Fourier transform algorithms for pricing and hedging discretely sampled exotic variance products and volatility derivatives under additive processes

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Keywords: variance products, volatility derivatives, Fourier transform algorithms, additive processes

Date: July 9, 2012
ABSTRACT

We develop efficient fast Fourier transform algorithms for pricing and hedging discretely sampled variance products and volatility derivatives under additive processes (time-inhomogeneous Lévy processes). Our numerical algorithms are non-trivial versions of the Fourier space time stepping method to nonlinear path dependent payoff structures, like those in variance products and volatility derivatives. The exotic path dependency associated with the discretely sampled realized variance is captured in the numerical procedure by updating two path dependent state variables across monitoring instants. The time stepping procedure between successive monitoring instants can be performed using fast Fourier transform calculations without the usual tedious time stepping calculations in typical finite difference algorithms. We also derive effective numerical procedures that compute the hedge parameters of variance products and volatility derivatives. Numerical tests on pricing various variance products and volatility derivatives were performed that illustrate efficiency, accuracy, reliability and robustness of the proposed Fourier transform algorithms.

1 Introduction

Volatility is an important risk measure in managing vega exposure in a portfolio of assets. Also, one may view volatility as the underlying state variable in the asset class of variance products and volatility derivatives. For example, investors can trade on the spread between the realized and implied volatility levels. Unlike equity options, volatility derivatives can provide pure exposure to volatility of the underlying equity. The volatility measure used to define the payoff structures in volatility derivatives may be either the implied volatility derived from option prices or the realized volatility observed from the logarithm return of realized prices of an underlying asset. The trading of volatility derivatives first appeared in 1993 in the form of variance swaps. In the past two decades, we have witnessed the proliferation of the types of variance products and volatility derivatives traded in the financial markets. A recent review of the market for these volatility derivatives can be found in Carr and Li (2009).

The nonparametric approach of developing various replicating strategies for continuously sampled variance swaps have been proposed in several pioneering works in 1990s. Under the assumption of continuous dynamics of the price process of the underlying stock and existence of the limit of the sum of squared returns, Neuberger (1994) shows that the hedging of a log contract provides a payoff that is related to the variance of the stock’s return. Dupire (1993) develops the first preference-free stochastic volatility model that can be used to price continuously sampled volatility derivatives that are more exotic than the vanilla variance swaps. Carr
and Madan (1999) demonstrate how to replicate the payoff of a continuously monitored variance swap by static position in a continuum of options plus dynamic position in the underlying asset. Later works on pricing variance options and volatility derivatives adopt the assumption of jumps in asset returns (Carr et al., 2005) or jumps in both returns and volatility (Sepp, 2008). In more recent works, Kallsen et al. (2011) consider pricing options on the quadratic variation of asset return under the affine stochastic volatility model. Drimus (2011) considers pricing and hedging options on continuously sampled realized variance under non-affine stochastic volatility models and the general class of Log-OU models.

The actual contractual specifications of variance products and volatility derivatives are based on discretely sampled realized variance of the underlying asset price process. Zhu and Lian (2011) obtain closed form pricing formulas for discretely sampled vanilla variance swaps under stochastic volatility. Zheng and Kwok (2011) derive pricing formulas for discretely sampled generalized variance swaps, like conditional variance swaps and corridor variance swaps, under the affine stochastic volatility model with simultaneous jumps. They also examine the convergence of the fair strikes of variance swaps under discretely sampled variance to their continuously sampled counterparts [see also a related study by Crosby and Davis (2011)]. Itkin and Carr (2010) obtain closed form pricing formulas of discretely monitored quadratic variation derivatives under a class of Lévy processes with stochastic time change. Broadie and Jain (2008) examine the effect of discrete sampling and asset price jump on the fair strikes of variance and volatility swaps. They show that the well known convexity correction formula may not provide a good approximation of the fair strikes of volatility swaps with jumps in the underlying asset price process. Jarrow et al. (2011) examine the sufficient conditions under which the fair strikes of swap products on discretely sampled realized variance converge to those of their continuous counterparts. Keller-Ressel and Muhle-Karbe (2011) propose two methods, one is analytic approximation while the other is exact, for pricing options on discretely sampled realized variance. Zheng and Kwok (2012) develop the saddlepoint approximation methods for pricing derivatives on discretely sampled realized variance under Lévy models and affine stochastic volatility models.

The analytic derivation of exact or approximation formulas for volatility derivatives under general type of stochastic processes requires high level of mathematical sophistication and the procedure is invariably quite tedious. Most often, analytic tractability is limited to payoff structures that are mostly linear on quadratic variation. For effective pricing and hedging of volatility derivatives, it would be highly desirable to derive versatile and reliable numerical pricing algorithms for computing prices and their hedge parameters for most types of payoff structures and underlying stochastic price processes.

In this paper, we propose various time stepping algorithms for pricing variance products and volatility derivatives in the Fourier domain. Our fast Fourier transform (FFT) algorithms
are distinctive from earlier pricing algorithms that perform numerical calculations in the real domain. Little and Pant (2001) propose a finite difference method for numerical valuation of discretely sampled variance swaps under the local volatility model. Windcliff et al. (2006) develop robust numerical schemes that solve the partial integral-differential option pricing equation under jump-diffusion asset price dynamics. The high level of path dependence in discretely sampled volatility derivatives is handled by tracking two stochastic state variables that capture the jump of the sampled variance across a monitoring date. Both of these numerical algorithms consider pricing volatility derivatives in the real domain. When we consider option pricing under Lévy processes, it is more effective to consider time stepping calculations in the Fourier domain. Lord et al. (2008) and Jackson et al. (2008) propose effective FFT algorithms for pricing derivatives with mild path dependence in the payoff structures, like barrier options and American options. A recent review of various FFT algorithms in option pricing can be found in Kwok et al. (2012). Our enhanced versions of the Fourier space time stepping algorithm can handle exotic path dependence associated with discretely sampled variance (through the incorporation of appropriate jump conditions on the path dependent state variables across monitoring dates). Moreover, we extend the class of the underlying price processes to exponential additive processes (time-inhomogeneous exponential Lévy processes) by relaxing the assumption of stationarity of increments in exponential Lévy processes. Also, the usual requirement of closed form expression for the characteristic function of the underlying asset price process may be relaxed. We simply require the availability of numerical values of the characteristic function at a set of discrete grid points. While most analytic approximation formulas in the literature do not include the evaluation of the hedge parameters of the price functions of variance products and volatility derivatives, common hedge parameters like delta and gamma can be computed efficiently using our Fourier transform algorithms.

The paper is organized as follows. In the next section, we present a brief review of additive processes and the model formulation of discretely sampled variance products and volatility derivatives under additive processes. In Section 3, we first review the FFT algorithms for option pricing using the CONV method (Lord et al., 2008). We then discuss the details of our enhanced FFT pricing algorithms that handle strong path dependence as shown in exotic variance products and volatility derivatives. We also show how to perform the calculations of hedge parameters efficiently. In Section 4, we demonstrate how to compute the fair price functions and their hedge parameters of various types of variance products and volatility derivatives under the piecewise double exponential model (a simple example of an additive process). We report the results from our numerical experiments that were performed to test for accuracy and convergence of the Fourier transform algorithms for pricing different types of variance products and volatility derivatives. Properties on the pricing functions and hedge parameters of some exotic variance products are also discussed. Conclusive remarks are presented in the last section.
2 Exponential additive processes and derivatives on discretely sampled realized variance

An exponential Lévy process can offer versatile asset return distribution for fitting the actual return distribution and volatility smiles. Together with the advantage of nice analytic tractability, Lévy processes have been widely adopted as the underlying asset price processes in pricing various types of exotic derivatives (Carr et al., 2005). However, Lévy processes are well known to have limitations in capturing the term structures of smiles, largely due to the stationarity of increments of Lévy processes. One remedy is to introduce stochastic volatility by time-changing a Lévy process (Carr and Wu, 2004), where the time-changed process is modeled by a non-decreasing process. Under the assumption that the time-changed process is a continuous process, one can introduce an activity rate process that is always positive. In this way, the time-changed process can be represented as an integral of the activity rate process. Though the time-changed Lévy models can fit the volatility smile surface better, an increase in dimensionality adds computational complexity and difficulties in calibration. An alternative approach is to remove stationarity in Lévy processes and take into account deterministic time inhomogeneities (Cont and Tankov, 2003). An advantage of this approach is that most of the nice analytic tractability of Lévy processes can be carried over to additive processes.

Additive processes

A short summary of the properties of additive processes is presented below.

**Definition.** Let $\mathfrak{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ be a complete stochastic basis. An additive process is an $\mathbb{R}$-valued, adapted, càdlàg process $\{X_t : t \geq 0\}$ such that:

1. For any $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent.

2. $X_0 = 0$ a.s.

3. $X_t$ is stochastically continuous.

**Remark:** When the distribution of $X_{t+s} - X_s$ is assumed to be independent of $s$, this corresponding category of additive processes are known as Lévy processes.

Additive processes belong to a subclass of semimartingales that can be fully characterized by its associated triplet $(B, C, \nu)$ of characteristics, where $B, C$ are predictable processes and $\nu$ is a predictable random measure on $\mathbb{R}_+ \times \mathbb{R}$. The first characteristic $B$ depends on a truncation function, say, $h(x) = x 1_{\{|x| \leq 1\}}$, which is chosen a priori. A semimartingale is an additive process if and only if the characteristics are non-random.
In practice, the characteristics are usually assumed to be absolutely continuous in time, where

\[ B_t = \int_0^t b_s \, ds, \quad C_t = \int_0^t c_s \, ds, \quad \nu([0,t] \times G) = \int_0^t F_s(G) \, ds, \quad \forall G \in \mathcal{B} \]

with predictable processes \( b, c \) and a transition kernel \( F \) from \((\Omega \times \mathbb{R}_+), \mathcal{P}) \) to \((\mathbb{R}, \mathcal{B})\). In this case, we call \((b, c, \lambda)\) to be the differential characteristics of \( X \). A semimartingale with these differential characteristics resembles locally a Lévy process with triplet \((b, c, F)\)(\(\omega, t\)). Hereafter, we implicitly assume \((b, c, F)\) to be a good version in the sense that \(c_s\) is nonnegative, \(F_s(\{0\}) = 0\) and the triplet satisfies

\[
\int_0^T \left( |b_s| + |c_s| + \int_{\mathbb{R}} (1 \wedge |x|^2)F_s(dx) \right) ds < \infty.
\]

In the literature, an additive process with absolutely continuous characteristics (also known as a time-inhomogeneous Lévy process) is frequently adopted as the building block for financial modeling (Cont and Tankov, 2003). In financial engineering, the existence of the exponential moments of the driving process is usually required, which naturally leads to the following assumption.

**Assumption.** There exists a constant \( M > 1 \), such that the Lévy measure \( \lambda_s \) satisfies

\[
\int_0^T \exp(uX_s)F_s(dx) \, ds < \infty, \quad \forall u \in [-M, M]. \tag{2.1}
\]

As a result of the above assumption, by the Lévy-Khintchine formula, the moment generating function is given by

\[
e^{\psi_t(u)} = E[e^{uX_t}] = \exp \left( \int_0^t \left[ b_s u + \frac{1}{2} c_s u^2 + \int_{\mathbb{R}} (e^{ux} - 1 - ux \mathbf{1}_{|x| \leq 1}) F_s(dx) \right] ds \right). \tag{2.2}
\]

Let \( \psi_{t,T}(u) \) denote the cumulant generating function of the increment \( X_T - X_t \). By the independence of the increments, we have

\[
\psi_{t,T}(u) = \ln E[e^{u(X_T-X_t)}] = \ln \frac{E[e^{uX_T}]}{E[e^{uX_t}]} = \psi_T(u) - \psi_t(u)
\]

\[
= \int_t^T \left[ b_s u + \frac{1}{2} c_s u^2 + \int_{\mathbb{R}} (e^{ux} - 1 - ux \mathbf{1}_{|x| \leq 1}) F_s(dx) \right] ds. \tag{2.3}
\]

Let \( S_t \) denote the asset price process with constant dividend yield \( d \) and \( r \) be the constant riskless interest rate. Suppose the risk neutral dynamics of asset price process is assumed to be an exponential time-inhomogeneous Lévy process under a risk neutral measure \( Q \), where

\[
S_t = S_0 e^{(r-d)t} e^{X_t - \psi_t(1)}, \quad t \geq 0, \tag{2.4}
\]
where \( X_t \) is a time-inhomogeneous Lévy process with the triplet of differential characteristics \((b, c, F)\) satisfying the conditions stated above. Note that the compensation term \( e^{-\psi(t)} \) is appended so that the discounted and dividend-stripped asset price process is a martingale under the risk neutral measure \( Q \).

**Derivatives on discretely sampled realized variance**

Let \( 0 = t_0 < t_1 < \cdots < t_M = T \) be the monitoring dates for the discretely sampled variance and \( T \) be the maturity date. We define \( R_m = \ln \frac{S_{tm}}{S_{tm-1}} \) to be the log return of the underlying asset price over \((t_{m-1}, t_m)\). The discretely sampled realized variance over \([0, T]\) based on log return is defined by

\[
V(0, T; M) = \frac{1}{T} \sum_{m=1}^{M} R_m^2 = \frac{1}{T} \sum_{m=1}^{M} \left( \ln \frac{S_{tm}}{S_{tm-1}} \right)^2, \tag{2.5}
\]

The simple return is sometimes used as an alternative definition, where the simple return is defined by \( R_m = \frac{S_{tm}}{S_{tm-1}} - 1 \).

The terminal payoff of the put option and the volatility swap on the discretely sampled realized variance are defined by

\[
\begin{align*}
p(S, V(0, T; M), T) &= \max(K_p - V(0, T; M), 0), \tag{2.6a} \\
w(S, V(0, T; M), T) &= \sqrt{V(0, T; M)} - K_w, \tag{2.6b}
\end{align*}
\]

respectively, where \( K_p \) is the strike of the put option and \( K_w \) is the strike of the volatility swap. An example of the third generation exotic variance swaps is the downside variance swap. Let \( U \) be the specified upper barrier, the discretely sampled downside realized variance is defined by

\[
D(0, T; M, U) = \sum_{m=1}^{M} \left( \ln \frac{S_{tm}}{S_{tm-1}} \right)^2 1_{\{S_{tm} \leq U\}}, \tag{2.6c}
\]

where \( 1_{\{\cdot\}} \) is the indicator function. The square of the log return over \((t_{m-1}, t_m]\) is counted toward the discretely sampled realized variance only when the asset price \( S_{tm} \) stays at or below \( U \). The corresponding terminal payoff of the downside variance swap is given by \( D(0, T; M, U) - K_D \), where \( K_D \) is the strike.
3 Fourier transform algorithms for pricing variance products and volatility derivatives

Our proposed Fourier transform algorithms for pricing variance products and volatility derivatives are visualized as a synthetic combination of the Convolution (CONV) method (Lord et al., 2008) and the Fourier space time stepping (FST) method (Jackson et al., 2008) for numerical option pricing in the Fourier domain. Also, we adopt the treatment of jump conditions in Windcliff et al.’s algorithm that deals with exotic path dependence associated with discretely sampled realized variance. The implementation of the CONV method requires the conditional probability density of the underlying asset process $f_{X_t}(y|X_t = x)$ to be dependent on $x$ and $y$ only via their difference $y - x$ so that

$$f_{X_t}(y|X_t = x) = f_{X_t-x_t}(y-x). \quad (3.1)$$

A sufficient condition for satisfying the above requirement is that the distribution of $X_T - X_t$ is independent of $X_t$.

FFT and CONV method

We let $w = \alpha + i\beta$ and define the generalized Fourier transform $\hat{f}$ of the given function $f$ as follows:

$$\hat{f}(w) = \int_{-\infty}^{\infty} f(x)e^{wx} \, dx, \quad (3.2a)$$

where $\alpha$ is a constant (known as the damping factor) that is properly chosen to ensure the existence of the generalized Fourier transform $\hat{f}$. The Fourier inversion formula is known to be

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(w)e^{-wx} \, d\beta. \quad (3.2b)$$

Next, we present a brief discussion of the use of the FFT techniques in the CONV method for pricing European options [refer to Lord et al. (2008) and Kwok et al. (2012) for details].

Let $X_t$ denote the process of the underlying stochastic state variable that defines the option payoff. For convenience, we consider the valuation of the undiscounted time-$t$ price function $u(x,t)$ of a European option with the terminal payoff $u(X_T, T)$ at maturity date $T$, where $X_t = x$. Let $p(x,t;y,T)$ be the transition density from $(x,t)$ to $(y,T)$. From the renowned
Parseval relation in Fourier transform, we have

\[ u(x,t) = E_Q[u(X_T,T)|\mathcal{F}_t] = \int_{-\infty}^{\infty} u(y,T) p(x,t;y,T) \, dy \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{p}(x,t;w,T) \hat{u}(w,T) \, d\beta. \quad (3.3) \]

Here, \( \hat{p}(x,t;w,T) \) and \( \hat{u}(w,T) \) denote the respective generalized Fourier and inversion transform of \( p(x,t;y,T) \) and \( u(y,T) \). Some special precaution on the choice of \( \alpha \) may be required to ensure the existence of \( \hat{u}(w,T) \). One may visualize \( \hat{p}(x,t;w,T) \) as the conditional moment generating function

\[ E_Q[ e^{wX_T} | X_t = x ] = e^{wx + \psi_t,T(w)} . \]

We may rewrite Eq. (3.3) into the following form:

\[ u(x,t) = \frac{e^{\alpha x}}{2\pi} \int_{-\infty}^{\infty} e^{i\beta x} e^{\psi_t,T(\alpha + i\beta)} \hat{u}(\alpha + i\beta, T) \, d\beta, \quad (3.4a) \]

where

\[ \hat{u}(\alpha + i\beta, T) = \int_{-\infty}^{\infty} e^{-i\beta y} e^{-\alpha y} u(y,T) \, dy. \quad (3.4b) \]

In the FFT procedure, the Fourier integrals are approximated by a discrete sum via numerical integration quadrature rules. We define uniform grids for \( \beta, x \) and \( y \), where

\[ \beta_n = \beta_0 + n\Delta\beta, \quad x_n = x_0 + n\Delta x, \quad y_n = y_0 + n\Delta y, \quad n = 0, 1, \ldots, N-1, \]

with \( N = 2^k \) for some positive integer \( k \). It is common to choose \( \beta_0 = -\frac{N}{2}\Delta\beta, \Delta x = \Delta y \) and \( x_0 = y_0 = -\frac{N}{2}\Delta x \) so that the grids are centered at the origin.

For notational convenience, we write \( u(x,t) \) and \( \hat{u}(w,T) \) as \( u_t(x) \) and \( \hat{u}_T(w) \), respectively. Suppose we approximate the Fourier integrals in Eqs. (3.4a) and (3.4b) by the trapezoidal rule, we obtain

\[ u_t(x_k) = \frac{e^{\alpha x_k}}{2\pi} \sum_{m=0}^{N-1} e^{i\beta_m x_k} e^{\psi_t,T(\alpha + i\beta_m)} \sum_{n=0}^{N-1} \gamma_n e^{-i\beta_m y_n} e^{-\alpha y_n} u_T(y_n) \Delta y \Delta\beta, \quad (3.5) \]

where \( \gamma_0 = \frac{1}{2}, \gamma_N = \frac{1}{2}, \gamma_n = 1 \) for \( n = 1, \ldots, N-2 \). Suppose we define the discrete Fourier transform and inverse transform by

\[ D_m \{ x_n \} = \sum_{n=0}^{N-1} e^{im\pi n/N} x_n \quad \text{and} \quad D_n^{-1} \{ x_m \} = \frac{1}{N} \sum_{m=0}^{N-1} e^{-im\pi n/N} x_m, \]
respectively, and take the usual Nyquist relation:
\[ \Delta \beta \Delta x = \frac{2\pi}{N}, \]
then Eq. (3.5) can be expressed as
\[ u_t(x_k) = e^{\alpha x_k} (-1)^k D_k \{ (-1)^n \gamma_n e^{-\alpha y_n} u_T(y_n) \} e^{\psi t,\tau (\alpha + i\beta_m)}. \] (3.6)
The above expression can be efficiently evaluated via two FFT calculations.

Path dependence and jump conditions across monitoring dates
We follow the approach in Windcliff et al.’s algorithm, where the original multi-state pricing problem is decomposed into a series of one-dimensional model problems indexed by two additional state variables which are only updated at monitoring dates.

Let \( P \) and \( Z \) denote the logarithm of the asset price on the previous monitoring date and the running average of the squared returns accumulated up to the current time, respectively. Apparently, \( P \) and \( Z \) are only updated on the monitoring dates and stay constant between two consecutive monitoring dates. The updating rules are given by
\[ P_{t_m^+} = X_{t_m}, \quad Z_{t_m^+} = Z_{t_m^-} - \frac{R_m^2 - Z_{t_m^-}}{m}, \] (3.7)
where \( t_m^- \) and \( t_m^+ \) represent the time instant immediately before and after the monitoring date \( t_m, m = 1, 2, \cdots, M \). We set \( P_0 = X_0 \) and \( Z_0 = 0 \). The time-\( t \) value of the volatility derivative can be regarded as a function of the logarithm of the underlying asset price \( X_t \) and time \( t \). Between two consecutive monitoring dates, the price function \( U = U(X, t; P, Z) \) is a function of \( X \) and \( t \) with the state variables \( P \) and \( Z \) treated as parameters.

Time stepping calculations between monitoring dates
Since \( P \) and \( Z \) remain constant between two consecutive monitoring dates, the martingale pricing theory gives
\[ U(X_{t_{m-1}}, t_{m-1}^+; P, Z) = e^{-r(t_{m-1}^--t_{m-1})} E[U(X_{t_{m-1}}, t_{m}^-; P, Z)|F_{t_{m-1}^-}]. \] (3.8)
The numerical calculation of the expectation can be done using the CONV method developed earlier. For the terminal condition, we initiate our time stepping calculations at the instant right before maturity \( T^- \), where
\[ U(X, T^-; P_{T^-}, Z_{T^-}) = f \left( \frac{1}{T} [(M - 1)Z_{T^-} + (X - P_{T^-})^2] \right) \] (3.9)
for some specified terminal payoff function \( f \).

The CONV method can be applied only if the generalized Fourier transform of the terminal payoff function exists. In fact, in order that \( \tilde{u}_T(\alpha + i\beta) \) is finite for all \( \beta \), the payoff is required to have at least one-sided boundedness with respect to \( X_T \). This is not an issue for equity options. However, it may become a problem for derivatives on discretely sampled realized variance. As seen in the above, the realized variance is unbounded when \( X_T \) goes to \( \pm \infty \) (corresponding to \( S_T \) approaching \( +\infty \) and 0, respectively). It can be shown that the payoff of a variance swap fails to meet this technical requirement of the FFT scheme. To resolve this problem, we impose a cap and floor on the underlying asset price, and consider the truncated form of the price function \( u(x, t)1_{|x| \leq L} \) instead of \( u(x, t) \). The parameter \( L \) is chosen to be sufficiently large so as to minimize the approximation error in this truncation procedure. This approximation can be justified as follows: given that the expectation is finite, the risk neutral density of the asset price distribution should decay faster than any polynomials in \( X \). That is, at a sufficiently large value of \( X \), the value of the density is close to zero. As a remark, similar type of truncation is commonly adopted in most numerical option pricing schemes that operate within the restriction of a finite computational domain.

**Jump condition across a monitoring date**

Since there no cash flow to the holder of the derivative across a monitoring date, by no arbitrage argument, the value of the derivative should be the same at the instants right before and after any monitoring date \( t_m \). The jump condition is illustrated by

\[
U(X, t_m^-, P_{t_m^-}, Z_{t_m^-}) = U(X, t_m^+, P_{t_m^+}, Z_{t_m^+}).
\]

(3.10)

**Backward induction calculations**

As in typical option pricing algorithms, we proceed the backward induction calculations from time \( t_m \) to \( t_{m-1} \), \( m = M, M-1, \ldots, 1 \), until we reach the initiation time \( t_0 \). The price of the derivative on the discretely sampled realized variance at initiation is then obtained.

**Choice of the grids**

For each state variable, we assign a grid of its respective truncated computational domain. Let \( N \) be the number of grid points of the \( X \)-grid. We take \( \Delta x = L/N \), where \( L \) is chosen to be a multiple of the standard derivation of \( X_T \). By the Nyquist relation, it then follows that \( \Delta \beta = 2\pi/L \). For the state variable \( P \), we may assign the same uniform grid as that of \( X \). The determination of the \( Z \)-grid is a subtle task. While a fine grid improves the accuracy at the expense of computational cost, it is observed that different types of variance and volatility products may have different sensitivity to the choice of the \( Z \)-grid. In our numerical tests, we find that derivatives with linear payoff on the (generalized) realized variance are less sensitive
to the Z-grid. By virtue of the linearity in the payoff, only two points are needed and the derivative values at intermediate value of Z can be well estimated by linear interpolation. However, nonlinear payoff structures such as volatility swaps and options on realized variance, the Z-grid should be chosen to be sufficiently dense. To develop the numerical algorithm that is more effective, a nonuniform grid structure (an example is shown in Figure 1) that allocates more points to the left side of the range seems to improve the convergence significantly for pricing put options on realized variance. The possible explanation is that most of the effective contribution to the put option value arises from a very small subinterval of the computational range of Z. Effectively, we strike a subtle balance between efficiency and accuracy in the choice of nonuniform grids. We present some numerical examples on the choice of the Z-grid in Section 4.

Given the grids \( x = (x_i)_{i=1}^N \), \( p = (p_j)_{j=1}^N \) and \( z = (z_k)_{k=1}^K \) for the discretization of \( X \), \( P \) and \( Z \), respectively (the adoption of the same uniform grid for \( x \) and \( p \) has been explained earlier), a naive approach would be to compute \( U(x, t_{m-1}; p_j, z_k) \) from \( U(x, t_m; p_j, z_k) \) via FFT for each pair \((p_j, z_k)\). This requires the storage of \( N^2 \times K \) values in total. When \( N \) is large, say \( N = 2^{10} \) which is a typical level in the actual computation, this requires prohibitively huge memory. We make an improvement on this memory usage issue by taking advantage of the updating rule \( P(t_m^+) = X_{t_m} \), for \( m = 1, \cdots, M \). In fact, not every value in the total \( N^2 \times K \) values computed is needed to start the next loop. We only need to keep track of those values that correspond to \( x_i = p_j \).

The key steps in our FFT algorithm are summarized as follows.

```
For \( m = M \) to 1
    For each \((p_j, z_k)\), determine \( U(x, t_m^+; p_j, z_k) \) using the updating rule Eq. (3.7).
        If \( m = M \)
            Apply the payoff function in Eq. (3.9) directly
        Else
            Use an interpolation method
        EndIf
    Compute \( U(x, t_{m-1}^+; p_j, z_k) \) via FFT in Eq. (3.6)
    If \( m = 1 \)
        Output \( U(x, t_0; P_0, Z_0) \); return
    Else
        Store \( U(p_j, t; p_j, z_k) \)
    EndIf
Next \( j, k \)
Next \( m \)
```
Calculations of the hedge parameters

Though the values of the derivatives on discretely sampled variance have no dependency on the spot price when evaluated at the initiation date $t_0$, the values do have sensitivities with respect to the spot price $S_t$ when we consider in-progress derivatives. Let $t \in (t_{k-1}, t_k]$, where $k = 1, 2, \ldots, M$ is fixed, the discretely sampled realized variance on $[0, T]$ can be decomposed into three terms as follows:

$$V(0, T; M) = \frac{1}{T} \left[ \sum_{i=1}^{k-1} \left( \ln \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 + \left( \ln \frac{S_{t_k}}{S_{t_{k-1}}} + \ln \frac{S_t}{S_{t_{k-1}}} \right)^2 + \sum_{i=k}^{M} \left( \ln \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 \right].$$

The first term is a known quantity by time $t$ with no dependency on $S_t$ while the last term is the future part of the realized variance whose distribution is independent of the spot price under the exponential additive model. The dependency on the spot price stems from the second term [see Drimus and Farkas (2011) for a similar analysis for variance swaps]. For options on realized variance, one cannot perform the same decomposition due to optionality in the payoff. The CONV method, however, can provide analytical expressions and numerical schemes for the two important hedge parameters $\Delta$ and $\Gamma$ as well. For convenience, we write these hedge parameters in terms of the log price $X$ as follows:

$$\Delta = \frac{\partial U}{\partial S} = e^{-X} \frac{\partial U}{\partial X}, \quad \Gamma = \frac{\partial^2 U}{\partial S^2} = e^{-2X} \left( -\frac{\partial U}{\partial X} + \frac{\partial^2 U}{\partial X^2} \right).$$

Since the differentiation can be performed analytically in Eq. (3.4a), we have

$$\Delta = \frac{e^{(\alpha-1)x}}{2\pi} \int_{-\infty}^{\infty} e^{i\beta x} (\alpha + i\beta) e^{\psi_{t,T}(\alpha+i\beta)} \hat{u}_T(\alpha + i\beta) \, d\beta,$$  \hspace{1cm} (3.11a)

and

$$\Gamma = \frac{e^{(\alpha-2)x}}{2\pi} \int_{-\infty}^{\infty} e^{i\beta x} (\alpha^2 - \alpha - \beta^2 + i(2\alpha - 1)\beta) e^{\psi_{t,T}(\alpha+i\beta)} \hat{u}_T(\alpha + i\beta) \, d\beta.$$  \hspace{1cm} (3.11b)

To compute the above greeks, one just need to perform two additional FFTs and inverse FFTs in the final step.

A direct alternative method is to compute the finite difference approximation of the values of the price function to obtain the values of the hedge parameters. One can make use of the option values at $X = x \pm \Delta x$, which are readily known when we have computed $u(x, t)$, and
apply the finite difference approximation formulas

\[
\Delta = \frac{u(x + \Delta x, t) - u(x - \Delta x, t)}{2\Delta x}, \quad (3.12a)
\]

\[
\Gamma = \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{\Delta x^2}. \quad (3.12b)
\]

4 Numerical examples

In this section, we report the numerical tests that were performed for the use of our proposed FFT algorithms for computing the price functions and their hedge parameters of various variance and volatility products under a “piecewise” double exponential model.

Piecewise double exponential model

In our sample calculations, we take the underlying asset price process to follow the piecewise double exponential model, which is seen as one simple example of Lévy process with time inhomogeneity. A “piecewise” double exponential model is constructed by modeling the underlying asset price process with different double exponential models on different time intervals. Consider a typical “piece” of the piecewise double exponential model over a specified time interval within the life of the derivative, the risk neutral dynamics of the asset price \( S_t \) observes the following jump diffusion process of double exponential type:

\[
\frac{dS_t}{S_t} = (r - d - m\lambda)dt + \sigma dW_t + (e^Y - 1) dN_t, \quad (4.1)
\]

where \( N_t \) is a Poisson process with intensity \( \lambda \) that is independent of \( W_t \), and \( Y \) denotes the independent random jump size and has an asymmetric double exponential distribution specified by

\[
Y = \begin{cases} 
\xi_+ & \text{with probability } p \\
-\xi_- & \text{with probability } 1 - p 
\end{cases}
\]

Here, \( \xi_{\pm} \) are exponential random variables with means \( 1/\eta_{\pm} \), respectively; and

\[
m = \frac{p}{\eta_+ - 1} - \frac{1 - p}{\eta_- + 1}.
\]

For simplicity, we adopt the two-piece double exponential model. We assume the asset price process to follow a double exponential jump diffusion process on \([0, t_0]\) and another double exponential jump diffusion process with a different set of parameters on \((t_0, T] \). The combi-
nation of these two pieces of separate double exponential jump diffusion processes is a time-
inhomogeneous Lévy process.

\[
\begin{array}{cccccc}
  t       & \sigma & \lambda & \eta_+ & \eta_- & p \\
\hline
  [0, 0.05] & 0.3   & 3.97   & 16.67 & 10    & 0.15  \\
  (0.05, T] & 0.18  & 1.43   & 10    & 6.25  & 0.01  \\
\end{array}
\]

Table 1: Model parameters of the two-piece double exponential model in which \( t_0 = 0.05 \).

In Table 1 we list the model parameters of the two-piece double exponential model used in our numerical examples. Both sets of parameters are calibrated to the DAX implied volatility on 5 July, 2002. The first set of parameters [taken from Sepp (2004)] are calibrated to options with the shortest maturity (2 weeks, \( T = 0.04 \)), while the second set [from Sepp and Skachkov (2003)] are calibrated to options with medium-term maturity (6 months, \( T = 0.46 \)). Correspondingly, the moment generating function of \( \ln \frac{S_t}{S_0} \) is a piecewise function, where

\[
e^{\psi_t(u)} = \exp \left\{ t(r - d)u + \min\{t, t_0\} \left[ \frac{\sigma^2}{2}(u^2 - u) + \lambda u \left( \frac{p}{\eta_+ - u} - \frac{1 - p}{\eta_- + u} + m \right) \right] \\
+ (t - t_0)^+ \left[ \frac{\bar{\sigma}^2}{2}(u^2 - u) + \bar{\lambda} u \left( \frac{\bar{p}}{\bar{\eta}_+ - u} - \frac{1 - \bar{p}}{\bar{\eta}_- + u} + \bar{m} \right) \right] \right\},
\]

for \(-\min(\eta_-, \bar{\eta}_-) < u < \min(\eta_+, \bar{\eta}_+) \). The corresponding Lévy measure in each piece is given by \( \lambda f_Y(x) \, dx \), where

\[
f_Y(x) = p \eta_+ e^{-\eta_+ x} 1_{\{x \geq 0\}} + (1 - p) \eta_- e^{\eta_- x} 1_{\{x < 0\}}.
\]

For simplicity, we choose \( r = d = 0 \) and \( S_0 = 1 \) in our sample calculations.

**Variance swaps**

Suppose the evaluation time \( t \in (t_{k-1}, t_k] \), \( k \) is fixed. The value of a variance swap with zero strike rate is given by

\[
V_t(0, T; M) = \frac{1}{T} \left\{ \sum_{i=1}^{k-1} \left( \ln \frac{S_{ti}}{S_{t_{i-1}}} \right)^2 + \left( \ln \frac{S_t}{S_{t_{k-1}}} + \psi''_{t,t_k}(0) \right)^2 + \psi''_{t,t_k}(0) \\
+ \sum_{i=k+1}^{M} \left[ \left( \psi'_{t_{i-1}, t_i}(0) \right)^2 + \psi''_{t_{i-1}, t_i}(0) \right] \right\}.
\] (4.2)
One can compute its delta and gamma readily by the following formulas:

\[
\Delta_V = \frac{1}{T} \cdot \frac{2}{S_t} \left( \ln \frac{S_t}{S_{t_{k-1}}} + \psi'_{t,t_k}(0) \right), \tag{4.3a}
\]

\[
\Gamma_V = \frac{1}{T^2} \cdot \frac{2}{S_t^2} \left( 1 - \ln \frac{S_t}{S_{t_{k-1}}} - \psi'_{t,t_k}(0) \right). \tag{4.3b}
\]

In Table 2, we show the comparison of the time-\(t\) value of a typical 3-month (\(M = 60\)) variance swap contract with daily sampling (\(\Delta t = 1/252\)) and zero strike as well as the greeks computed using our FFT algorithm with the exact values computed by analytical pricing formulas (4.3a, 4.3b). We take \(t = \Delta t/2\) and \(S_t = S_0 = 1\). The values of \(L\) and \(N\) (which are the corresponding number of \(X\)-grid points) chosen in our sample calculations are shown in the first and second columns in Table 2, respectively. Here, \(N\) is taken to be a power of 2 for the effective implementation of the FFT algorithms. Our numerical experiments demonstrate that a fixed value of \(L\) for all choices of \(N\) is inappropriate as this causes oscillation and instability to the numerical results. By choosing an increasing sequence for \(L\) as \(N\) increases, we manage to keep \(\Delta x\) and \(\Delta \beta\) to be in par. That is, both \(\Delta x\) and \(\Delta \beta\) are reduced to attain a more accurate discretization of the Fourier integrals. Unfortunately, such a procedure may implicitly force us to choose an unrealistic small value of \(L\) (like \(L = 2\) when \(N = 2^7\)). This may cause significant truncation error, as demonstrated by the numerical results in the row of \(L = 2\) and \(N = 2^7\) in Table 2. In the third column, we list the numerical results of the time-\(\Delta t/2\) price of the variance swap corresponding to an increasing number of \(X\)-grid points. In the columns labelled “delta” and “gamma”, we present the respective hedge parameter values computed using direct differentiation of the Fourier integral [Eqs. (3.11a) and (3.11b)] and the finite difference formulas [Eqs. (3.12a) and (3.12b)], respectively. We observe the convergence of the numerical results of the price function and its hedge parameters to the respective exact value (shown in the last row) up to 4 significant figures. However, one has to be alerted that \(N\) must be chosen to be sufficiently large (\(N \geq 2^8\) in our sample calculations) in order to achieve a reliable approximation to the exact value, especially in the calculations of delta and gamma.

**Options on the realized variance and volatility swaps**

We now apply our FFT algorithm to pricing and hedging of nonlinear contingent claims on the discretely sampled realized variance, namely, put options and volatility swaps (volatility is the positive square root of the realized variance). Note that the payoff of a put drops to a low value at very high or low underlying asset price since the realized variance would achieve a high value. This phenomenon is highly desirable since the truncation of the real space that is sufficiently large enough to cover the nonzero payoff range will hardly cause any substantial error. For call options on discretely sampled realized variance, one can make use of the well known put-
Table 2: The time-$\Delta t/2$ price and the corresponding greek values of a 3-month ($M = 60$) daily sampled variance swap with zero strike rate under the two-piece double exponential model. All prices and greek values are multiplied by notional value of 100.

call parity. For a volatility swap, the growth rate of its value is apparently limited by that of a variance swap at the extreme values of the underlying asset price. Thus, our algorithm should have no problem to deal with these products as long as the algorithm performs well for variance swaps. As a remark, choosing proper values for the parameters in the numerical scheme is of vital importance. From our experience in pricing variance swaps, we observe that $L = 8$ and $N = 2^{10}$ represent an appropriate set of parameters to ensure sufficient accuracy of the numerical results.

Discretization error of the quadratic variation approximation

It is known that accuracy of the quadratic variation approximation deteriorates for short-dated options on realized variance. Keller-Ressel and Muhle-Karbe (2010) have successfully quantified the discretization error using their asymptotic approximation method and confirmed that the discretization effect cannot be ignored for short-dated options on realized variance when the underlying model has nonzero diffusion term. In Figures 2(a,b), we present the plots of the prices of put options and volatility swaps based on daily sampled realized variance as well as the prices obtained based on quadratic variation against days to maturity ranging from a small number of days to 3 months (60 days). Specifically, the prices of the put options obtained based on quadratic variation are computed using the saddlepoint approximation formula in Zheng and Kwok (2012). On the other hand, the price of the volatility swaps based on quadratic variation is computed by the numerical quadrature method. Besides, we also provide the Monte Carlo simulation results as a benchmark. From Figure 2(a), we observe the significant discretization error arising from the quadratic variation approximation for short-dated put options on the daily sampled realized variance. However, the error shrinks for options with longer maturities. Similar phenomena are observed for short-dated volatility swaps [as shown in Figure 2(b)] though the discretization error is not as substantial as those in put options. Our calculations indicate that inadequacy of the quadratic variation approximation is common among nonlinear contingent claims on discretely sampled realized variance with short maturities.
**Prices and Hedge parameters**

Though options on the quadratic variation have no direct dependence on the spot value of the underlying asset price, it is not true for their discretely sampled realized variance counterparts when the valuation time is not at the initiation time. Consequently, it is useful to compute the delta and gamma of those products for hedging purposes. In our calculations, we assume the valuation time to be $\Delta t/2$, which lies in the middle of the first and second monitoring dates. We would like to investigate the properties of the hedge parameters of a typical one-month (20-day) put option on daily sampled realized variance. Figures 3(a,b,c) show the plots of the price, delta and gamma of the put option versus the spot price $S_t$. The price of the put option versus the spot price exhibits a bell shape. When $S_t = S_0$, the next squared return to be accumulated is expected to be the smallest among all scenarios. This would lead to the smallest expected realized variance, so the highest put option price. On the other hand, when $S_t$ is far away from $S_0$, a larger squared return is expected to be accumulated. This then drives the put option price down to a very small value. When the spot price $S_t$ is less than the initial stock price $S_0$, the delta value stays positive. This is expected from the plot of the price function against $S_t$ in Figure 3(a). The reverse effect on the delta value holds when the spot price is larger than $S_0$. However, when the spot price is sufficiently far away from $S_0$, the put option is bound to be out-of-the-money at maturity and any small change in the spot price will have little effect on the option price. As a result, the delta value is close to zero at the two ends far from $S_0$. Moreover, since gamma is the rate of change of delta with respect to the spot value, the curvature pattern in the plot of gamma against $S_t$ in Figure 3(c) can be fully inferred from the plot of delta in Figure 3(b). Note that the minimum value of gamma is realized when $S_t = S_0$ at which the gamma is negative, indicating that any significant change in the spot price will be unfavorable to the put option holder. We also observe good agreements between numerical results for the hedge parameters obtained using FFT calculations of the Fourier integral formulas and finite difference formulas.

**Choice of the Z-grid**

Lastly, we would like to illustrate the vital importance of the choice of the Z-grid for the convergence of the numerical results in our FFT algorithms. In Table 3, we present the numerical results obtained using different choices of the Z-grid. In the first column, an increasing sequence of the number of Z-grid is presented. The numerical values of the price of a 1-month (20-day) put option using the uniform Z-grid are shown in the second column with increasing value of $K$. The third and fourth columns show similar numerical values of the put option price obtained based on two nonuniform Z-grids that put more points on the left hand side of the range of $Z$. The graphic plots of the various choices of the distribution of the grid points are shown in Figure 1. Note that the “nonuniform-2” grid is the best performer and it has been adopted in our previous calculations of option prices. From Table 3 we observe that the
uniform Z-grid fails to give any reasonable approximation value to the true price while the nonuniform-1 Z-grid exhibits oscillations around the true value (estimated to be 0.0623).

<table>
<thead>
<tr>
<th>$K$</th>
<th>uniform</th>
<th>nonuniform-1</th>
<th>nonuniform-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^7$</td>
<td>0.1381</td>
<td>0.0675</td>
<td>0.0622</td>
</tr>
<tr>
<td>$2^8$</td>
<td>0.1359</td>
<td>0.0593</td>
<td>0.0622</td>
</tr>
<tr>
<td>$2^9$</td>
<td>0.1157</td>
<td>0.0618</td>
<td>0.0623</td>
</tr>
</tbody>
</table>

Table 3: The numerical results of the time-$\Delta t/2$ price of the 3-month daily sampled downside variance swap with zero strike rate under the two-piece double exponential model computed using different choices of Z-grid distribution and number of Z-grids, $K$.

**Downside variance swaps**

We apply our FFT algorithms to price the downside variance swaps, whose product nature and some of its intriguing pricing properties have been discussed in Zheng and Kwok (2011). Despite the slight difference between the generalized realized variance and realized variance with no restriction, we can handle the generalized variance swaps in a similar manner by modifying the state variable $Z$ in the numerical algorithm accordingly. Numerical tests show that both the variance swaps and downside variance swaps are insensitive to the degree of fineness of the Z-grid due to linearity in $Z$ in the payoff structure. We may simply choose a two-point gird of $Z$ and use linear interpolation for other points.

<table>
<thead>
<tr>
<th>$L$</th>
<th>$N$</th>
<th>$U = 0.9$</th>
<th>$U = 1$</th>
<th>$U = 1.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$2^7$</td>
<td>7.5308</td>
<td>9.6388</td>
<td>11.3789</td>
</tr>
<tr>
<td>4</td>
<td>$2^8$</td>
<td>7.1206</td>
<td>9.2287</td>
<td>10.9687</td>
</tr>
<tr>
<td>6</td>
<td>$2^9$</td>
<td>7.1396</td>
<td>9.1777</td>
<td>10.9439</td>
</tr>
<tr>
<td>8</td>
<td>$2^{10}$</td>
<td>7.0640</td>
<td>9.1325</td>
<td>10.9233</td>
</tr>
<tr>
<td>10</td>
<td>$2^{11}$</td>
<td>7.0704</td>
<td>9.0987</td>
<td>10.8970</td>
</tr>
<tr>
<td>exact</td>
<td></td>
<td>7.0746</td>
<td>9.0419</td>
<td>10.8804</td>
</tr>
</tbody>
</table>

Table 4: The numerical results of the time-$\Delta t/2$ price of the 3-month daily sampled downside variance swap with zero strike rate under the two-piece double exponential model with varying values of the upper barrier $U$. The convergence of the numerical results to the exact values with increasing value of $N$ is apparent, though the rate of convergence appears to be relatively slow. All prices are multiplied by notional value of 100.

In Table 4 we show the comparison of the price at time $\Delta t/2$ computed using our Fourier
transform algorithms of various 3-month (60-day) downside variance swaps with varying values of the upper barrier $U$ and explore the convergence to the exact values. Similar to the variance swaps, apparent convergence to the exact values with increasing value of $N$ is observed. For the downside variance swap with $U < 1$, we observe oscillation of the prices around the true value as $N$ increases.

Next, we focus on the examination of the pricing properties of the downside variance swap with $U = 1.1$ and examine how the price as well as the hedge parameters change with respect to the spot price of the underlying asset. In Figures 4(a,b,c), we show the plots of the prices of the downside variance swap and its greeks with varying spot price. When $S_t$ is less than $S_0$, the price of the downside variance swap tends to decrease as $S_t$ increases since the squared return to be accumulated is expected to decrease as a result. When $S_t$ is above $S_0$, the price of the downside variance swap first increases in value as $S_t$ does, like the variance swap. However, this trend cannot persist since we have the upper barrier $U$ (in this example, $U$ is chosen to be slightly larger than $S_0$). In fact, when $S_t$ approaches $U$, the probability that the squared return will not be accumulated increases rapidly due to the violation of the range restriction of the underlying price. For some time before $S_t$ reaches $U$, this ‘knock-out’ effect becomes more dominant and drives down the price of the downside variance swap. The plot of the time-$\Delta t/2$ price of the downside variance swap against $S_t$ shown in Figure 4(a) agrees with the above intuitive arguments on the pricing properties. As for the greeks, the numerical results obtained using FFT calculations of the Fourier integral formulas and finite difference formulas again show good agreement. The deviation becomes slightly larger at the spot price that is far away from $S_0$ [see Figures 4(b,c)]. At low values of $S_t$, the delta is negative and increases in value until it becomes close to zero at $S_t = S_0$. It then changes sign after increasing beyond $S_0$. Interestingly, the delta value starts to drop as the ‘knock-out’ effect becomes more significant, consistent with the earlier discussion on the price function. It reaches a local minimum at $S_t = U$ where the price of the downside variance swap is extremely sensitive to the spot price. When $S_t$ is close to the upper barrier $U$, hedging of the downside variance swap becomes more difficult. This is a phenomenon that is commonly shared by derivatives with embedded barrier feature. Further increase in the spot price causes less dramatic drop in the price of the downside variance swap. This is because when $S_t$ is sufficiently large, the ‘knock-out’ is almost sure to happen. This explains why the delta value asymptotically approaches zero from below [see Figure 4(b)]. Lastly, the plot of the gamma value in Figure 4(c) can be well inferred from that of the delta. The gamma value exhibits a high level of oscillation at $S_t$ close to the upper barrier $U$. 
5 Conclusion

We illustrate how to develop and apply effective fast Fourier transform algorithms for calculating the price functions and their hedge parameters of exotic variance products and volatility derivatives under time-inhomogeneous Lévy models (additive processes). We adopt the efficient procedure in the CONV method in the FFT calculations of the price function expressed as a convolution integral. Our enhanced versions of Fourier transform algorithm represent non-trivial extension to the Fourier space time stepping algorithm. We show how special precautions have been taken in order to incorporate the exotic path dependence associated with updating of the discretely sampled realized variance across monitoring instants. Also, we illustrate how to use truncated computational domain in order to avoid unboundedness of solution values at the far ends of the computational domain and choose a nonuniform set of grids for the path dependent variable of the running average of squared returns. The usual difficulties in computing hedge parameters in most option pricing algorithms are well resolved under FFT calculations. The efficiency, accuracy, reliability and robustness of our FFT algorithms are demonstrated through various numerical tests in pricing different types of exotic variance products and volatility derivatives. Lastly, we explore various properties of the price functions and their hedge parameters of put options on realized variance, volatility swaps and downside variance swaps. For future works, one may consider the extension to pricing more exotic path dependent variance products, like the timer options and target volatility options, and also deal with pricing under other types of asset price models, like the stochastic volatility model with jumps.

ACKNOWLEDGEMENT

This research was supported by the Hong Kong Research Grants Council under Project 642110 of the General Research Funds.

REFERENCES


Figure 1: Plots of uniform and nonuniform Z-grids that are adopted for pricing put options on realized variance.

Figure 2: The prices of at-the-money put options and volatility swaps on daily sampled realized variance with various maturities calculated by the Fourier transform algorithm (labelled “FT”) are compared to those based on quadratic variation (labelled “QV”). The numerical results obtained by Monte Carlo simulation (labelled “MC”) are seen to agree favorably well with those of the FFT calculations.
Figure 3: Plots of the price, delta and gamma of the one-month put option on daily sampled realized variance at time $\Delta t$ versus the spot price $S_t$. Good agreement between the numerical results on the delta and gamma obtained using FFT calculations of the Fourier integral formulas (labelled “FFT”) and finite difference formulas (labelled “FD”) is observed.
Figure 4: Plots of the price, delta and gamma of the 3-month daily sampled downside-variance swap with $U = 1.1$ at time $\Delta t/2$ versus the spot price $S_t$. Good agreement between the numerical results on the delta and gamma obtained using FFT calculations of the Fourier integral formulas (labelled “FFT”) and finite difference formulas (labelled “FD”) is observed.