Closed Form Pricing Formulas for Discretely Sampled Generalized Variance Swaps

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Abstract

Most of the existing pricing models of variance derivative products assume continuous sampling of the realized variance processes, though actual contractual specifications compute the realized variance based on sampling at discrete times. We present a general analytic approach for pricing discretely sampled exotic variance swaps under the stochastic volatility models with simultaneous jumps in the asset price and variance processes. The resulting pricing formula of the gamma swap is in closed form while those of the corridor variance swaps and conditional variance swaps take the form of one-dimensional Fourier integrals. We examine the exposure to convexity (volatility of volatility) and skew (correlation between the equity returns and variance process) of these third generation variance swaps under the more realistic framework of discrete sampling of the realized variance process. We explore the impact on the fair strike prices of these exotic variance swaps with respect to different sets of parameter values, like varying sampling frequencies, jump intensity, and width of the corridor. We also examine the greek sensitivities of these exotic variance swaps and the convergence of the fair strike prices of discretely monitored generalized variance swaps to those of their continuously monitored counterparts.

Key words: exotic variance swaps, stochastic volatility models, Fourier transform, discrete sampling

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1 Introduction

Volatility measures the standard deviation of the logarithm of returns of an underlying asset, thus it gives a measure of the risk of holding that asset. Volatility risk has drawn a wider attention in the financial markets in recent years, especially after the global financial crisis. Volatility trading becomes an important topic of risk management. In a bearish market environment, volatility typically stays at a high level, so holding a long position of volatility may be useful in hedging an equity portfolio. Indeed, volatility can be viewed as an asset class in its own right. Investors can use volatility derivatives to perform directional trading of volatility levels, say, trading the spread between the realized and implied volatility levels, or hedging an implicit volatility exposure. These volatility derivatives are investment tools for investors with specific views on the future market volatility or with particular risk exposures by allowing them to deal with these views or risks without taking a direct position in the underlying asset and/or delta-hedging their position (Brockhaus and Long, 2000).

Volatility products can be generally classified into two types. The historical-variance-based volatility derivatives include products whose payoff depends on the realized variance of the underlying asset. Another class of volatility products are the implied-volatility-based products, like the VIX futures traded in the Chicago Board of Exchange (CBOE). The VIX stands for the CBOE Volatility Index, and it measures the implied volatility of the S&P 500 index options with maturity of 30 days (Carr and Wu, 2006). In recent years, the third generation of volatility products, like the corridor variance swaps, conditional variance swaps and gamma swaps have gained wider popularity as volatility trading instruments. These exotic variance swaps can offer investors a more finely tuned volatility exposure than the traditional variance swaps [see the review articles by Carr and Lewis (2004), Bouzoubaa and Osseiran (2010) and Lee (2010)]. Their product specifications and potential uses in hedging or betting the various forms of volatility exposures will be presented in later sections. One of the objectives of this paper is to present a systematic and efficient analytic approach for pricing these third generation exotic variance swaps under the realistic framework of discrete sampling of the realized variance process.

Assuming that the stock prices evolve without jumps, Neuberger (1994) shows how a continuously sampled variance swap can be theoretically equivalent to a dynamically adjusted, constant dollar exposure to the stock, in combination with a static long position in a portfolio of options and a forward contract that replicate the payoff of a log contract. Carr and Madan (1998) propose various methods of trading the realized volatility, like taking a static position in options, delta-hedging an option position, etc. They demonstrate that the delta-hedged option approach exhibits a large amount of path dependency in the underlying in the final profit/loss. On the other hand, volatility derivatives are shown to provide the pure exposure to realized market volatility independent of an inherent price path dependency. Demeterfi et al. (1999)
provide a nice review on the pricing behavior and theory of both variance and volatility swaps.

As a common approximation assumption in the pricing models of volatility derivatives in the literature, the discretely sampled realized variance in the actual contractual specification is approximated by a continuously sampled variance (as quantified by the quadratic variation of the log asset price process). The assumption of continuous sampling falls short of providing pricing results with sufficient accuracy when the actual discrete sampling becomes less frequent. Since it is not so straightforward to estimate the approximation errors in a unified framework, practitioners trading on contracts that are based on the realized variance with a low sampling frequency cannot properly assess the pricing errors caused by the continuous sampling assumption. A review on the replication errors for the discretely monitored variance swaps can be found in Carr and Lee (2009). Keller-Ressel and Muhle-Karbe (2010) discuss the rate of convergence of the approximation of the realized variance via quadratic variation and examine the errors of the approximation in pricing short-dated options with non-linear payoffs.

There have been numerous papers that consider the pricing of variance contracts on the discretely sampled realized variance. Little and Pant (2001) develop a finite difference approach for the valuation of the discretely sampled variance swaps in an extended Black-Scholes framework with a local volatility function. They adopt an effective numerical technique to capture the jumps in the realized variance across the sampling dates. Windcliff et al. (2006) improve on the pricing algorithm for the discretely sampled volatility derivatives by allowing jumps in the asset price process. Using the Monte Carlo simulation method, Broadie and Jain (2008) investigate the effect of discrete sampling and asset price jumps on the fair variance and volatility swap strikes under various stochastic volatility models. Carr and Lee (2009) consider the replication of discretely sampled variance products (including exotic path dependent payoff structures) using options, futures, and bonds with the same sampling frequency of the variance products. Itkin and Carr (2010) use a forward characteristic function approach to price discretely monitored variance and volatility swaps under various Lévy models with stochastic time change. Crosby and Davis (2011) consider the pricing of generalized variance swaps, such as self-quantoed variance swaps, gamma swaps, skewness swaps and proportional variance swaps under the time-changed Lévy processes. They show that the prices of discretely monitored variance swaps and their generalizations all converge to the prices of continuously monitored counterparts as $O(1/N)$, where $N$ is the number of monitoring instants. Sepp (2011) analyzes the impact of discrete sampling on the pricing of options on the realized variance under Heston’s stochastic volatility model. He proposes a method of mixing of the discrete variance in a log-normal model and the quadratic variance in a stochastic volatility model that approximates well the distribution of the discrete variance. Drimus and Farkas (2010) show that conditional on the realization of the instantaneous variance process, the residual randomness arising from discrete sampling follows a normal distribution. They also provide a practical analysis of the greeks of options on discretely sampled variance. Following the Little-Pant pricing formulation,
Zhu and Lian (2011) manage to derive closed form pricing formulas for the vanilla variance swaps under Heston’s two-factor stochastic volatility model for the underlying asset price process by solving a coupled system of partial differential equations. Lian (2010) extends the above analytic pricing approach to the underlying asset price process that allows stochastic volatility with simultaneous jumps in both the asset price and variance process (SVSJ model). The success of analytic tractability in the Zhu-Lian approach lies on the exponential affine structures of the SVSJ model, where the corresponding analytic expression of the marginal characteristic functions can be derived. When the payoff structures of the variance swap contracts become more exotic, like those of the corridor variance swaps, conditional variance swaps and gamma swaps, the knowledge of the marginal characteristic functions alone may not be enough in the derivation of the closed form pricing formulas. Instead, the joint moment generating functions play a vital role in deriving the pricing formulas for exotic variance swaps.

In this paper, we propose a general analytic approach for pricing various types of discretely sampled generalized variance swaps, thanks to the availability of the analytical expression of the joint moment generating function of the underlying processes. The analytic derivation of the associated moment generating function under the SVSJ model can be accomplished via the solution to a Riccati system of ordinary differential equations. Provided that the payoff function can be transformed into an exponential function of the state variables, closed form or semi-analytic (in terms of one-dimensional Fourier integrals) pricing formula of the derivative product can be derived. Duffie et al. (2000) and Chacko and Das (2002) demonstrate the versatility of this analytic approach in pricing various types of fixed income derivatives. These papers explore various invariant properties of the solutions to the Riccati systems and manage to express the pricing formulas in terms of these solutions. Sepp (2007) applies similar techniques to price continuously-sampled variance derivatives and conditional variance swaps via the derivation of the analytic representation of the Green function associated with the governing partial integro-differential equation under the SVSJ model.

This paper is organized as follows. In the next section, we present the formulation of the SVSJ model with a discussion on various possible extensions of the underlying joint dynamics of the asset returns and its variance. Thanks to the exponential affine structures of the SVSJ model, we manage to obtain an analytic representation of the corresponding joint moment generating function by solving a Riccati system of ordinary differential equations. In Section 3, we present the product specification and potential uses of various generalized variance swaps. We then show how to derive the closed form pricing formula of each of these discretely sampled generalized variance swaps under the SVSJ model. The continuously sampled gamma swap is known to provide a constant share gamma exposure. We illustrate how this gamma exposure property is modified under discrete sampling. We show that the pricing of the conditional variance swap requires the computation of the expected occupation time of the asset price below a specified barrier. For the formulas of the fair strikes of the discretely monitored
variance swaps and gamma swaps, we take the asymptotic limit by letting the monitoring time interval approach zero and illustrate that one can recover the same set of formulas of their continuously sampled counterparts (Sepp, 2008). Also, we manage to obtain the pricing formulas of corridor and conditional variance swaps under continuous sampling. In Section 4, we report the numerical tests that verify the accuracy of the pricing formulas by (i) comparing the fair strike prices at various sampling frequencies obtained from the pricing formulas and Monte Carlo simulation, (ii) examining the convergence of the fair strike prices with increasing sampling frequencies to the fair strike price of the continuously sampled counterpart. Our numerical tests show that a linear rate of convergence with respect to the sampling time interval of the fair strike price under discrete sampling to that under continuously sampling is revealed for variance swaps. However, such convergence behavior may not always be observed for gamma swaps. We examine the exposure to convexity (volatility of volatility), skew (correlation between equity returns and variance process) and jump intensities of these exotic swap products under the more realistic framework of discrete sampling of the realized variance. We also demonstrate how the fair strike prices in these generalized variance swaps depend on their contractual specifications. Summary and conclusive remarks are presented in the last section.

2 Stochastic volatility models with simultaneous jumps and joint moment generating function

There have been numerous empirical studies on the dynamics of asset returns that illustrate evidence for both jumps in the price level and its volatility. A prominent continuous time model that has been widely adopted is the affine simultaneous jump model (Duffie et al., 2000) where the asset returns and its variance follow the jump-diffusion process for which the drift, covariance and jump intensities are assumed to have an affine dependence on the state vector. Due to the assumed affine structures, the affine simultaneous jump model admits the reduced form solutions when one solves the pricing models of equity and fixed income derivatives. The valuation of these solutions can be performed via the solutions of the corresponding Riccati system of ordinary differential equations and numerical Fourier inversion algorithms (Chacko and Das, 2002).

In this paper, we adopt the following stochastic volatility model with simultaneous jumps (SVSJ) to describe the joint dynamics of the non-dividend paying stock price $S_t$ and its instantaneous variance $V_t$. Under the pricing measure $Q$, the joint dynamics of $S_t$ and $V_t$ assumes
the form

\[
\begin{align*}
\frac{dS_t}{S_t} &= (r - \lambda m) dt + \sqrt{V_t} dW_t^S + (e^{J^S} - 1) dN_t, \\
\frac{dV_t}{V_t} &= \kappa(\theta - V_t) dt + \varepsilon \sqrt{V_t} dW_t^V + J^V dN_t, \\
\frac{dI_t}{I_t} &= V_t dt + (J^S)^2 dN_t,
\end{align*}
\]

where \(I_t\) represents the instantaneous cumulative realized variance, \(W_t^S\) and \(W_t^V\) are a pair of correlated standard Brownian motions with \(dW_t^S dW_t^V = \rho dt\), and \(N_t\) is a Poisson process with constant intensity \(\lambda\) that is independent of the two Brownian motions. We let \(J^S\) and \(J^V\) denote the random jump sizes of the log price and variance, respectively, and these random jump sizes are assumed to be independent of \(W_t^S\), \(W_t^V\) and \(N_t\). Also, \(r\) denotes the riskless interest rate and \(m = E^Q_t[e^{J^S} - 1]\). Throughout the paper, all the expectation calculations \(E^Q_t[\cdot]\) are done under the risk neutral pricing measure \(Q\) and conditional on filtration \(\mathcal{F}_t\) at the current time \(t\). In the sequel, we suppress the superscript and subscript in the expectation operator for notational convenience.

It is well known that jumps in the stock price provide a more realistic description of the short term behavior of the stock price while jumps in the variance give the more accurate modeling of the volatility skew. Various empirical studies reveal that jumps in the price level and variance in general occur together, and they are strongly dependent and have opposite sign. One may argue that the above SVSJ model with specific affine forms of the parameter functions may be somewhat restrictive. Some recent non-parametric studies of the high frequency movements in stock market volatility reveal that volatility may follow quite different forms of jump behavior (Todorov and Tauchen, 2010). Various extended versions of the stochastic volatility models have also been proposed. For example, Kangro et al. (2004) propose to generalize the intensity of the Poisson process to be a non-reverting stochastic process. Carr and Wu (2007) assume stochastic hazard rate for the Poisson process, where the stochastic hazard rate parameter is assumed to be the sum of the instantaneous variance and a latent risk factor that follows a diffusion process with mean reversion drift rate. Cont and Kokholm (2008) model directly the forward variance swap rates for a discrete tenor of maturities, somewhat analogous to the LIBOR market model in interest rates modeling. In some of these extended models, analytic tractability that is similar to the SVSJ model can be maintained though the analytic procedures tend to become more involved. In this paper, we illustrate the set of analytic procedures of deriving the pricing formulas of discretely sampled exotic variance products under the popular SVSJ model and relegate the research on analytic pricing under other types of stochastic volatility models to future works.

Joint moment generating function

For convenience, we let \(X_t = \ln S_t\). The joint moment generating function of \(X_t\), \(V_t\) and \(I_t\) is defined to be

\[E[\exp(\phi X_T + bV_T + \psi I_T + \gamma)],\]
where \( \phi, b, \psi \) and \( \gamma \) are constant parameters. Let \( U(X_t, V_t, I_t, t) \) denote the non-discounted time-\( t \) value of a contingent claim with the terminal payoff function: \( U_T(X_T, V_T, I_T) \), where \( T \) is the maturity date. By adopting the temporal variable, \( \tau = T - t \), it can be deduced from the Feynman-Kac theorem that \( U(X,V,I,\tau) \) is governed by the following partial integro-differential equation (PIDE):

\[
\frac{\partial U}{\partial \tau} = \left( r - m\lambda - \frac{V}{2} \right) \frac{\partial U}{\partial X} + \kappa(\theta - V) \frac{\partial U}{\partial V} + V \frac{\partial U}{\partial I}
\]

\[
+ \frac{V}{2} \frac{\partial^2 U}{\partial X^2} + \frac{\varepsilon^2 V}{2} \frac{\partial^2 U}{\partial V^2} + \rho\varepsilon V \frac{\partial^2 U}{\partial X \partial V} + \lambda E \left[ U(X + J^S, V + J^V, I + (J^S)^2, \tau) - U(X, V, I, \tau) \right].
\]

The terminal payoff function of the contingent claim becomes the initial condition of the PIDE. Note that the joint moment generating function (MGF) can be regarded as the time-\( t \) forward value of the contingent claim with the terminal payoff: \( \exp(\phi X_T + bV_T + \psi I_T + \gamma) \), so that the MGF also satisfies the PIDE (2.2).

Using the analytic procedures similar to those in interest rate derivatives pricing under the stochastic volatility with simultaneous jumps model (Chacko and Das, 2002), analytic solution to the joint MGF can be obtained via the solution of a Riccati system of ordinary differential equations. Due to the exotic payoff structure of the generalized variance swaps, analytic form of the joint MGF is required in deriving analytic pricing formulas for these discretely sampled variance products. As a remark, it suffices to use the marginal MGFs to price discretely sampled vanilla variance swaps (Zhu and Lian, 2011) due to their simpler payoff structure. Once the joint MGF is known, the respective marginal MGF can be obtained easily by setting the irrelevant parameters in the joint MGF to be zero. For example, the marginal MGF with respect to the state variable \( V \) can be obtained by setting \( \phi = \psi = 0 \).

Thanks to the affine structure in the SVSJ model, \( U(X, V, I, \tau) \) admits an analytic solution of the following form (Duffie et al., 2002):

\[
U(X, V, I, \tau) = \exp \left( \phi X + B(\Theta; \tau, q)V + \psi I + \Gamma(\Theta; \tau, q) + \Lambda(\Theta; \tau, q) \right),
\]

where the parameter functions \( B(\Theta; \tau, q), \Gamma(\Theta; \tau, q) \) and \( \Lambda(\Theta; \tau, q) \) satisfy the following Riccati system of ordinary differential equations:

\[
\left\{ \begin{array}{l}
\frac{\partial B}{\partial \tau} = -\frac{1}{2}(\phi - \phi^2) - (\kappa - \rho \varepsilon \phi)B + \frac{\varepsilon^2}{2}B^2 + \psi \\
\frac{\partial \phi}{\partial \tau} = r\phi + \kappa \theta B \\
\frac{\partial \Gamma}{\partial \tau} = \lambda \left( E[\exp(\phi J^S + \psi (J^S)^2 + BJ^V) - 1] - m\phi \right) \\
\end{array} \right.
\]

with the initial conditions: \( B(0) = b, \Gamma(0) = \gamma \) and \( \Lambda(0) = 0 \). Here, \( q = (\phi \ b \ \psi \ \gamma)^T \) and we use \( \Theta \) to indicate the dependence of these parameter functions on the model parameters in the SVSJ model. One has to specify the distributions for \( J^S \) and \( J^V \) in order to obtain a complete
solution to these parameter functions. For example, suppose we assume that \( J^V \sim \exp(\eta) \) and \( J^S \) follows
\[
J^S | J^V \sim \text{Normal}(\nu + \rho_J J^V, \delta^2),
\]
which is the Gaussian distribution with mean \( \nu + \rho_J J^V \) and variance \( \delta^2 \), we obtain
\[
m = E[e^{J^S} - 1] = \frac{e^{\nu + \delta^2/2} - 1}{1 - \eta \rho_J} - 1,
\]
provided that \( \eta \rho_J < 1 \). In our subsequent discussion, we only require the joint MGF with \( \psi = 0 \). Under the above assumption on \( J^S \) and \( J^V \), and setting \( \psi = 0 \), the parameter functions can be found to be
\[
\begin{align*}
B(\Theta; \tau, q) &= \frac{b(\xi e^{-\zeta \tau} + \xi) - (\phi - \phi^2)(1 - e^{-\zeta \tau})}{(\xi + \varepsilon^2 b)e^{-\zeta \tau} + \xi - \varepsilon^2 b}, \\
\Gamma(\Theta; \tau, q) &= r \phi \tau + \gamma - \frac{k b}{\varepsilon^2} \left[ \xi_+ + 2 \ln \left( \frac{\xi_+ + \varepsilon^2 b}{\varepsilon^2} \right) + \frac{2 \xi}{2 \zeta} \right], \\
\Lambda(\Theta; \tau, q) &= -\lambda(m \phi + 1) \tau + \frac{\lambda e^{\phi + \delta^2 \phi^2/2}}{2} \left[ \frac{k_2}{\kappa^2} - 1 \left( \frac{k_1}{k_3} - \frac{k_2}{k_4} \right) \ln \frac{k_3 e^{-\zeta \tau} + k_4}{k_3 + k_4} \right],
\end{align*}
\]
with \( q = (\phi \ b \ 0 \ \gamma)^T \) and
\[
\begin{align*}
\zeta &= \sqrt{(\kappa - \rho \varepsilon) \hat{\phi}^2 + \varepsilon^2 (\phi - \phi^2)}, \\
\xi_\pm &= \zeta \mp (\kappa - \rho \varepsilon) \phi, \\
k_1 &= \xi_+ + \varepsilon^2 b, \\
k_2 &= \xi_ - \varepsilon^2 b, \\
k_3 &= (1 - \phi \rho_J \eta) k_1 - \eta (\phi - \phi^2 + \xi_ - b), \\
k_4 &= (1 - \phi \rho_J \eta) k_2 - \eta [\xi_+ b - (\phi - \phi^2)].
\end{align*}
\]

Once the joint MGF is available, an effective analytic pricing approach can be constructed to derive the pricing formulas of the various types of discretely sampled generalized variance swaps. As an illustration, we demonstrate how the pricing formula of the vanilla variance swap can be readily derived. Let the sampling dates be denoted by \( 0 = t_0 < t_1 < \cdots < t_N = T \). At the maturity \( T \), the payoff of the vanilla variance swap is defined to be
\[
\frac{A}{N} \sum_{k=1}^{N} \left( \ln \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2 - K,
\]
where \( K \) is the strike price of the variance swap and \( A \) is the annualized factor (say, we take \( A = 252 \) for daily sampling). We write \( \Delta t_k = t_k - t_{k-1} \) and express the time interval \( t_{k-1} - t_0 \) simply as \( t_{k-1} \) since \( t_0 \) is taken to be zero. The pricing problem amounts to finding the fair
strike price $K$ such that the value of the variance swap at initiation is zero. The fair strike price $K$ is then given by

$$ K = E \left[ \frac{A}{N} \sum_{k=1}^{N} \left( \ln \frac{S_{tk}}{S_{tk-1}} \right)^2 \right]. $$

We now show how to evaluate each term in the above summation. Using the known analytic expression of the marginal MGFs, we apply the tower rule in conditional expectation to give

$$ E\left[ \left( \ln \frac{S_{tk}}{S_{tk-1}} \right)^2 \right] = E\left[ \frac{\partial^2}{\partial \phi^2} e^{\phi(X_{tk} - X_{tk-1})} \right] \bigg|_{\phi=0} $$

$$ = \frac{\partial^2}{\partial \phi^2} E\left[ e^{\phi X_{tk}} \bigg| X_{tk-1}, V_{tk-1} \right] e^{-\phi X_{tk-1}} \bigg|_{\phi=0} $$

$$ = \frac{\partial^2}{\partial \phi^2} E\left[ e^{B(\Theta; \Delta t_k, q_1) V_{tk-1}^2 + \Gamma(\Theta; \Delta t_k, q_1) + \Lambda(\Theta; \Delta t_k, q_1)} \right] \bigg|_{\phi=0} $$

$$ = \frac{\partial^2}{\partial \phi^2} e^{B(\Theta; t_{k-1}, q_2) V_0^2 + \Gamma(\Theta; t_{k-1}, q_2) + \Lambda(\Theta; t_{k-1}, q_2)} \bigg|_{\phi=0}, $$

where $q_1 = (\phi 0 0 0)^T$, and

$$ q_2 = \begin{pmatrix} 0 \\ B(\Theta; \Delta t_k, q_1) \\ 0 \\ \Gamma(\Theta; \Delta t_k, q_1) + \Lambda(\Theta; \Delta t_k, q_1) \end{pmatrix}. $$

The fair strike price of the variance swap is then given by

$$ K_V(T, N) = \frac{A}{N} \sum_{k=1}^{N} \frac{\partial^2}{\partial \phi^2} e^{B(\Theta; t_{k-1}, q_2) V_0 + \Gamma(\Theta; t_{k-1}, q_2) + \Lambda(\Theta; t_{k-1}, q_2)} \bigg|_{\phi=0}. $$

Note that the derivation procedure is less involved compared to the pricing approach used by Zhu and Lian (2011).

**Asymptotic limit of vanishing sampling interval**

It would be instructive to examine whether we can deduce the formula for the fair strike price of the continuously sampled variance swap by taking the asymptotic limit as $\Delta t \to 0$, where $\Delta t = \max_k \Delta t_k$, in formula (2.7). By expanding the parameter functions $B$, $\Gamma$ and $\Lambda$ in powers of $\Delta t_k$ and taking $\Delta t \to 0$ subsequently, we manage to obtain the following closed form formula for the fair strike of the continuously sampled variance swap

$$ K_V(T, \infty) = \frac{1}{\kappa} \left\{ \frac{1 - e^{-\kappa T}}{\kappa} V_0 - \frac{\lambda \eta}{\kappa^2} (1 - e^{-\kappa T} - \kappa T) \right. $$

$$ \left. + \lambda \left[ \delta^2 + \rho^2 \eta^2 + (\nu + \rho \eta)^2 \right] T + \frac{\theta}{\kappa} (\kappa T - 1 + e^{-\kappa T}) \right\}. $$

The above formula is in agreement with a similar pricing formula in Sepp (2008). The proof of formula (2.8) is presented in Appendix A.

In the next section, we illustrate how to generalize the above pricing approach to find the pricing formulas for the various types of generalized variance swaps.
3 Generalized variance swaps

Vanilla variance swaps are known to be the appropriate instruments to provide investors with pure volatility exposure. In recent years, a new generation of variance products have been introduced in the financial markets to enhance volatility trading by providing asymmetric bets or hedges on volatility. Given a tenor structure \( \{t_0, t_1, \ldots, t_N\} \) as above, the generalized realized variance over the period \( [t_0, t_N] \) is defined to be

\[
\frac{A}{N} \sum_{k=1}^{N} w_k \left( \ln \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2,
\]

where \( w_k \) is some discrete weight process chosen so as to target a specific form of volatility exposure. In this section, analytic pricing of discretely sampled gamma swaps, corridor variance swaps and conditional variance swaps would be considered. For each class of these exotic variance products, we start with the description of their product nature and potential uses in volatility trading and hedging.

3.1 Gamma swaps

In a gamma swap, the weight \( w_k \) is chosen to be \( \frac{S_{t_k}}{S_{t_0}}, k = 1, 2, \ldots, N \). Accordingly, the terminal payoff of the gamma swap is defined by

\[
\frac{A}{N} \sum_{k=1}^{N} \frac{S_{t_k}}{S_{t_0}} \left( \ln \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2 - K.
\]

The motivation of choosing the weight to be the underlying level is to provide the embedded damping of the large downside variance when the stock price falls close to zero. This feature serves to protect the swap seller from crash risk, thus provides an advantage over the non-capped variance swaps. Another motivation is related to variance dispersion trade, which refers to trading the difference between the realized index volatility and the market-cap weighted sum of the realized volatilities of its constituents. Gamma swaps provides better means to trade dispersion than vanilla variance swaps since the risk associated with changes in weights of the constituent stocks over the life of the variance swap is reduced (Bouzoubaa and Osseiran, 2010).

To find the analytic fair strike price of a discretely sampled gamma swap, we compute the
expectation of a typical term in the floating leg of the gamma swap as follows:

\[
E\left[\frac{S_{tk}}{S_{t_0}}\left(\ln \frac{S_{tk}}{S_{tk-1}}\right)^2\right] = e^{-X_0}E\left[e^{X_{tk}-X_{tk-1}}(X_{tk} - X_{tk-1})^2 e^{X_{tk-1}}\right]
\]

\[
= e^{-X_0}E\left[\frac{\partial^2}{\partial \phi^2}e^{\phi(X_{tk}-X_{tk-1})}X_{tk-1}\right]\bigg|_{\phi=1}
\]

\[
= e^{-X_0}\frac{\partial^2}{\partial \phi^2}E\left[e^{\phi X_{tk}}X_{tk-1}, V_{tk-1}\right]e^{(1-\phi)X_{tk-1}}\bigg|_{\phi=1}
\]

\[
= \frac{\partial^2}{\partial \phi^2}e^{B(\Theta; \Delta t_k, q_2)V_0+\Gamma(\Theta; \Delta t_k, q_1)+\Lambda(\Theta; \Delta t_k, q_1)}\bigg|_{\phi=1},
\]

where \(q_1 = (\phi 0 0 0)^T\), and

\[
q_2 = \begin{pmatrix}
1 \\
B(\Theta; \Delta t_k, q_1) \\
0 \\
\Gamma(\Theta; \Delta t_k, q_1) + \Lambda(\Theta; \Delta t_k, q_1)
\end{pmatrix}.
\]

In the above derivation procedure, we have made use of the analytic form of the joint moment generating function of \(X_t\) and \(V_t\). The fair strike price of the gamma swap is then given by

\[
K_{\Gamma}(T, N) = \frac{A}{N} \sum_{k=1}^{N} \frac{\partial^2}{\partial \phi^2}e^{B(\Theta; \Delta t_k, q_2)V_0+\Gamma(\Theta; \Delta t_k, q_1)+\Lambda(\Theta; \Delta t_k, q_1)}\bigg|_{\phi=1},
\]

which is seen to have no dependence on the initial price level \(S_{t_0}\).

It is well known that the gamma exposure of vanilla variance swaps is insensitive to the underlying level (Demeterifi et al., 1999), a property known as constant cash gamma exposure. Suppose an investor is interested in the gamma exposure in the number of portfolio units rather than the initial cash value of the portfolio, the gamma swaps provide constant share gamma exposure by choosing weights that are set equal to the underlying level at the sampling dates. The proof of this property of constant (almost constant) share gamma exposure for the continuously (discretely) sampled gamma swaps will be shown next.

**Constant share gamma exposure**

We would like to compute the share gamma exposure of an in-progress gamma swap at time
where \( t_{i-1} < t \leq t_i, \ i \geq 1 \). The time-\( t \) value of the gamma swap is given by

\[
V_{\Gamma}^{t,t_N} = e^{-r(t_N-t)} \left\{ \frac{A}{N} \sum_{k=1}^{i-1} \frac{S_{tk}}{S_{t0}} \left( \log \frac{S_{tk}}{S_{tk-1}} \right)^2 + \frac{A}{N} E_t \left[ \frac{S_{tk}}{S_{t0}} \left( \log \frac{S_{tk}}{S_{tk-1}} \right)^2 \right] \right\} + \frac{A}{N} E_t \sum_{k=i+1}^{N} \frac{S_{tk}}{S_{t0}} \left( \log \frac{S_{tk}}{S_{tk-1}} \right)^2 - K_{\Gamma}^{t_0,t_N}
\]

\[
(3.2) \ 
\frac{A}{N} \frac{S_{t_{i-1}}}{S_{t0}} \frac{\partial^2}{\partial \phi^2} \left[ \left( \frac{S_t}{S_{t_{i-1}}} \right)^\phi e^{B(\Theta; t_i-t, \phi_1)V_t + \Gamma(\Theta; t_i-t, \phi_1) + \Lambda(\Theta; t_i-t, \phi_1)} \right] \bigg|_{\phi=1} + \frac{A}{N} \frac{S_{t_i}}{S_{t0}} \sum_{k=i+1}^{N} \frac{\partial^2}{\partial \phi^2} e^{B(\Theta; t_{k-1}-t, \phi_2)V_t + \Gamma(\Theta; t_{k-1}-t, \phi_2) + \Lambda(\Theta; t_{k-1}-t, \phi_2)} \bigg|_{\phi=1} - K_{\Gamma}^{t_0,t_N}
\]

where \( \phi_1 = (\phi \ 0 \ 0 \ 0)^T \), and

\[
\phi_2 = \begin{pmatrix} 1 \\ B(\Theta; \Delta t_k, \phi_1) \\ 0 \\ \Gamma(\Theta; \Delta t_k, \phi_1) + \Lambda(\Theta; \Delta t_k, \phi_1) \end{pmatrix}
\]

The share gamma of the discretely sampled gamma swap is found to be

\[
(3.3a) \ 
\Gamma_S = e^{-r(t_N-t)} \frac{A}{N} \frac{S_t}{S_{t_i-1}} \left[ \left( \log \frac{S_t}{S_{t_{i-1}}} + 1 \right) F(1) + F'(1) \right],
\]

where

\[
F(\phi) = e^{B(\Theta; t_i-t, \phi_1)V_t + \Gamma(\Theta; t_i-t, \phi_1) + \Lambda(\Theta; t_i-t, \phi_1)}.
\]

The dependence of \( \Gamma_S \) on \( S_t \) appears in the term \( \log \frac{S_t}{S_{t_{i-1}}} \). Suppose we take the limit of continuous sampling, \( S_t \to S_t_{i-1} \), we then obtain

\[
(3.3b) \ 
\Gamma_S \to e^{-r(t_N-t)} \frac{A}{N} \frac{S_t}{S_{t_i-1}} \left[ F(1) + F'(1) \right] = e^{-r(t_N-t)} \frac{A}{N} \frac{2}{S_{t0}}.
\]

The above limit has no dependence on \( S_t \), so it verifies the property of constant gamma exposure under continuous sampling. A similar result has been obtained by Jacquier and Slaoui (2010) using a replication argument.

**Fair strike prices of continuously monitored gamma swaps**

By following a similar procedure of taking the limit of vanishing sampling time interval (see Appendix A), we can obtain the following formula for the fair strike price of a continuously monitored gamma swap

\[
(3.4) \ 
K_{\Gamma}(T, \infty) = \frac{1}{T} \left[ \left( V_0 - \frac{\kappa \theta}{\kappa - \rho \varepsilon} - C_2 \right) e^{(r-\kappa+\rho \varepsilon)T} - \frac{1}{r-\kappa+\rho \varepsilon} + \left( \frac{\kappa \theta}{\kappa - \rho \varepsilon} + C_1 + C_2 \right) e^{\frac{1}{r} - 1} \right],
\]

12
where

\[
C_1 = \frac{\lambda e^{\nu + \delta^2/2}}{1 - \rho J} \left[ \left( \nu + \delta^2 + \frac{\rho J \eta}{1 - \rho J \eta} \right)^2 + \delta^2 + \left( \frac{\rho J \eta}{1 - \rho J \eta} \right)^2 \right],
\]

\[
C_2 = \frac{\lambda \eta e^{\nu + \delta^2/2}}{(1 - \rho J \eta)^2 (\kappa - \rho \varepsilon)}.
\]

### 3.2 Corridor variance swaps

A corridor variance swap differs from the vanilla variance swap in that the underlying price must fall inside a specified corridor \((L, U]\) \((L \geq 0, \ U < \infty)\), in order for its squared return to be included in the floating leg of the corridor variance swap. For a discretely sampled corridor variance swap with the tenor \(0 = t_0 < t_1 < t_2 < \cdots < t_N = T\), suppose the corridor is monitored on the underlying price at the old time level \(t_{k-1}\) for the \(k\)th squared log return, the floating leg with the corridor \((L, U]\) is given by

\[
\frac{A}{N} \sum_{k=1}^{N} \left( \ln \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2 1_{\{L < S_{t_{k-1}} \leq U\}}.
\]

Here, \(1_{\cdot} \) denotes the indicator function. Corridor variance swaps with a one-sided bound are also widely traded in the financial markets, where the downside-variance swap and upside-variance swap can be obtained by taking \(L = 0\) and \(U \to \infty\), respectively. As further generalizations, one can choose to have the corridor being monitored on the underlying price at the new time level \(t_k\) (Sepp, 2007) or even at both time levels (Carr and Lewis, 2004).

Corridor variance swaps allow the investors to take their views on the implied volatility skew. Suppose the implied volatility skew is expected to steepen, the investor may benefit from buying a downside-variance swap and selling an upside-variance swap if this view is realized. Also, investors seeking crash protection may buy the downside-variance swap since it can provide almost the same level of crash protection as the vanilla variance swap but at a lower premium.

It suffices to consider the pricing of downside-variance swaps since the payoffs of downside-variance swaps and vanilla variance swaps are sufficient to span all different payoffs of various corridor variance swaps. We would like to find the fair strike price of a downside-variance swap with an upper bound \(U\) whose payoff at maturity \(T\) is given by

\[
\frac{A}{N} \sum_{k=1}^{N} \left( \ln \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2 1_{\{S_{t_{k-1}} \leq U\}} - K.
\]
Let us consider the expectation calculation of a typical term:

\[
E \left[ \left( \ln \frac{S_{tk}}{S_{tk-1}} \right)^2 1_{\{s_{tk-1} \leq U\}} \right]
\]

\[
= E \left[ E \left[ \frac{\partial^2}{\partial \phi^2} e^{\phi(X_{tk} - X_{tk-1})} X_{tk-1}, V_{tk-1} \right] 1_{\{X_{tk-1} \leq \ln U\}} \right] \bigg|_{\phi=0}
\]

\[
= E \left[ \frac{\partial^2}{\partial \phi^2} e^{B(\Theta; \Delta t_k, \mathbf{q}_1) V_{tk-1} + \Gamma(\Theta; \Delta t_k, \mathbf{q}_1) + \Lambda(\Theta; \Delta t_k, \mathbf{q}_1)} 1_{\{X_{tk-1} \leq \ln U\}} \right] \bigg|_{\phi=0}
\]

(3.5)

where \( \mathbf{q}_1 = (\phi \ 0 \ 0)^T \). For \( k = 1 \), the above expectation is readily seen to be

(3.6a) \[
E \left[ \left( \ln \frac{S_t}{S_0} \right)^2 1_{\{s_0 \leq U\}} \right] = \frac{\partial^2}{\partial \phi^2} e^{B(\Theta; \Delta t_1, \mathbf{q}_1) V_0 + \Gamma(\Theta; \Delta t_1, \mathbf{q}_1) + \Lambda(\Theta; \Delta t_1, \mathbf{q}_1)} 1_{\{X_0 \leq \ln U\}} \bigg|_{\phi=0}.
\]

For \( k \geq 2 \), the evaluation of expectation in formula (3.5) requires the representation of the indicator function \( 1_{\{X_{tk-1} \leq \ln U\}} \) to be in terms of inverse Fourier transform. As a result, formula (3.5) can be expressed in terms of a Fourier integral as follows:

(3.6b) \[
E \left[ \left( \ln \frac{S_{tk}}{S_{tk-1}} \right)^2 1_{\{s_{tk-1} \leq U\}} \right] = \frac{\partial^2}{\partial \phi^2} e^{B(\Theta; \Delta t_k, \mathbf{q}_2) V_{tk-1} + \Gamma(\Theta; \Delta t_k, \mathbf{q}_2) + \Lambda(\Theta; \Delta t_k, \mathbf{q}_2)} 1_{\{X_0 \leq \ln U\}} \bigg|_{\phi=0}, \ k \geq 2,
\]

where \( w = w_r + iw_i, u = \ln U \), and

\[
F_k(w) = \frac{\partial^2}{\partial \phi^2} e^{B(\Theta; \Delta t_k, \mathbf{q}_2) V_{tk-1} + \Gamma(\Theta; \Delta t_k, \mathbf{q}_2) + \Lambda(\Theta; \Delta t_k, \mathbf{q}_2)} \bigg|_{\phi=0}, \ k \geq 2,
\]

with

\[
\mathbf{q}_2 = \begin{pmatrix}
-iw \\
B(\Theta; \Delta t_k, \mathbf{q}_1) \\
\Gamma(\Theta; \Delta t_k, \mathbf{q}_1) + \Lambda(\Theta; \Delta t_k, \mathbf{q}_1)
\end{pmatrix}.
\]

The above Fourier integral is regular provided that \( w_i \) is chosen to lie within \((-\infty, 0)\). The proof of Eq. (3.6b) is presented in the Appendix B. The fair strike price of the downside-variance swap is then given by

(3.7) \[
K_D(T, N) = \frac{\partial^2}{\partial \phi^2} e^{B(\Theta; \Delta t_1, \mathbf{q}_1) V_0 + \Gamma(\Theta; \Delta t_1, \mathbf{q}_1) + \Lambda(\Theta; \Delta t_1, \mathbf{q}_1)} 1_{\{X_0 \leq \ln U\}} \bigg|_{\phi=0}
\]

\[
+ \frac{e^{u_1(X_0 - u)}}{\pi} \int_0^\infty \text{Re} \left( e^{-i w_r (X_0 - u)} \sum_{k=2}^N F_k(w_r + iw_i) \right) dw_r.
\]
The evaluation of the Fourier integral in Eq. (3.7) can be effected by adopting the fast Fourier transform (FFT) algorithm. Actually, by following a similar FFT calculation approach as in Carr and Madan (1999), one can produce the fair strike prices for all downside-variance swaps with varying values of the upper bound using one single FFT calculation.

**Fair strike prices for continuously monitored downside-variance swaps**

By taking the asymptotic limit of vanishing sampling time interval, the fair strike price of the continuously sampled downside-variance swaps is given by (see Appendix A)

\[
K_D(T, \infty) = e^{w_i(X_0-u)} \int_0^\infty \int_0^T \text{Re} \left( e^{-iw_r(X_0-u)} \frac{F(w_r + iw_i, t)}{iw_r - w_i} \right) dt \, dw_r,
\]

where

\[
F(w, t) = e^{B^0(-iw,t)V_0 + \Gamma^0(-iw,t) + \Lambda^0(-iw,t) + \lambda \left[ (\nu + \rho J\eta)^2 + \delta^2 + \rho_J^2 \eta^2 \right]},
\]

and the coefficient functions \(B^0(\phi, t_{k-1}), B^1(\phi, t_{k-1}), \Gamma^0(\phi, t_{k-1}), \Gamma^1(\phi, t_{k-1}), \Lambda^0(\phi, t_{k-1})\) and \(\Lambda^1(\phi, t_{k-1})\) are defined in Appendix A [see Eq. (A.3)].

**3.3 Conditional variance swaps**

A conditional variance swap is similar to a corridor variance swap, but they differ in the following two aspects:

(i) The accumulated sum of squared returns is divided by the number of observations \(D\) that the underlying price stays within the corridor instead of the total number of sampling observations \(N\);

(ii) The final payoff to the holder is scaled by the ratio \(D/N\).

Let \(K\) be the strike price of a conditional downside-variance swap and \(K'\) be the strike price of its corridor variance swap counterpart. The holder’s payoff of the conditional downside-variance swap with corridor’s upper bound \(U\) is given by

\[
\frac{D}{N} \left[ A \sum_{k=1}^N \left( \ln \frac{S_{tk}}{S_{tk-1}} \right)^2 \mathbf{1}_{\{S_{tk-1} \leq U\}} - K \right]
\]

\[
= \left[ A \sum_{k=1}^N \left( \ln \frac{S_{tk}}{S_{tk-1}} \right)^2 \mathbf{1}_{\{S_{tk-1} \leq U\}} - K' \right] + \left( K' - K \frac{D}{N} \right),
\]

where \(D = \sum_{k=1}^N \mathbf{1}_{\{S_{tk-1} \leq U\}}\).
The above formula reveals that the conditional variance swap can be decomposed into a corridor variance swap with the same upper bound plus a range accrual note. Since we have shown how to find the fair strike price of a downside-variance swap, it suffices to compute the expected value of the discretely sampled occupation time that the underlying stock price stays below the upper bound $U$. For a typical term in $E[D]$, the expectation calculation involves

$$E[1_{\{s_{k-1} \leq u\}}] = E[1_{\{x_{k-1} \leq u\}}],$$

where $u = \ln U$.

Following similar calculations as in Sec. 3.2, we obtain

$$E[1_{\{x_{k-1} \leq u\}}] = \frac{e^{w_i(X_0-u)}}{\pi} \int_0^\infty \text{Re} \left( e^{-iw_0(X_0-u)} \frac{G_k(w_r + iw_i)}{iw_r - w_i} \right) dw_r, \ k \geq 2,$$

where $w_i \in (-\infty, 0)$ and

$$G_k(w) = e^{B(t_{k-1}, q_1)}V_0 + \Gamma(t_{k-1}, q_1) + \Lambda(t_{k-1}, q_1),$$

with $q_1 = (-iw 0 0 0)^T$. Similarly, the numerical evaluation of the above Fourier integral can be done via the FFT algorithm. Finally, the fair strike of the conditional downside-variance swap is given by

$$K_C(T, N) = K_D(T, N) \left[ \frac{e^{w_i(X_0-u)}}{\pi N} \int_0^\infty \text{Re} \left( e^{-iX_0-u} \sum_{k=2}^{N} G_k(w_r + iw_i) \right) dw_r \right]^{-1}.$$

The payoff in a conditional variance swap counts only the sampling dates at which the realized variance does accumulate (conditional on the underlying price lying within the corridor). Compared to the corridor variance swaps, the conditional variance swaps are structured specifically for investors who would like to be exposed only to volatility risk within a prespecified corridor. In a corridor variance swap, the actual amount of the occupation time that the underlying price falls within the corridor over the whole life of the swap has a significant effect on the profit and loss to its holder. However, the conditional variance swap is immunized from this risk since only the realized variance within the corridor matters. Indeed, the decomposition formula (3.9) shows that the holder of a conditional variance swap receives compensation from the range accrual note when the occupation time attains a lower value leading to a lower payoff in the corridor variance swap counterpart.

**Fair strike prices for continuously monitored conditional variance swaps**

The fair strike price of the continuously sampled conditional downside-variance swap is related to that of the downside-variance swap given in the previous section. The explicit formula is given by

$$K_C(T, \infty) = K_D(T, \infty) \left[ \frac{e^{w_i(X_0-u)}}{\pi T} \int_0^T \int_0^T \text{Re} \left( e^{-iX_0-u} \frac{G(w_r + iw_i, t)}{iw_r - w_i} \right) dt dw_r \right]^{-1},$$

16
where
\[ G(w, t) = e^{B(\Theta; t, q_1)V_0 + \Gamma(\Theta; t, q_1) + \Lambda(\Theta; t, q_1)}, \]

with \( q_1 = (-iw\ 0\ 0\ 0)^T \).

4 Numerical examples

In this section, we report the numerical calculations that have been performed for testing the accuracy of the analytic pricing formulas obtained for the various types of discretely sampled generalized variance swaps. The numerical results obtained from the analytic pricing formulas are compared to those obtained from Monte Carlo simulation. We examine the impact of the sampling frequency of the realized variance on the fair strike prices of gamma swaps, corridor variance swaps and conditional variance swaps, and the convergence of the fair strike prices of these discretely monitored generalized variance swaps to those of their continuously monitored counterparts. We also explore the pricing behavior of these generalized variance swaps with various contractual specifications and varying values of the model parameters, like the correlation coefficient, volatility of volatility and jump intensity. In our numerical examples, we adopt the set of parameter values shown in Table 1 that are calibrated to S&P500 option prices on November 2, 1993 (Duffie et al., 2000). In addition, we take \( r = 0.0319 \), \( S_0 = 1 \), and consider one-year swap contracts so that \( A = N \). Unless otherwise stated, we take \( U = 1 \) as the upper bound in the downside corridor.

<table>
<thead>
<tr>
<th>( \kappa )</th>
<th>3.46</th>
<th>( \nu )</th>
<th>-0.086</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta )</td>
<td>(0.0894)^2</td>
<td>( \eta )</td>
<td>0.05</td>
</tr>
<tr>
<td>( \varepsilon )</td>
<td>0.14</td>
<td>( \lambda )</td>
<td>0.47</td>
</tr>
<tr>
<td>( \rho )</td>
<td>-0.82</td>
<td>( \rho_J )</td>
<td>-0.38</td>
</tr>
<tr>
<td>( \sqrt{V_0} )</td>
<td>0.087</td>
<td>( \delta )</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

Table 1: Model parameter values of the SVSJ model.

4.1 Comparison of numerical results from pricing formulas and Monte Carlo simulation

We would like to verify the analytic pricing formulas obtained for the different generalized variance swaps by comparing the numerical results obtained from the analytic pricing formulas and direct Monte Carlo simulation. In our Monte Carlo simulation, we adopt the well known Euler-Maruyama discretization scheme for the SVSJ model, where the evolution dynamics of
the stochastic processes for $V_t$ and $X_t$ are approximated by

$$
\begin{align*}
V(t_k) &= V(t_{k-1}) + \kappa[\theta - V(t_{k-1})]\delta t + \varepsilon \sqrt{|V(t_{k-1})|}\delta t Z_1 + \sum_{i=1}^{N(\delta t)} J_i^V \\
X(t_k) &= X(t_{k-1}) + \left( r - \frac{V(t_{k-1})}{2} - \lambda m \right) \delta t + \sqrt{|V(t_{k-1})|}\delta t (\rho Z_1 + \sqrt{1-\rho^2} Z_2) + \sum_{i=1}^{N(\delta t)} J_i^S,
\end{align*}
$$

(4.1)

where $\delta t$ is the uniform time step, $J_i^S$ and $J_i^V$ are independently sampled random jump sizes, $Z_1$ and $Z_2$ are a pair of independent standard normal random variables. We choose the time discretization in the Monte Carlo simulation such that the sampling dates of the realized variance form a subset of the discretized time points. Our choice of this simulation scheme is based more on the consideration of ease of the implementation of the Monte Carlo simulation, where the benchmark numerical results are provided for comparison. We relegate the development of more efficient simulation schemes similar to the exact Monte Carlo simulation scheme of Broadie and Kaya (2006) for the SVSJ model to future work. In our Monte Carlo simulation, we divide the one year time span of the life of the generalized variance swaps into $M$ subintervals, where $M$ is taken to be $252 \times 26 = 3276$. The number of simulation paths is taken to be 500,000.

In Table 2, we show the comparison of the numerical values of the fair strike prices of gamma swaps, downside-variance swaps and conditional downside-variance swaps with varying values of the sampling frequency. The standard deviations in the Monte Carlo simulation are given in the parentheses. Again, good agreement between the two sets of numerical results is demonstrated.

<table>
<thead>
<tr>
<th>Sampling frequency</th>
<th>Gamma swaps</th>
<th>Downside variance swaps</th>
<th>Conditional variance swaps</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Analytic formula</td>
<td>Monte Carlo simulation</td>
<td>Analytic formula</td>
</tr>
<tr>
<td>4 (quarterly)</td>
<td>171.0</td>
<td>171.0 (0.3)</td>
<td>110.5</td>
</tr>
<tr>
<td>12 (monthly)</td>
<td>170.0</td>
<td>169.8 (0.2)</td>
<td>101.0</td>
</tr>
<tr>
<td>26 (biweekly)</td>
<td>169.9</td>
<td>169.9 (0.2)</td>
<td>99.7</td>
</tr>
<tr>
<td>52 (weekly)</td>
<td>169.9</td>
<td>169.8 (0.2)</td>
<td>99.2</td>
</tr>
<tr>
<td>252 (daily)</td>
<td>169.8</td>
<td>169.9 (0.2)</td>
<td>99.0</td>
</tr>
</tbody>
</table>

Table 2: Comparison of the numerical values of the fair strike prices of the gamma swaps, downside-variance swaps and conditional downside-variance swaps with varying values of the sampling frequency.
4.2 Rate of convergence of discrete sampling to the asymptotic limit of continuous sampling

Several earlier papers have examined the convergence behavior of the fair strike prices of discretely monitored variance swaps to those of their continuously monitored counterparts as the monitoring time interval tends to zero. The numerical tests performed by Broadie and Jain (2008) reveal linear rate of convergence of vanilla variance swaps under various stochastic volatility models of the asset price process. Under the time-changed Lévy processes, Crosby and Davis (2011) manage to establish mathematically (with certain assumptions) the linear rate of convergence of various generalized variance swaps. However, due to the limitation of their approach, they have not been able to perform a similar analysis for the generalized variance swaps with corridor restriction on the realized variance.

We perform numerical calculations to explore the convergence behavior of the fair strike prices of various discretely monitored generalized variance swaps, including vanilla variance swap, gamma swap, downside-variance swap and conditional downside-variance swap, with varying values of $\Delta t$ under the SVSJ model. In Figures 1(a-d), we show the respective plot of the fair prices of the above four types of generalized variance swaps with varying values of $\Delta t$ (in units of year). The values of the fair strike prices are all presented in variance points, which is the realized variance multiplied by $100^2$. We assume the number of trading days in one year to be 252. It is observed that the vanilla variance swap, downside-variance swap and conditional downside-variance swap exhibit a linear rate of convergence while the gamma swap does not show a similar convergence behavior. This non-linear rate of convergence occurs when the correlation coefficient $\rho$ is chosen to be close to $-1$ (strong leverage effect). Our extensive numerical tests show that the departure from linear rate of convergence is less apparent when $\rho$ is chosen to be positive. The analysis of the convergence behavior of the limit of vanishing sampling time interval under different types of asset price processes is relegated to later works.

4.3 Sensitivity to model parameters

We would like to examine the sensitivity of the fair strike prices of various discretely sampled generalized variance swaps to varying values of the model parameters. First, we consider the impact on the fair strike price of the monthly sampled downside-variance swap with varying values of the corridor’s upper bound. In Sec. 3.2, we consider the discretely sampled downside-variance swaps with the breaching of the downside corridor $(0, U]$ being monitored on the stock price at the old time level. We would like to investigate how the fair strike price may change
with an alternative definition, where the floating leg payoff is defined to be

\[
\frac{A}{N} \sum_{k=1}^{N} \left( \ln \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2 1_{\{S_{t_k} \leq U\}}.
\]

In this new definition, the breaching of the downside corridor for the \(k^{th}\) squared log return is monitored on the stock price at the new time level. For \(k = 1, 2, \cdots, N\), by following a similar procedure as shown in Eq. (3.5), we manage to obtain

\[
E \left[ \left( \ln \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2 1_{\{S_{t_k} \leq U\}} \right] = e^{w_i(X_0-u)} \pi \int_{0}^{\infty} \text{Re} \left( e^{-iw_r(X_0-u)} \frac{F_k(w_r+iw_i)}{iw_r-w_i} \right) dw_r,
\]

where \(w = w_r + iw_i\), \(w_i\) is chosen to lie in \((-\infty, 0)\) as before, \(u = \ln U\), and

\[
F_k(w) = \frac{\partial^2}{\partial \phi^2} e^{B(\Theta; \Delta t_k, q_2) V_0 + \Gamma(\Theta; \Delta t_k, q_2) + \Lambda(\Theta; \Delta t_k, q_2)} \bigg|_{\phi=0},
\]

with \(q_1 = (\phi - iw 0 0 0)^T\) and

\[
q_2 = \begin{pmatrix}
-iw \\
B(\Theta; \Delta t_k, q_1) \\
0 \\
\Gamma(\Theta; \Delta t_k, q_1) + \Lambda(\Theta; \Delta t_k, q_1)
\end{pmatrix}.
\]

In Figure 2, we show the plot of the fair strike price of the monthly sampled downside-variance swap with varying values of the corridor’s upper bound \(U\). The difference in the fair strike prices of the two different types of downside-variance swaps, corresponding to the corridor’s upper bound \(U\) being monitored on the stock price at the old time level or new time level, can be quite substantial when the upper bound is below the current stock price \(S_0\) (set equal to 1). When \(U\) is less than \(S_0\), the chance that the upper bound is breached at the subsequent sampling dates is relatively high. Therefore, the choice of the stock price either at the old or new time level that is used for monitoring becomes more significant when \(U\) is close to but below \(S_0\). The plotted points in Figure 2 reveal the significant impact of the choice of upper bound \(U\) on the fair strike prices of the downside-variance swaps, in particular when \(U\) is chosen close to the current stock price \(S_0\).

Next, we examine the sensitivity of the fair strike price of various discretely sampled swaps on the following model parameters: (i) correlation coefficient \(\rho\), (ii) volatility of volatility \(\varepsilon\), (iii) jump intensity \(\lambda\). The comparison of the fair strike prices of various generalized variance swaps with varying values of the above model parameters are shown in Figure 3 to Figure 5. These three figures reveal the different degrees of impact on the different types of generalized variance swaps and various sampling frequencies with respect to these three model parameters.
Figure 3(a) shows that the fair strike price of the gamma swap increases with increasing correlation coefficient $\rho$ while Figures 3(b,c) show the reverse dependence on $\rho$ for the downside-variance swaps and conditional variance swaps. In the event of volatility running high, the gamma swap assigns lower weights to the realized variance when the underlying price declines in value in view of the negative correlation. This explains the drop in the fair strike price of the gamma swap when $\rho$ becomes more negative. The reverse effect works for the downside-variance swaps and conditional variance swaps since more realized variance will be accumulated when the underlying price falls within the corridor with a negative value of $\rho$. Also, the sampling frequencies are seen to exhibit different forms of impact on the fair strike prices of various types of variance swap contracts. In order for a continuously-sampling approximation to be acceptable, the discrete sampling has to be weekly or at a higher frequency.

As shown in Figures 4(a,b,c), the volatility of volatility $\varepsilon$ gives the opposite effects on the fair strike prices of the gamma swap and variance swaps with a downside corridor. Normally, we expect that a higher $\varepsilon$ would lead to a higher level of accumulation of the realized variance. The plots in Figures 4(b,c) for the downside corridor and conditional variance swaps reveal the increase in the fair strike price with an increase in $\varepsilon$. The reverse dependence of the fair strike prices on $\varepsilon$, as shown in Figure 4(a) for the gamma swap, may be attributed to the choice of a negative value of correlation in the calculations.

Broadie and Jain (2001) and other papers report the strong dependence of the fair strike prices of variance swaps on jumps. The jumps in the underlying price and variance under the SVSJ model can be characterized by a set of jump parameters, including $\lambda$, $\nu$, $\eta$, $\rho_J$ and $\delta$. The jump intensity $\lambda$ is considered to be the most crucial parameter. In our calculations, we take $\nu$ and $\rho_J$ to be negative. This would mean each jump most likely leads to a decline in the underlying price. Actually, a larger value of $\lambda$ leads to a higher chance of crash in the underlying price.

Figure 5(a) shows comparison of the fair strike prices of the vanilla variance swaps and gamma swaps. While both products are highly sensitive to $\lambda$, we see the slope for the variance swap is slightly steeper than that of the gamma swap. On the other hand, the presence of the downside corridor in a downside-variance swap has significant effect on the increase of the fair strike price with an increasing jump intensity $\lambda$. For example, the rate of increase for the downside-variance swap with $U = 1$ is reduced by almost half compared to that of the variance swap. Also shown in Figure 5(b), the fair strike price is highly sensitive to the choice of the upper bound $U$ of the downside corridor. At $S_0 = 1$ and $\lambda = 2$, the fair strike price increases almost by a ratio of 3 when we change $U$ from 0.9 to 1.1.
5 Conclusions

In this paper, we demonstrate an analytic approach of deriving closed form pricing formulas of the discretely sampled generalized variance swaps under the dynamics of stochastic volatility with simultaneous jumps in the underlying price and its variance. The success of the analytic approach relies on the availability of the analytic expression of the joint moment generating functions of the SVSJ model. We manage to derive analytic pricing formulas for the gamma swaps, corridor variance swaps, and conditional variance swaps whose terminal payoffs involve the stochastic occupation time that the underlying price lies within a specified corridor. The semi-analytic pricing formulas for the corridor and conditional variance swaps are expressed in terms of Fourier integrals. The numerical evaluation of these Fourier integral can be performed effectively, thanks to the fast Fourier transform algorithm. We perform numerical evaluation of these pricing formulas for the examination of the impact of sampling frequency on the fair strike prices of the gamma swaps, corridor variance swaps and conditional variance swaps. We also perform various numerical calculations to demonstrate the convergence of the fair strike prices of the discretely monitored generalized variance swaps to those of their continuously monitored counterparts, and the sensitivity of the fair strike price to different sets of model parameters in the SVSJ model. The fair strike prices of the variance swaps are seen to be highly sensitive to the contractual terms in the swap contracts and the choices of model parameter values. Our studies also show that the impact of the sampling frequency on the fair strike price is secondary when compared to the choices of the parameter values in the pricing model.
References


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Appendix A Proof of formulas (2.7), (3.4) and (3.8)

For notational convenience, we write $B_{\Delta t_k}$ as $B(\Theta; \Delta t_k, q_1)$, and similar interpretation for other parameter functions $\Gamma(\Theta; \Delta t_k, q_1)$ and $\Lambda(\Theta; \Delta t_k, q_1)$. When $q_1 = (\alpha \ 0 \ 0 \ 0)^T$, we expand $B_{\Delta t_k}$, $\Gamma_{\Delta t_k}$ and $\Lambda_{\Delta t_k}$ in powers of $\Delta t_k$, where

$$
B_{\Delta t_k} = \frac{1}{2}(\alpha^2 - \alpha)\Delta t_k + O(\Delta t_k^2)
$$

$$
\Gamma_{\Delta t_k} = r\alpha \Delta t_k + O(\Delta t_k^2)
$$

$$
\Lambda_{\Delta t_k} = -\lambda(m\alpha + 1)\Delta t_k + \frac{\lambda e^{\alpha \theta + \delta^2 \alpha^2/2}}{1 - \alpha \rho \eta} \Delta t_k + O(\Delta t_k^2).
$$

Also, we write $B_{t_{k-1}}$ as $B(\Theta; t_{k-1}, q_2)$, and similar notational interpretation for $\Gamma_{t_{k-1}}$ and $\Lambda_{t_{k-1}}$. Now, we expand $B_{t_{k-1}}, \Gamma_{t_{k-1}}$ and $\Lambda_{t_{k-1}}$ in powers of $B_{\Delta t_k}$, $\Gamma_{\Delta t_k}$ and $\Lambda_{\Delta t_k}$, and keep only the linear terms. This gives

$$
\begin{align*}
B_{t_{k-1}} &= \frac{\phi^2 - \phi}{\xi_+ e^{-\zeta t_{k-1}} + \xi_-} \\
&\quad + \left[\frac{\xi_+ e^{-\zeta t_{k-1}} + \xi_-}{\xi_+ e^{-\zeta t_{k-1}} + \xi_-} + (\phi^2 - \phi) \frac{(1 - e^{-\zeta t_{k-1}})\bar{\varepsilon}^2}{\xi_+ e^{-\zeta t_{k-1}} + \xi_-}\right] B_{\Delta t_k} \\
\Gamma_{t_{k-1}} &= r\phi_{t_{k-1}} - \frac{k\theta}{\bar{\varepsilon}^2} \left(\frac{\xi_+ t_{k-1} + 2\ln \frac{\xi_+ e^{-\zeta t_{k-1}} + \xi_-}{2\zeta} + \Gamma_{\Delta t_k} + \Lambda_{\Delta t_k} + \frac{2k\theta}{\xi_+ e^{-\zeta t_{k-1}} + \xi_-} \xi_+ e^{-\zeta t_{k-1}} + \xi_- \right) B_{\Delta t_k} \\
\Lambda_{t_{k-1}} &= \lambda(\bar{\varepsilon}^2 e^{\omega \theta + \delta^2 \alpha^2/2}/J_2 - \phi m - 1) t_{k-1} - 2\eta \lambda \bar{\varepsilon}^2 e^{\omega \theta + \delta^2 \alpha^2/2}/(J_1 J_2) \\
&\quad \left[\ln \frac{I_1 e^{-\zeta t_{k-1}} + I_2}{2\zeta (1 - \phi \rho \eta)} + \frac{(J_1 e^{-\zeta t_{k-1}} - J_2)}{I_1 e^{-\zeta t_{k-1}} + I_2} + \frac{\eta}{1 - \phi \rho \eta}\right] B_{\Delta t_k},
\end{align*}
$$

where $\zeta, \xi_+$ and $\xi_-$ are defined in Eq. (2.6) and

$$
\begin{align*}
I_1 &= (1 - \phi \rho \eta) \xi_+ - \eta (\phi - \phi^2) \\
I_2 &= (1 - \phi \rho \eta) \xi_- + \eta (\phi - \phi^2) \\
J_1 &= (1 - \phi \rho \eta) e^2 - \eta \xi_- \\
J_2 &= (1 - \phi \rho \eta) e^2 + \eta \xi_+.
\end{align*}
$$

in the above expansion procedure, we still maintain first order accuracy with respect to $\Delta t_k$. The second order derivative of $\exp(B_{t_{k-1}} V_0 + \Gamma_{t_{k-1}} + \Lambda_{t_{k-1}})$ [see Eq. (2.7)] with respect to $\alpha$
can be expressed as

$$\frac{\partial^2}{\partial \alpha^2} e^{B_{t_{k-1}}v_0 + \Gamma_{t_{k-1}} + \Lambda_{t_{k-1}}} \bigg|_{\alpha=0}$$

(A.2)

$$= e^{B_{t_{k-1}}v_0 + \Gamma_{t_{k-1}} + \Lambda_{t_{k-1}}} \left( V_0 \frac{\partial^2 B_{t_{k-1}}}{\partial \alpha^2} + \frac{\partial^2 \Gamma_{t_{k-1}}}{\partial \alpha^2} + \frac{\partial^2 \Lambda_{t_{k-1}}}{\partial \alpha^2} \right) + O(\Delta t_k^2).$$

For the variance swap, we set $\phi = 0$ in (A.2). By substituting all the relations between $B_{\Delta t_k}$, $\Gamma_{\Delta t_k}$, $\Lambda_{\Delta t_k}$, $B_{t_{k-1}}$, $\Gamma_{t_{k-1}}$ and $\Lambda_{t_{k-1}}$, and setting $\alpha = 0$, we obtain

$$\frac{\partial^2}{\partial \alpha^2} e^{B_{t_{k-1}}v_0 + \Gamma_{t_{k-1}} + \Lambda_{t_{k-1}}} \bigg|_{\alpha=0} = \left\{ e^{-\kappa t_{k-1}} V_0 + \frac{\lambda \eta}{\kappa} (1 - e^{-\kappa t_{k-1}}) + \lambda [\delta^2 + \rho^2 \eta^2 + (\nu + \rho \eta)^2] \right. \\
+ \theta (1 - e^{-\kappa t_{k-1}}) \bigg\} \Delta t_k + O(\Delta t_k^2).$$

The fair strike price of the variance swap is then given by

$$K_V(T, N) = \frac{1}{T} \sum_{k=1}^{N} \left\{ e^{-\kappa t_{k-1}} V_0 + \frac{\lambda \eta}{\kappa} (1 - e^{-\kappa t_{k-1}}) + \lambda [\delta^2 + \rho^2 \eta^2 + (\nu + \rho \eta)^2] \right. \\
+ \theta (1 - e^{-\kappa t_{k-1}}) \bigg\} \Delta t_k + O(\Delta t^2),$$

which is visualized as the Riemann sum of an integral. Lastly, by taking the limit $\Delta t \to 0$, we obtain the fair strike price as shown in formula (2.7).

For the gamma swap, we set $\phi = 1$ in (A.2). By repeating similar calculations as above, we can obtain the pricing formula (3.4).

For notional convenience, we express the relations in (A.1) in terms of the coefficient functions $B^0(\phi, t_{k-1})$, $B^1(\phi, t_{k-1})$, $\Gamma^0(\phi, t_{k-1})$, $\Gamma^1(\phi, t_{k-1})$, $\Lambda^0(\phi, t_{k-1})$ and $\Lambda^1(\phi, t_{k-1})$ as follows:

$$\begin{cases}
B_{t_{k-1}} = B^0(\phi, t_{k-1}) + B^1(\phi, t_{k-1}) B_{\Delta t_k}, \\
\Gamma_{t_{k-1}} = \Gamma^0(\phi, t_{k-1}) + \Gamma^1(\phi, t_{k-1}) \Gamma_{\Delta t_k} + \Gamma_{\Delta t_k} + \Lambda_{\Delta t_k}, \\
\Lambda_{t_{k-1}} = \Lambda^0(\phi, t_{k-1}) + \Lambda^1(\phi, t_{k-1}) \Lambda_{\Delta t_k}.
\end{cases}
$$

(A.3)

To derive the fair strike price of the continuously sampled downside-variance swap, we set $\phi = -i\omega$ in (A.2). Again, by repeating a similar asymptotic analysis as above, we obtain the fair strike price as shown in formula (3.8).

**Appendix B  Proof of Eq. (3.6b)**

We consider the generalized Fourier transform of the indicator function $1_{\{X_{t_{k-1}} \leq u\}}$ treated as a function of $u$, where

$$\int_{-\infty}^{\infty} 1_{\{X_{t_{k-1}} \leq u\}} e^{-i\omega u} du = \int_{X_{t_{k-1}}}^{\infty} e^{-i\omega u} du = \frac{e^{-1} X_{t_{k-1}} \omega}{i\omega}.$$
Here, the Fourier variable \( w \) is taken to be complex and we write \( w = w_r + iw_i \). Provided that \( w_i \in (-\infty, 0) \), the above generalized Fourier transform exists. By taking the corresponding generalized inverse Fourier transform, we obtain

\[
1_{\{x_{tk-1} \leq u\}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iw} e^{-iw_{tk-1}w} \frac{e^{-iX_{tk-1}w}}{iw} dw_r.
\]

This analytic representation of the indicator function expressed in terms of a generalized Fourier integral is then substituted into Eq. (3.5). By interchanging the order of performing integration with the two operations of differentiation and expectation, we manage to obtain

\[
E \left[ \left( \ln \frac{S_{tk}}{S_{tk-1}} \right)^2 1_{\{S_{tk-1} \leq U\}} \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial^2}{\partial \phi^2} E \left[ e^{-iwX_{tk-1} + B(\Theta,\Delta t_k,q_1)V_{tk-1} + \Gamma(\Theta,\Delta t_k,q_1) + \Lambda(\Theta,\Delta t_k,q_1)} \right] \bigg|_{\phi=0} \frac{e^{iw}}{iw} dw_r
\]

\[
= e^{w_i (X_0 - u)} \pi \int_0^{\infty} \text{Re} \left( e^{-iw_i (X_0 - u)} F_k(w_r + iw_i) \frac{e^{-iw_i w_r}}{iw_r - w_i} \right) dw_r, \ k \geq 2,
\]

as shown in Eq. (3.6b).
Figure 1: Plot of the fair strike of various discretely sampled generalized variance swaps against sampling time interval (in units of year) under the SVSJ model: (a) variance swaps, (b) gamma swaps, (c) downside-variance swaps, (d) conditional downside-variance swaps.
Figure 2: Comparison of the fair strike prices of the monthly sampled downside-variance swap with varying values of the corridor’s upper bound when the breaching of the corridor is monitored on the stock price at the old time level or new time level.
Figure 3: Plot of the fair strike of various discretely sampled generalized variance swaps against correlation coefficient $\rho$ with varying values of the sampling frequency: (a) gamma swaps, (b) downside-variance swaps, (c) conditional downside-variance swaps.
Figure 4: Plot of the fair strike of various discretely sampled generalized variance swaps against vol-vol $\varepsilon$ with varying values of the sampling frequency: (a) gamma swaps, (b) downside-variance swaps, (c) conditional downside-variance swaps.
Figure 5: Comparison of the fair strike prices of the vanilla variance swaps with various generalized variance swaps under varying values of the jump intensity $\lambda$: (a) gamma swaps, (b) downside-variance swaps with different choices of the corridor’s upper bound $U$. 