(r + \frac{\sigma^2}{2}\right)^2$, then $\widetilde{U}(x,\tau)$ is governed by

$$
\frac{\partial \widetilde{U}}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 \widetilde{U}}{\partial x^2}, \quad x > 0, \tau > 0 \tag{B.1}
$$

$$
\left(\frac{\partial \widetilde{U}}{\partial x} + \widetilde{\alpha} \widetilde{U} \right) \Big|_{x=0} = 0, \quad \tau > 0, \tag{B.2a}
$$

$$
\widetilde{U}(x,0) = e^{-\widetilde{\alpha}x} f(e^x) = h_+(x), \quad x > 0. \tag{B.2b}
$$

Let $h_-(x)$ denote the continuation of the initial condition for $x < 0$, $\widetilde{U}(x,\tau)$ can then be formally represented by

$$
\widetilde{U}(x,\tau) = \int_{-\infty}^0 \psi(x-\xi,\tau) h_-(\xi)\ d\xi + \int_0^\infty \psi(x-\xi,\tau) h_+(\xi)\ d\xi, \tag{B.3}
$$

where $\psi(x,\tau)$ is defined in Eq. (2.4b). The function $h_-(x)$ is determined by enforcing the satisfaction of the Robin boundary condition (B.2a) by the solution $\widetilde{U}(x,\tau)$ in Eq. (B.3). We then obtain the following governing differential equation for $h_-(x)$:

$$
\begin{aligned}
& h'_-(x) + \widetilde{\alpha} h_-(x) + h'_+(-x) + \widetilde{\alpha} h_+(-x) = 0, \\
& h_+(0) = h_-(0).
\end{aligned} \tag{B.4}
$$

For example, suppose $f(e^x) = e^x - 1$, then $h_+(x) = e^{-\widetilde{\alpha}x}(e^x - 1)$. By solving Eq. (B.4), we obtain

$$
h_-(x) = \frac{e^{-\widetilde{\alpha}x} - e^{(\widetilde{\alpha}-1)x}}{2\widetilde{\alpha} - 1}.
$$

In general, the solution to Eq. (B.4) is found to be

$$
h_-(x) = h_+(-x) + 2\widetilde{\alpha} e^{-\widetilde{\alpha}x} \int_0^x e^{\widetilde{\alpha}\xi} h(-\xi)\ d\xi.
$$

Substituting the above expression for $h_-(x)$ into Eq. (B.3) and performing some simplification, we obtain Eq. (3.3).

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