Pricing options on discrete realized variance with partially exact and bounded approximations

Wendong Zheng and Yue Kuen Kwok*

Department of Mathematics
Hong Kong University of Science and Technology
Clear Water Bay, Hong Kong

Abstract

We derive efficient and accurate analytic approximation formulas for pricing options on discrete realized variance under stochastic volatility models using the partially exact and bounded (PEB) approximations. The PEB method is an enhanced extension of the conditioning variable approach commonly used in deriving analytic approximation formulas for pricing discrete Asian style options. By adopting either the normal or gamma distribution approximation based on some asymptotic behavior of the discrete realized variance of the underlying asset price process, we manage to obtain PEB approximation formulas that achieve high level of numerical accuracy in option values even for short-maturity options on discrete realized variance.

Keywords: options on discrete variance, stochastic volatility, conditioning variable method, partially exact and bounded approximations.

1 Introduction

We consider pricing of options on discrete realized variance of the price process of an underlying risky asset. Let \( S_t \) denote the price process of the risky asset. Given \( N \) monitoring dates \( 0 = t_0 < t_1 < \cdots < t_N = T \) over the time period \([0, T]\), the discrete realized variance of \( S_t \)

*Correspondence author; e-mail: maykwok@ust.hk
with respect to the above tenor is defined to be

\[ I_T^{(N)} = \sum_{k=1}^{N} \left( \ln \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2 = \sum_{k=1}^{N} (X_{t_k} - X_{t_{k-1}})^2, \tag{1.1} \]

where \( X_t = \ln S_t \) is the log asset price process. It is common to define the annualized discrete realized variance by multiplying \( I_T^{(N)} \) by \( A/N \), where the annualization factor \( A \) is taken to be 252 for daily monitoring. In this paper, under the dynamics of stochastic volatility for \( S_t \), we derive analytic approximation formulas using the enhanced conditioning variable approach for pricing financial options on \( I_T^{(N)} \). The terminal payoff functions of these options take the form: \( \max(I_T^{(N)} - K, 0) \) and \( \max(K - I_T^{(N)}, 0) \), where \( K \) is the strike price.

Analytic approximation of the prices of options on discrete realized variance poses mathematical challenges due to the exotic path dependence of \( I_T^{(N)} \), as exemplified by the sum of the quadratic terms involving \( (X_{t_k} - X_{t_{k-1}})^2 \). There have been several recent papers on pricing options on \( I_T^{(N)} \). For the asymptotic approach proposed by Keller-Ressel and Muhle-Karbe (2013), the prices of options on continuous realized variance are adjusted by the asymptotic formula for the short-time limit of the discretization gap between the continuous realized variance and discrete realized variance. They also develop the Laplace transform pricing method for options on discrete realized variance under exponential Lévy models without any use of approximation. Sepp (2013) considers an analytic approximation for the characteristic function of the discrete realized variance when the price dynamics is governed by the Heston model via the combination of the distribution of the continuous realized variance under the Heston model with that of the discrete realized variance under the Black-Scholes model. Good numerical accuracy is achieved for pricing near-the-money options on discrete realized variance over varying maturities. Zheng and Kwok (2014) develop the saddlepoint approximation formulas for pricing options on discrete realized variance and volatility derivatives under both Lévy models and stochastic volatility models with jumps. As part of their procedure, they use the small time asymptotic expansion of the Laplace transform of the discrete realized variance (Keller-Ressel and Muhle-Karbe, 2013) as a control in the approximation of the cumulants of the discrete realized variance. The numerical accuracy of their saddlepoint approximation formulas is shown to be within a few percents of relative errors. Drimus and Farkas (2013) derive the discretization adjustment term added to the price of an option on continuously sampled realized variance that serves to adjust the discretization effect in the discrete sampling of the realized variance. They show that conditional on the realization of the instantaneous variance process, the residual randomness arising from discrete sampling can be approximated by a normal random variable. Their treatment of the discretization effect arising from discrete sampling is applicable for general stochastic volatility processes. Motivated by the similarities between options on continuous realized variance and Asian options, Drimus (2012) adopts the conditioning variable approach (Rogers and Shi, 1995) to derive the lower bound for the prices
of options on continuous realized variance under log-OU models.

In this paper, we derive efficient and accurate analytic approximation formulas for pricing options on discrete realized variance under stochastic volatility using an enhanced version of the conditioning variable approach. We consider the PEB approximation method (Lord, 2006), which adds adjustment terms by finding an analytic approximation to the residual component in the conditioning variable method. A similar approach has also been used by Zeng and Kwok (2014) to derive pricing bounds and approximation for discrete arithmetic Asian options under time-changed Lévy processes. The application of the PEB method in pricing options on discrete realized variance relies on the adoption of either the normal distribution approximation (Drimus and Farkas, 2013) or gamma distribution approximation (Keller-Ressel and Muhle-Karbe, 2013) that is based on some asymptotic behavior of the discrete realized variance of the underlying asset price process.

The rest of the paper is organized as follows. In the next section, we derive the lower bound of the price of a call option on $I_T^{(N)}$ using the conditioning variable approach. The lower bound is expressible in terms of an integral that involves the joint characteristic functions of $(X_{t_k} - X_{t_{k-1}})$ and $I_T$, where $I_T$ is the terminal value of the continuous realized variance of the underlying asset price process. Numerical tests reveal that the lower bound based on the conditioning variable approach does not provide sufficiently accurate approximation to the price of an option on discrete realized variance. In Section 3, we derive the PEB approximation for the residual terms in the conditioning variable approximation. Instead of following the usual procedure of constructing an approximation to the distribution of $I_T^{(N)}|I$ (dependency on $T$ is suppressed for notational convenience) by matching the conditional moments (Lord, 2006; Zeng and Kwok, 2014), we propose to approximate $I_T^{(N)}$ by $\hat{I}_T^{(N)}$ such that $\hat{I}_T^{(N)}|I$ follows some common type of distribution that is highly tractable. One choice is based on the generalized Central Limit Theorem, where $\hat{I}_T^{(N)}|I$ is normally distributed with conditional mean $I$ and conditional variance $\frac{2}{N}I^2$. The other choice is based on the small-time asymptotic behavior of the discrete realized variance, where $\hat{I}_T^{(N)}|I$ follows a gamma distribution with shape parameter $N/2$ and scale parameter $2I/N$. In Section 4, we show how to apply the PEB approximation formulas for pricing options on discrete realized variance whose underlying asset price process follows the Heston stochastic volatility model. By virtue of the affine structure of the Heston stochastic volatility model, the joint characteristic function of $X_{t_k} - X_{t_{k-1}}$ and $I_T$ is readily available in an affine form. This leads to nice computational efficiency in the evaluation of the Fourier integrals in the PEB approximation formulas. The effective implementation of the PEB approximation for pricing options on discrete realized variance under the Heston model are illustrated via various numerical tests presented in Section 5. The PEB approximation based on the gamma distribution is seen to provide high level of accuracy even for short maturity options. The last section contains summary and conclusive remarks.
2 Optimal lower bound based on conditioning on continuous realized variance

Our PEB approximation consists of two steps. The first step is to derive the lower bound based on the conditioning variable approach. In the second step, we derive the adjustment terms that approximate the residual component in the lower bound. In this section, we focus on the derivation of the lower bound, where the conditioning variable is chosen to be $I_T = [\ln S, \ln S]^T$, the time-$T$ continuous realized variance of the underlying asset price process $S_t$.

Let $\{F_t\}_{0 \leq t \leq T}$ be the natural filtration generated by the asset price process $S_t$ and $A$ be an event of the form $\{I_T > c\}$ with $c > 0$ such that $A \in F_T$. For the undiscounted price of the call option on the discrete realized variance with strike price $K$, we have (Drimus, 2012)

$$\mathbb{E}[\max(0, I_T^{(N)} - K)] = \mathbb{E}[\max(0, I_T^{(N)} + 1A)] = \mathbb{E}[\max(0, K - I_T^{(N)} + 1A)] + \mathbb{E}[(I_T^{(N)} - K)1A] \geq \mathbb{E}[(I_T^{(N)} - K)1_{\{I_T > c\}}].$$

(2.1)

The last term gives a lower bound for the undiscounted call option price and the corresponding largest value among all nonnegative values of $c$ provides the best lower bound. Note that $I_T$ is chosen to be the conditioning variable since $I_T^{(N)}$ and $I_T$ are highly correlated and $I_T$ is tractable. For convenience, we write

$$g(c) = \mathbb{E}[(I_T^{(N)} - K)1_{\{I_T > c\}}] = \int_c^\infty \mathbb{E}[(I_T^{(N)} - K)1_{I_T = y}]f_I(y)\,dy,$n

(2.2)

where $f_I$ is the density function of $I_T$. We observe that the above conditional expectation increases in value with $c$ when $c$ is small and eventually drops to zero as $c$ becomes sufficiently large due to the rapid decay of $f_I$. Therefore, we expect that $g(c)$ achieves its maximum value at some finite value $c^*$ that is close to $K$. The critical value $c^*$ satisfies the first order condition:

$$g'(c) = -\mathbb{E}[(I_T^{(N)} - K)1_{I_T = c}]f_I(c) = 0.$n

(2.3)

For a typical quadratic term $(X_{t_k} - X_{t_{k-1}})^2$ in $I_T^{(N)}$, $k = 1, 2, \ldots, N$, we consider the evaluation of the conditional expectation via the following transformation:

$$\mathbb{E}[(X_{t_k} - X_{t_{k-1}})^2|I_T = c] = -\frac{\partial^2}{\partial \phi^2} \mathbb{E}[e^{i\phi(X_{t_k} - X_{t_{k-1}})}|I_T = c]|_{\phi=0}.$n

Denote $\Delta_k = X_{t_k} - X_{t_{k-1}}$ and let $f_{\Delta_k, I}$ and $\Phi_{\Delta_k, I}$ be the joint density function and joint characteristic function of $\Delta_k$ and $I_T$, respectively. By virtue of the Parseval identity and
interchanging order of integration, we obtain

\[
E[e^{i\phi k} | I_T = c] = \frac{1}{f_I(c)} \int_{-\infty}^{\infty} e^{i\phi x} f_{\Delta_k, I}(x, c) \, dx
\]

\[
= \frac{1}{f_I(c)} \int_{-\infty}^{\infty} e^{i\phi x} \frac{1}{4\pi^2} \int_{i\beta + \infty}^{i\beta - \infty} \int_{i\alpha + \infty}^{i\alpha - \infty} \Phi_{\Delta_k, I}(\alpha, \beta)e^{-i\alpha x - i\beta c} \, d\alpha \, d\beta \, dx
\]

\[
= \frac{1}{f_I(c)} \int_{i\beta - \infty}^{i\beta + \infty} \int_{i\alpha - \infty}^{i\alpha + \infty} \frac{1}{4\pi^2} \Phi_{\Delta_k, I}(\alpha, \beta) \int e^{i(\phi - \alpha)x} \, dx \, e^{-i\beta c} \, d\alpha \, d\beta
\]

\[
= \frac{1}{f_I(c)} \int_{i\beta - \infty}^{i\beta + \infty} \frac{1}{2\pi} \Phi_{\Delta_k, I}(\phi, \beta)e^{-i\beta c} \, d\beta,
\]

where \( \alpha = \alpha_r + i\alpha_i \) and \( \beta = \beta_r + i\beta_i \) are complex Fourier transform variables. The respective imaginary part \( \alpha_i \) and \( \beta_i \) of the pair of transform variables \( \alpha \) and \( \beta \) are chosen to be some appropriate fixed constants to ensure convergence of the generalized Fourier transform integral. Summing all the individual expectations of \( e^{i\phi \Delta_k} \) conditional on \( I_T = c \), where \( k = 1, 2, \cdots , N \), and substituting into eq. \((2.3)\), the first order condition can be expressed as

\[
g'(c) = \int_{0}^{\infty} \frac{1}{\pi} \sum_{k=1}^{N} \Re \left[ \frac{\partial^2}{\partial \phi^2} \Phi_{\Delta_k, I}(\phi, \beta_r + i\beta_i) \right] e^{-i(\beta_r + i\beta_i)c} \, d\beta_r + K f_I(c) = 0.
\]

where \( \Re(\cdot) \) stands for the real part. It is known that most stochastic volatility models do not admit closed form analytic expression for \( f_I \). However, the characteristic function \( \Phi_I \) of \( I \) may exist in an analytic form. We may express \( f_I(c) \) as a Fourier inversion integral of \( \Phi_I \), where

\[
f_I(c) = \frac{1}{\pi} \int_{0}^{\infty} \Re \left[ \Phi_I(\beta_r + i\beta_i)e^{-i(\beta_r + i\beta_i)c} \right] \, d\beta_r.
\]

As a result, the first order condition can be written in the following compact form:

\[
g'(c) = \frac{1}{\pi} \int_{0}^{\infty} \Re \left[ \Psi(\beta_r + i\beta_i)e^{-i(\beta_r + i\beta_i)c} \right] \, d\beta_r = 0,
\]

where

\[
\Psi(\beta) = \sum_{k=1}^{N} \frac{\partial^2}{\partial \phi^2} \Phi_{\Delta_k, I}(\phi, \beta) \bigg|_{\phi=0} + K \Phi_I(\beta).
\]

Since \( g''(c) \) is readily available, where

\[
g''(c) = \frac{1}{\pi} \int_{0}^{\infty} \Re \left[ -i(\beta_r + i\beta_i) \Psi(\beta_r + i\beta_i)e^{-i(\beta_r + i\beta_i)c} \right] \, d\beta_r,
\]

one may solve eq. \((2.4)\) for the critical value \( c^* \) via Newton’s iteration method with the initial guess \( c = K \). One may check for \( g''(c) < 0 \) to ensure that \( c^* \) is a maximizer of \( g(c) \).
Finally, we can calculate the largest lower bound for the option price as follows:

\[ g(c^*) = \sum_{k=1}^{N} E[\Delta_k^2 1_{\{I_T > c^*\}}] - K \mathbb{P}(I_T > c^*) \]

\[ = -\frac{1}{\pi} \int_0^\infty \Re \left[ \sum_{k=1}^{N} \frac{\partial^2}{\partial \phi^2} \Phi_{\Delta_k, I} (\phi, \beta_r + i \beta_i) \bigg|_{\phi=0} \right] \frac{e^{-ic^* (\beta_r + i \beta_i)}}{i(\beta_r + i \beta_i)} d\beta_r \\
- K \frac{1}{\pi} \int_0^\infty \Re \left[ \Phi_I (\beta_r + i \beta_i) \frac{e^{-ic^* (\beta_r + i \beta_i)}}{i(\beta_r + i \beta_i)} \right] d\beta_r \\
= -\frac{1}{\pi} \int_0^\infty \Re \left[ \Psi(\beta_r + i \beta_i) \frac{e^{-ic^* (\beta_r + i \beta_i)}}{i(\beta_r + i \beta_i)} \right] d\beta_r. \tag{2.5} \]

Note that it is necessary to restrict \( \beta_i < 0. \)

**Alternative derivation**

It is relatively easy to establish a lower bound for the undiscounted call option price via an approximation of \( I_T^{(N)} \) by the conditional variable \( E[I_T^{(N)} | I_T] \). Indeed, by the Jensen inequality, we deduce that

\[ \mathbb{E}[(I_T^{(N)} - K)^+] = \mathbb{E}[\mathbb{E}[(I_T^{(N)} - K)^+ | I_T]] \geq \mathbb{E}[(\mathbb{E}[I_T^{(N)} | I_T] - K)^+]. \]

Interestingly, the lower bound on the right hand side is simply the optimal (largest) lower bound \( g(c^*) \) defined in eq. (2.5).

To show the claim, it is necessary to apply the following analytic properties of \( g'(c) \). It can be shown that \( g'(c) \) starts at the zero value at \( c = 0 \), increases in value to remain positive and decreases at increasing value of \( c \) so that \( g'(c) \) hits the zero value at \( c^* \). Finally, \( g'(c) \) approaches the zero value from below at asymptotically large value of \( c \). An illustrative plot of \( g'(c) \) is shown in Figure [1]. One can then obtain

\[ \mathbb{E}[\mathbb{E}[I_T^{(N)} | I_T] - K)^+] = \int_0^\infty (\mathbb{E}[I_T^{(N)} | I_T = c] - K)^+ f_I(c) \, dc \\
= \int_0^\infty \left( \mathbb{E}[I_T^{(N)} | I_T = c] f_I(c) - K f_I(c) \right)^+ \, dc \\
= \int_0^\infty [g'(c)]^+ \, dc = g(c^*). \]
3 Partially exact and bounded approximation

While the lower bound derived from the conditioning variable approach in the last section works quite well for arithmetic Asian options with conditioning on the geometric average counterpart, the lower bound $g(c^*)$ defined in eq. (2.5) fails to provide sufficiently accurate approximation formulas for short-maturity options on discrete realized variance. While we observe dominance of arithmetic average over geometric average, there is a lack of strict dominance of the discrete realized variance over the continuous counterpart or vice versa. Due to this lack of dominance, optionality on the continuous realized variance may not be carried over to optionality on the discrete counterpart. This explains the significant gap between the lower bound and the exact price of an option on discrete realized variance. Indeed, the discrepancy between the discrete and continuous realized variance becomes more profound when maturity or sampling period becomes shorter. Therefore, the lower bound approximation becomes more unreliable for short-maturity options on discrete realized variance. As a remark, the crude approximation of $I_T^{(N)}$ by $I_T$ in the option valuation provides an even worse approximation than the lower bound $g(c^*)$ derived by conditioning.

Henceforth, we drop the subscript $T$ in both $I_T^{(N)}$ and $I_T$ in our later exposition when no ambiguity arises. To provide a better approximation, it is natural to consider analytic
approximation to the residual terms
\[ E[(I^{(N)} - K)^+ 1_{\{I \leq c^*\}}] + E[(K - I^{(N)})^+ 1_{\{I > c^*\}}] \]
(3.1)
in the decomposition of the option price shown in eq. (2.1). In the literature on pricing arithmetic Asian options, this approach is termed the partially exact and bounded (PEB) approximation. The essence of the PEB approximation is to consider an approximation to the conditional distribution of \( I^{(N)} | I \) so that evaluation of the two residual terms can be performed efficiently. The common technique in the PEB approximation for pricing arithmetic Asian options is to fit a lognormal or normal distribution to the difference of \( I^{(N)} | I - I \) by matching the respective conditional moments (Lord, 2006; Zeng and Kwok, 2014). Instead, we propose two analytic approximations based on the normal distribution and gamma distribution approximation derived from some asymptotic behavior of the realized variance of the underlying asset price process.

### 3.1 Conditional normal distribution approximation

Based on the generalized Central Limit Theorem and asymptotic analysis of the discrete realized variance of an asset price process under stochastic volatility, Drimus and Farkas (2013) show that one may approximate \( I^{(N)} | I \) by \( \hat{I}^{(N)} | I \) for a sufficiently large value of \( N \), where
\[ \hat{I}^{(N)} | I \sim N(I, \frac{2}{N}I^2). \]
(3.2)
Here, \( N(\mu, \sigma^2) \) denotes a normal distribution with mean \( \mu \) and variance \( \sigma^2 \). Though their result is derived under the stochastic volatility framework, it can be seen that it remains to work well under stochastic volatility with jumps.

In our derivation of the PEB approximation, it is necessary to introduce another approximation with order of approximation consistent with that of the Drimus-Farkas approximation. Let \( \Phi_{\hat{I}^{(N)}, I}(\alpha, \beta) \) denote the joint characteristic function of \( \hat{I}^{(N)} \) and \( I \). As \( \hat{I}^{(N)} | I \) is given by eq. (3.2), by introducing the approximation: \( e^{-\alpha^2 I^2/N} \approx 1 - \frac{\alpha^2 I^2}{N} \) that is \( O(N^{-2}) \), we have
\[
\Phi_{\hat{I}^{(N)}, I}(\alpha, \beta) = \mathbb{E}[e^{i\alpha \hat{I}^{(N)} + i\beta I}] = \mathbb{E}[\mathbb{E}[e^{i\alpha \hat{I}^{(N)} + i\beta I} | I]] = \mathbb{E}[e^{i\alpha I - \frac{\alpha^2 I^2}{N}} e^{i\beta I}]
\approx \mathbb{E} \left[ e^{i(\alpha + \beta) I} \left( 1 - \frac{\alpha^2 I^2}{N} \right) \right] = \Phi_I(\alpha + \beta) + \frac{\alpha^2}{N} \Phi_I^{(2)}(\alpha + \beta),
\]
(3.3)
where \( \Phi_I \) denotes the characteristic function of \( I \), and \( \Phi_I^{(2)} \) refers to the second order derivative of \( \Phi_I \). The above approximation has the same order as that of the Drimus-Farkas approximation.
Next, we derive the analytic approximation of the two residual terms of Fourier transform integrals via the Parseval Theorem. For the first residual term, we consider the approximation

\[ \mathbb{E}[(I^{(N)} - K)^+ 1_{I \leq c^*}] \approx \mathbb{E}[(\hat{I}^{(N)} - K)^+ 1_{I \leq c^*}] \]

\[ = \frac{1}{4\pi^2} \int_{ib-\infty}^{ib+\infty} \int_{ia-\infty}^{ia+\infty} e^{-i\alpha K - i\beta c^*} \frac{\Phi_{\hat{I}^{(N)}, I}(\alpha, \beta)}{i\beta \alpha^2} \, d\alpha \, d\beta, \]

where \( a < 0 \) and \( b > 0 \) are chosen such that the integration contours are within the domain of convergence of the two-dimensional generalized Fourier transform. In the next step, we apply the analytic approximation of the joint characteristic function \( \Phi_{\hat{I}^{(N)}, I}(\alpha, \beta) \) given by eq. (3.3).

For convenience, we write \( z = \alpha + \beta \) so that

\[ \mathbb{E}[(I^{(N)} - K)^+ 1_{I \leq c^*}] \approx \frac{1}{2\pi} \int_{iu-\infty}^{iu+\infty} e^{-icz^*} \Phi_I(z) \frac{1}{2\pi} \int_{ia-\infty}^{ia+\infty} e^{-i\alpha(K-c^*)} \frac{1}{i(z-\alpha)\alpha^2} \, d\alpha \, dz \]

\[ + \frac{1}{2\pi} \int_{iu-\infty}^{iu+\infty} e^{-icz^*} \Phi_I^{(2)}(z) \frac{1}{N} \int_{ia-\infty}^{ia+\infty} e^{-i\alpha(K-c^*)} \frac{1}{i(z-\alpha)} \, d\alpha \, dz, \quad (3.4) \]

where \( u = a + b > a \) determines the contour of the complex integral with respect to \( z \). In a similar manner, we may approximate the second residual term by

\[ \mathbb{E}[(K - I^{(N)})^+ 1_{I > c^*}] \approx \mathbb{E}[(\hat{K} - \hat{I}^{(N)})^+ 1_{I > c^*}] \]

\[ = \frac{1}{4\pi^2} \int_{ib-\infty}^{ib+\infty} \int_{ia-\infty}^{ia+\infty} e^{-i\alpha K - i\beta c^*} \frac{\Phi_{\hat{I}^{(N)}, I}(\alpha, \beta)}{-i\beta \alpha^2} \, d\alpha \, d\beta, \]

where \( \hat{a} > 0 \) and \( \hat{b} < 0 \) are chosen to ensure that the integration contours are within the domain of convergence of the two-dimensional generalized Fourier transform. Again, by applying the approximation in eq. (3.3) and letting \( z = \alpha + \beta \), we obtain

\[ \mathbb{E}[(K - I^{(N)})^+ 1_{I > c^*}] \approx \frac{1}{2\pi} \int_{iu-\infty}^{iu+\infty} e^{-icz^*} \Phi_I(z) \frac{1}{2\pi} \int_{ia-\infty}^{ia+\infty} e^{-i\alpha(K-c^*)} \frac{1}{i(z-\alpha)\alpha^2} \, d\alpha \, dz \]

\[ - \frac{1}{2\pi} \int_{iu-\infty}^{iu+\infty} e^{-icz^*} \Phi_I^{(2)}(z) \frac{1}{N} \int_{ia-\infty}^{ia+\infty} e^{-i\alpha(K-c^*)} \frac{1}{i(z-\alpha)} \, d\alpha \, dz, \quad (3.5) \]

where \( \hat{u} = \hat{a} + \hat{b} < \hat{a} \) determines the contour of the complex integral with respect to \( z \).

Interestingly, the corresponding integrands in the Fourier transform integrals in eqs. (3.4) and (3.5) are identical. The two Fourier transform integrals differ only in the choices of the contours, where one is along a horizontal line below the real axis in the positive direction while the other is along a horizontal line above the real axis in the negative direction. This is not surprising since the two quantities in the two residual terms have the same analytic form but differ in sign. We include the vertical contours at the two extreme ends on the right and left sides of the complex plane that join the two horizontal contours to form a closed contour \( C \). The values of the contour integrals along the two far-end vertical contours are seen to assume zero value in the limit.
We now combine the Fourier integrals in eqs. (3.4) and (3.5) that approximate the two residual terms. We choose a common contour for the integral with respect to $z$. That is, we choose the horizontal path to be from $i\tilde{u} - \infty$ to $i\tilde{u} + \infty$, where $a < \tilde{u} < \hat{a}$. Also, we use the Cauchy Residue Theorem to evaluate the inner contour integral with respect to the closed contour $C$. Since we have chosen $a < \tilde{u} < \hat{a}$, where $a < 0$ and $\hat{a} > 0$, the poles are included inside the closed contour $C$. By combining the first terms in eqs. (3.4) and (3.5), we obtain

\[
A = \frac{1}{2\pi} \int_{i\tilde{u} - \infty}^{i\tilde{u} + \infty} e^{-izc^*} \Phi_I(z) \frac{1}{2\pi} \oint_{C} \frac{e^{-i\alpha(K - c^*)}}{i(z - \alpha)\alpha^2} \, d\alpha \, dz
\]

In a similar manner, by combining the second terms in eqs. (3.4) and (3.5), we obtain

\[
B = \frac{1}{2\pi} \int_{i\tilde{u} - \infty}^{i\tilde{u} + \infty} e^{-izc^*} \Phi_I^{(2)}(z) \frac{1}{2\pi} \oint_{C} \frac{e^{-i\alpha(K - c^*)}}{i(z - \alpha)} \, d\alpha \, dz
\]

We manage to express the approximation of the two residual terms as the sum of an one-dimensional integral and an explicitly known term.

Last but not least, we would like to discuss the financial interpretation of the two terms above. It is easily visualized that the term $A$ is simply equal to the following quantity:

\[
E[(I - K)^+1_{\{I \leq c^*\}}] + E[(K - I)^+1_{\{I > c^*\}}].
\]

In other words, keeping the single term $A$ alone in the analytic approximation would be equivalent to approximating the two residual terms by simply replacing $I^{(N)}$ by $I$. Since the optimal solution $c^* \approx K$, we expect that both $\{K < I \leq c^*\}$ and $\{K \geq I > c^*\}$ are small probability events. Therefore, the correction contributed by $A$ would be small and secondary. The second term $B$ is seen to be identical to the discretization adjustment term presented in Drimus and Farkas (2013). This discretization adjustment arises when Drimus and Farkas try to account for the discrete sampling effect of realized variance in the approximation of $E[(I^{(N)} - K)^+]$ by $E[(I - K)^+]$. It is interesting to observe that $B$ has dependence on $N$ but no dependence on $c^*$ while $A$ has the reverse properties of dependence. The term $B$ provides the discretization gap between $I^{(N)}$ and $I$ that is not captured by the optimal lower bound. In general, the contribution of $B$ as an adjustment added to the optimal lower bound is more significant compared to that of $A$. 

10
3.2 Conditional gamma distribution approximation

The conditional normal distribution is based on the asymptotic behavior of $I^{(N)}$ as $N \to \infty$. When we consider pricing of short-maturity options on discrete realized variance, the asymptotic behavior of the discrete realized variance as $T \to 0$ is more relevant. In this regard, Keller-Ressel and Mulhe-Karbe (2011) propose the asymptotic gamma distribution of the discrete realized variance as $T \to 0$. More specifically, it can be shown that the annualized continuous realized variance tends to $V_0$ as $T \to 0$ while the discrete realized variance converges in distribution to a gamma distribution with shape parameter $N/2$ and scale parameter $2V_0/N$, where $V_0$ is the initial value of the instantaneous variance. Motivated by this elegant theoretical result, we propose to approximate $I^{(N)}$ by $\hat{I}^{(N)}$, which has a gamma distribution with shape parameter $N/2$ and scale parameter $2I/N$ conditional on $I$, where

$$\hat{I}^{(N)}|I \sim \text{gamma}(N/2, 2I/N). \quad (3.7)$$

The above gamma approximation has the same conditional mean and variance as the normal approximation in the previous subsection. The gamma approximation is advantageous over the normal distribution in the following two aspects. Firstly, it becomes exact in asymptotic limit as $T \to 0$. Secondly, the gamma approximation retains nonnegativity of $I^{(N)}|I$.

As the first step in deriving the analytic approximation of the residual terms using the conditional gamma distribution approximation, we express the residual terms as nested conditional expectation:

$$E\left[ E\left[ (K - I^{(N)})^+ | I \right] 1_{\{I > c^*\}} \right] + E\left[ E\left[ (I^{(N)} - K)^+ | I \right] 1_{\{I \leq c^*\}} \right]. \quad (3.8)$$

Substituting the explicit form of the gamma density function and applying the put-call parity, the inner expectation can be evaluated as follows:

$$E[(K - I^{(N)})^+ | I] \approx \int_0^K (K - y) \frac{y^{N/2-1} e^{-Ny/2}}{\Gamma(N/2)(2I/N)^{N/2}} dy = \frac{1}{\Gamma(N/2)} \left( (K - I) \gamma\left( \frac{N}{2}, \frac{KN}{2I} \right) + K \exp\left( \left( \frac{N}{2} - 1 \right) \ln \frac{KN}{2I} - \frac{KN}{2I} \right) \right)$$

$$E[(I^{(N)} - K)^+ | I] = E[I^{(N)} | I] - K + E[(K - I^{(N)})^+ | I] \approx I - K + \frac{1}{\Gamma(N/2)} \left( (K - I) \gamma\left( \frac{N}{2}, \frac{KN}{2I} \right) + K \exp\left( \left( \frac{N}{2} - 1 \right) \ln \frac{KN}{2I} - \frac{KN}{2I} \right) \right),$$

where $\Gamma(\cdot)$ is the gamma function and $\gamma(s, x) = \int_0^x z^{s-1} e^{-t} dz$ is the lower incomplete gamma function. Putting the above results together, the correction term $C_g$ added to the optimal lower bound based on the conditional gamma distribution approximation is given by

$$C_g = \int_0^{\infty} G(y) f_I(y) dy + \int_{0}^{c^*} (y - K) f_I(y) dy, \quad (3.9)$$
where

\[ G(y) = \frac{1}{\Gamma\left(\frac{N}{2}\right)} \left[ (K - y) \gamma\left(\frac{N}{2}, \frac{KN}{2y}\right) + K \exp\left(\left(\frac{N}{2} - 1\right) \ln \frac{KN}{2y} - \frac{KN}{2y}\right) \right]. \]

The above correction formula does conform well with financial intuition. The first integral in \( C_g \) is seen to be \( \mathbb{E}[(K - \tilde{I}(N))^+] \) under the conditional gamma distribution approximation. The second term can be interpreted as \( \mathbb{E}[(\tilde{I}(N) - K)^+] - \mathbb{E}[(\tilde{I}(N) - K)1_{t>c^*}] \) under the same approximate distribution. The sum gives \( \mathbb{E}[(\tilde{I}(N) - K)^+] - \mathbb{E}[(\tilde{I}(N) - K)1_{t>c^*}] \), which is exactly the residual given by eq. (3.1) with \( I(N) \) being replaced by \( \tilde{I}(N) \) under the conditional gamma distribution. The small-time asymptotic approximation approach by Keller-Ressel and Mulhe-Karbe (2013) attempts to approximate the “discretization gap” between the price of an option on discrete realized variance and that of the continuous counterpart. Our PEB approximation considers approximating \( I(N) \) by \( \tilde{I}(N) \) under the approximate gamma distribution in the residual terms. As a result, while the small time asymptotic approximation is expected to perform well only for small \( T \), our PEB approximation would provide high accuracy over any value of \( T \).

4 Stochastic volatility models

The success of the PEB approximation procedure relies on the availability of the joint characteristic function of \( \Delta_k \) and \( I \) in analytic closed form (or in an integral representation). Thanks to the affine structure of the Heston stochastic volatility model, it is possible to express \( \Phi_{\Delta_k, I}(\alpha, \beta) \) in an exponential affine form (details shown below). For the general stochastic volatility model, we let \( X_t = \ln S_t \) be the log asset price, \( V_t \) be the instantaneous variance, and \( I_t \) be the continuous realized variance process, where \( I_t = [\ln S, \ln S]_t \). We manage to express the joint characteristic function \( \Phi_{\Delta_k, I}(\alpha, \beta) \) in an integral form, provided that the partial transform of the transition density of the triple \((X_t, I_t, V_t)\) is available in closed form [see eq. (4.4a)].

4.1 Heston stochastic volatility model

In the Heston stochastic volatility model with jumps in asset price, the joint dynamics of the asset price \( S_t \) and the instantaneous variance \( V_t \) are specified by

\[
\begin{align*}
\frac{dS_t}{S_t} &= (r - q) \, dt + \sqrt{V_t} (\rho \, dW^2_t + \sqrt{1 - \rho^2} \, dW^1_t) + (e^J - 1) \, dN_t, \\
\frac{dV_t}{V_t} &= \kappa (\theta - V_t) \, dt + \varepsilon \sqrt{V_t} \, dW^2_t,
\end{align*}
\]

where \( W^1_t \) and \( W^2_t \) are two independent Brownian motions, \( N_t \) is a Poisson process with intensity \( \lambda \), the jump size \( J \) is assumed to have a normal distribution with mean \( \nu \) and variance \( \delta^2 \), \( \rho \) is the correlation coefficient, \( r \) and \( q \) are the constant riskfree rate and dividend yield,
respectively. The Heston model is well known for its affine structure in which the characteristic function of the triplet \((X_t, I_t, V_t)\) has an exponential affine form (Kallsen et al., 2011). The joint characteristic function of the triplet is given by

\[
\mathbb{E}_t[e^{uX_t+wI_t+bV_t+c}] = \exp(uX_t + wI_t + B(\tau, q)V_t + D(\tau, q)),
\]

where the parameter functions \(B\) and \(D\) are determined by a Riccati system of ordinary differential equations, the details of which can be found in Appendix A. Here, \(q = (u, w, b, c)^T\) denotes the initial parameters. It then follows that

\[
\Phi_{\Delta k, t}(\alpha, \beta) = \mathbb{E}\left[\mathbb{E}_{t_k}[e^{i\beta I_{t_k}}e^{i\alpha V_{t_k}}]\right] = \mathbb{E}\left[e^{i\alpha V_{t_k} + B(T - t_k, q_1)V_{t_k} + D(T - t_k, q_1)}\right] = e^{B(t_k, q_3)V_0 + D(t_k, q_3)},
\]

where

\[
q_1 = (0, 0, i\beta, 0)^T,
q_2 = (i\alpha, B(T - t_k, q_1), i\beta, D(T - t_k, q_1))^T,
q_3 = (0, B(\Delta t_k, q_2), i\beta, D(\Delta t_k, q_2))^T.
\]

### 4.2 General stochastic volatility models

Consider the following general class of stochastic volatility models as characterized by

\[
\frac{dS_t}{S_t} = (r - q) dt + \sqrt{V_t} \left(\rho dW^2_t + \sqrt{1 - \rho^2} dW^1_t\right),
\]

\[
dV_t = \mu(t, V_t) dt + \sigma(V_t) dW^2_t,
\]

where \(\mu(t, V_t)\) and \(\sigma(V_t)\) denote the drift and volatility of variance, respectively. For example, the 3/2 stochastic volatility model takes the form (Zheng and Zeng, 2014):

\[
dV_t = V_t(\theta_t - \kappa V_t) dt + \varepsilon V_t^{3/2} dW^2_t.
\]

We consider the partial transform of the transition density of the triplet \((X_t, I_t, V_t)\) as defined by

\[
\tilde{G}(t, x, y, t', \omega, \eta, v') = \int_0^\infty \int_{-\infty}^\infty e^{i\omega x' + i\eta y'} G(t, x, y; t', x, y') dx' dy',
\]

where \(G(t, x, y; t', x', y', v')\) is the joint transition density. Zheng and Zeng (2014) show that it is possible to express \(\tilde{G}\) in the following form

\[
\tilde{G}(t, x, y, t', \omega, \eta, v') = e^{i\omega x + i\eta y} g(t, v; t', \omega, \eta, v'),
\]

13
where the function $g$ is the solution to an one-dimensional partial differential equation. By further integrating out $v'$ in $\hat{G}$, one can obtain the bivariate characteristic function of $(X, I)$ as follows:

$$E_t[e^{i\omega X_t + i\eta I_t}] = e^{i\omega X_t + i\eta I_t} \int_0^\infty g(t, V_t; t', \omega, \eta, v') \, dv' = e^{i\omega X_t + i\eta I_t} h(t, V_t; t', \omega, \eta). \quad (4.5)$$

Provided that the partial transform is available in closed form, one can derive the joint characteristic function of $(\Delta_k, I)$ as follows:

$$\Phi_{\Delta_k, I}(\alpha, \beta) = E[E_{t_k}[e^{i\beta I_{T_k}}] e^{i\alpha \Delta_k}] = E[E_{t_{k-1}}[e^{i\alpha X_{T_{k-1}}} + i\beta I_{T_{k-1}} h(t_{k-1}, V_{T_{k-1}}; T, 0, \beta)] e^{-i\alpha X_{T_k}}] = E \left[ e^{i\beta I_{T_{k-1}}} \int_0^\infty h(t_{k-1}, v'; T, 0, \beta) g(t_{k-1}, V_{T_{k-1}}; t_k, \alpha, \beta, v') \, dv' \right] = \int_0^\infty \int_0^\infty h(t_k, v'; T, 0, \beta) g(t_{k-1}, v; t_k, \alpha, \beta, v') g(t_0, V_0; t_{k-1}, 0, \beta, v) \, dv' \, dv'. \quad (4.6)$$

For the 3/2 model, the analytic expressions for $g$ and $h$ are available, which involve the modified Bessel function with complex argument and order and the confluent hypergeometric function of the first kind (Zheng and Zeng, 2014). As a result, numerical evaluation of the above double integral may be notoriously tedious and time consuming.

## 5 Numerical calculations

In this section, we present numerical calculations that were performed to examine accuracy of the proposed partially exact and bounded approximations. Recall that the PEB method can be applied for pricing options on discrete realized variance under a general stochastic volatility model, provided that the partial transform is known in closed form. The PEB method is particularly effective for the affine stochastic volatility models, where the joint characteristic function of the triplet $(X_t, V_t, I_t)$ exhibits an exponential affine structure. For illustrative purposes, our numerical examples are confined to the Heston stochastic volatility model with compound Poisson jumps in the asset price.

The set of model parameters for the Heston model [see eq. (4.1)] in our numerical calculations are shown in Table 1. Furthermore, we choose $r = 0.0319$, $q = 0$ and $S_0 = 1$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>3.46</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$(0.0894)^2$</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>0.14</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$-0.82$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>-0.086</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.47</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.0001</td>
</tr>
<tr>
<td>$\sqrt{V_0}$</td>
<td>0.087</td>
</tr>
</tbody>
</table>

Table 1: The basic set of parameter values of the Heston model with jumps in asset price.
Monte Carlo simulation

Since there is no exact pricing formulas for options on discrete realized variance under the Heston model, we use the numerical results from Monte Carlo simulation for the benchmark comparison. The most straightforward approach to implement the simulation is the use of the first-order Euler scheme to simulate the joint dynamics of the underlying price process. However, it is well known that the Euler discretization scheme of the instantaneous variance process may possibly generate negative values and improper handling of the negative values may lead to severely biased results. This effect becomes particularly noticeable since the price of a variance option is typically quite small in magnitude. To reduce the bias and generate reliable benchmark results, we adopt the following modified Euler scheme proposed by Lord et al. (2010)

\[
\ln S_{t+\Delta t} = \ln S_t + \left( r - q - \frac{V_t^+}{2} \right) \Delta t + \sqrt{V_t^+ \Delta t} \left( \rho Z_2 + \sqrt{1 - \rho^2} Z_1 \right) + \sum_{i=1}^{N_{\Delta t}} J_i
\]

\[
V_{t+\Delta t} = V_t + \kappa \Delta t (\theta - V_t^+) + \sqrt{V_t^+ \Delta t} Z_2,
\]

where \( V_t^+ = \max(V_t, 0) \), \( Z_2 \) and \( Z_1 \) are two independent standard normal random variables, and \( J_i \) are independent copies of the random jump size. To hasten the rate of convergence of the simulation, we use the discrete realized variance as a control variate. The details of this technique can be found in Broadie and Jain (2008).

Analysis of numerical accuracy

We present the numerical results for testing accuracy of the lower bound approximation and the partially exact and bounded approximation. We calculate the prices of the call options on daily sampled realized variance with varying sampling periods and strike prices. We investigate three maturities \( N = 20, N = 126 \) and \( N = 252 \) which represent one month (short), half a year (intermediate) and a year (long), respectively. For each maturity, we choose three representative strike prices that correspond to deep in-the-money (ITM), at-the-money (ATM) and deep out-of-the-money (OTM) call options. We also list the prices of the call options on the continuous realized variance, which can be regarded as a crude approximation to the prices of the discrete counterparts.
Table 2: All the option prices are interpreted as basis points. That is, the calculated results have been multiplied by $10^4$. “Cont” means the prices of the call options on the continuous realized variance, “LB” means the lower bound approximation given by eq. (2.5), “PEBn” means the PEB approximation with normal distribution, “PEBg” means the PEB approximation with gamma distribution, and “MC” means Monte Carlo simulation using the Euler scheme eq. (5.1). The numbers in brackets represent the relative errors (RE) with the Monte Carlo simulation results as benchmark.

The numerical results in Table 2 reveal that the performance of the lower bound approximation is quite similar to the crude approximation using the price of the call option on the continuous realized variance. For short-maturity options, though numerical accuracy deteriorates, the lower bound approximation slightly outperforms the “Cont” approximation. Both the PEB approximation methods with the normal or gamma distribution approximation have shown significant improvement over the lower bound approximation. However, the PEB method with the normal distribution approximation fails to deliver a consistent accurate approximation for the one-month call options. On the other hand, the PEB method with the gamma distribution approximation provides very accurate results for the short-maturity options. This is expected since the gamma distribution approximation is exact in the asymptotic limit when $T \rightarrow 0$. Surprisingly, the gamma distribution approximation remains to perform equally well for relatively long maturities. The numerical experiment once again confirms the discrepancy between the discrete and continuous realized variance when the time to maturity is small. The two PEB approximation methods, especially the gamma distribution approximation, prove to be an efficient and accurate analytic approximation method for pricing options on discrete realized variance under all ranges of maturities.

Figure 2 shows that the percentage error in numerical pricing of options on discrete realized variance of the LB and PEB approximation methods for varying moneyness and maturities. The volatility of variance is set to be a relatively large value 0.9. For short-maturity options
(\(N = 20\)), the normal and gamma distribution approximations are seen to exhibit comparable performance, while the lower bound approximation remains to be inferior. When the maturity of the option is lengthened to be half a year, the percentage errors in all three approximations are within 1%. In general, we find that it is reliable to use the gamma distribution approximation for short-maturity options and the normal distribution approximation for long-maturity options.

Figure 2: Plot of percentage error again moneyness for short-dated and long-dated call options on daily sampled realized variance. The volatility of variance parameter \(\varepsilon\) is set to be 0.9.

6 Conclusion

The conditioning variable approach with PEB approximation is known to be an effective analytic approximation method for pricing path dependent options. We propose an ingenious extension of the PEB approximation for pricing options on discrete realized variance. Our numerical tests demonstrate that the PEB approximation formulas provide very good performance for pricing options on discrete realized variance under the Heston model, without the shortcoming exhibited in other analytic approximation methods where accuracy may deteriorate significantly in pricing options with short maturities. The high level of numerical accuracy is attributed to the adoption of either the normal or gamma approximation of the distribution of discrete realized variance conditional on the quadratic variation. The PEB approximation works well when the partial transform of the triple transition density is available in closed form. Thanks to the affine structure of the Heston model, the PEB approximation is seen to
be particularly effective for pricing options on discrete realized variance under the Heston s-
tochastic volatility model. Since the gamma distribution approximation is exact in asymptotic
limit as maturity tends to zero, it is more reliable to use the PEB method with the gamma
distribution approximation when pricing short-maturity options on discrete realized variance.
For options with medium to long maturities, the normal distribution approximation is seen
to be very reliable to give numerical accuracy within 1% error for most reasonable ranges of
model parameter values.

Acknowledgement

This work was supported by the Hong Kong Research Grants Council under the General
Research Funds.

References

variance swaps. *International Journal of Theoretical and Applied Finance*, 11(8), p.761-
797.


general analytic approximation. Working paper of University of Copenhagen and Univer-
sity of Zurich.


Appendix A  Derivation of parameter functions $B, D$

Let $U(X_t, I_t, V_t, \tau) = \mathbb{E}_t[e^{uX_t + wI_t + bV_t + c}],$ where $\tau = T - t$. It is seen that $U$ satisfies the following Kolmogorov backward equation:

$$
\frac{\partial U}{\partial \tau} = \left(r - q - m\lambda - \frac{V}{2}\right) \frac{\partial U}{\partial X} + \kappa(\theta - V) \frac{\partial U}{\partial V} + \frac{\varepsilon^2 V}{2} \frac{\partial^2 U}{\partial V^2} + V \frac{\partial U}{\partial I} + \rho \varepsilon V \frac{\partial^2 U}{\partial X \partial V} + \lambda \mathbb{E}[U(X + J, V, I + J^2, \tau) - U(X, V, I, \tau)].
$$  \hspace{1cm} (A.1)

By substituting the solution form: $U(X_t, I_t, V_t, \tau) = e^{uX_t + wI_t + B(\tau, q)}V_t^2 + D(\tau, q)$ into eq. (A.1), we obtain the following Riccati system of ordinary differential equations (ODEs):

$$
\frac{\partial B}{\partial \tau} = -\frac{1}{2}(u - u^2) - (\kappa - \rho \varepsilon u)B + \frac{\varepsilon^2}{2} B^2 + w \\
\frac{\partial D}{\partial \tau} = (r - q)u + \kappa \theta B + \lambda \{\mathbb{E}[\exp(uJ + wJ^2) - 1] - mu\}.
$$  \hspace{1cm} (A.2)

Using a similar technique as in Zheng and Kwok (2012), we can obtain the solutions to the above ODEs as follows:

$$
B(\tau, q) = \frac{b(\xi_+ - e^{-\zeta \tau} + \xi_+) - (u - u^2 - 2w)(1 - e^{-\zeta \tau})}{(\xi_+ + \varepsilon^2 b)e^{-\zeta \tau} + \xi_- - \varepsilon^2 b},
$$

$$
D(\tau, q) = (r - q)u \tau + c - \frac{\kappa \theta}{\varepsilon^2} \left[\xi_+ \tau + 2 \ln \frac{(\xi_+ + \varepsilon^2 b)e^{-\zeta \tau} + \xi_- - \varepsilon^2 b}{2\zeta}\right]
$$

$$
- \lambda (mu + 1) \tau + \frac{\lambda \tau}{\sqrt{1 - 2\delta^2 w}} \exp \left(\frac{\delta^2 u^2 + 2\nu(u + \nu w)}{2(1 - 2\delta^2 w)}\right),
$$

where

$$
\zeta = \sqrt{(\kappa - \rho \varepsilon u)^2 + \varepsilon^2 (u - u^2 - 2w)},
$$

$$
\xi_+ = \zeta \mp (\kappa - \rho \varepsilon u).
$$

Here, the expression of $D$ is valid provided that $\Re(w) < \frac{1}{2\delta^2}$. 

20