Valuing employee reload options under time vesting requirement

Min Dai
Department of Mathematics, National University of Singapore, Singapore, 117543
and
Department of Financial Mathematics, Peking University, Beijing 100871, China
E-mail: matdm@nus.edu.sg

Yue Kuen Kwok[1]
Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong, China
E-mail: maykwok@ust.hk

Abstract
Upon the exercise of an employee stock option, the embedded reload provision entitles the holder to receive additional units of new options from the employer. The number of units of new options received is equal to the number of shares tendered as payment of strike and the new strike is set at the prevailing stock price. The reload provision may be subject to time vesting requirement, that is, after each exercise the employee is prohibited from exercising the reload until the end of a vesting period. In this paper, we construct an efficient numerical algorithm that computes the market value of the employee reload options under time vesting requirement. Also, we explore the analytic properties of the price functions and optimal exercise policies of the employee reload options.

1 Introduction

Employee (executive) stock options commonly contain non-traditional features that are not found in other conventional options traded in the financial

[1]Author to whom any correspondence should be addressed.

markets. Johnson and Tian's paper (2000) contains an interesting account of various types of exotic features in employee stock options. The reload provision in employee stock options has been gaining increasing popularity since its first launch in late 1980s. Suppose that at the exercise of an employee stock option prior to expiry, the employee pays the strike of the option with his owned shares. The embedded reload feature entitles the employee to receive one new option for each share tendered for the strike payment, in addition to receiving one share for the option exercised. The new reload option has the same expiration date as the original option but with a new strike price set at the prevailing stock price. Let X be the strike of the original option and S be the prevailing stock price at the exercise moment. Upon exercising the empolyee stock option, the employee receives $1 - X/S$ units of shares and X/S units of new reload option. Furthermore, the new option received may continue to have future reload rights, and the number of allowable reloads can be unlimited or finite. In addition, there may be vesting restrictions on exercising the reload right. The performance vesting requires a minimum level of stock price appreciation before exercise is allowed. As for the time vesting requirement, the first reload can be exercised only after an initial vesting period (say, 6-month period). The new reload options received after each exercise are subject to the same vesting period. More details of the product nature of different types of reload options can be found in the report by Frederic W. Cook & Co. (1998).

It is important for firms to estimate the compensation costs associated with the granting of such reload options to their employees. The valuation of the cost to the employer seems to be quite difficult, as noted by the Financial Accounting Standards Board (see Statement of Financial Accounting Standards No. 123, 1995). Since the employee options are non-transferable and the employees cannot short sell the stocks to hedge the risk, the exercise decision cannot be preference free. Also, the wealth of the holder depends highly on the fortune of the firm. Therefore, a risk averse holder would have higher tendency to exercise the options prematurely in order to guard against potential acute drop in stock price. Heath *et al.* (1999) and Core and Guay (2001) document that early exercise decisions made by employees are strongly dependent on recent stock returns, where employee exercise activity roughly doubles when the stock price exceeds the maximum price attained during the previous year. Since early exercise decisions are influenced by liquidity, risk aversion and recent stock price trends, it seems impossible to derive a general preference free pricing method of finding the cost to the issuer of such reload options. Besides, employee options are essentially warrants, the exercise of which would lead to dilution of stocks. In addition, since option compensation has incentive effects on the employees that drive the stock price, so the stock price process may be affected by the exercise decisions. These features add to the complexity in the valuation procedure of employee stock options.

When the number of reload rights granted is unlimited, Dybvig and Loewenstein (2003) prove by dominance argument that there exists a simple exercise policy - the holder should exercise whenever a new maxima of the stock price is realized. Such optimal policy is independent of the risk preference of the holder. Though the utility value of the cashflows at exercise depends on the utility function adopted by the risk averse holder, the exercise policy remains the same as that of a risk neutral holder. Since the employee reload option can be hedged by the issuer, the expected value of the replicating portfolio discounted under the riskless interest rate should represent the market value of the option. Further, as the employer is aware of such optimal exercise policy adopted by the holder, and he has no constraint on hedging, so the market value of the options also represents the cost to the employer granting such options. These arguments pave the way for the use of the Black-Scholes risk neutral pricing paradigm for valuation of the cost to the issuer of employee options with unlimited reload rights.

Analytic properties of the pricing models for finding the market value of reload options without vesting requirements have been analyzed in our other paper (Dai and Kwok, 2004). Dybvig and Loewenstein (2003) argue that since the valuation problem with time vesting exhibits close proximity to that without time vesting, therefore as an approximation, the market value of the option computed by the Black-Scholes model is taken to be the cost to the employer. In this paper, we use the Black-Scholes pricing paradigm to construct the pricing models of time vesting reload options. We also develop an efficient valuation algorithm for computing the market value of reload option and examine the behaviors of the price functions. Several earlier papers have attempted to solve similar valuation problems of reload options. Hemmer *et al.* (1998) bypass the complexity of valuing a time vesting infinite-reload option through approximating the time vesting requirement by an alternative restriction where exercise is allowed only at preset times. If the time interval between these allowable exercise instants is the same as the time vesting period, then the value of the option under the alternative restriction is always less than that under time vesting since the exercise at preset instants can only be sub-optimal. Saly *et al.* (1999) develop binomial algorithms for finite-reload options and consider the impact of various types of exercise payoff functions. In the trinomial scheme developed by Dybvig and Loewenstein (2003) for valuation of time vesting infinite-reload options, they append an extra index for tracking the vesting time lapsed from the last exercise instant.

The paper is organized as follows. In the next section, we explore some analytic properties of the price functions of reload options. Once the pricing models are well formulated, it becomes quite straightforward to construct efficient binomial scheme for numerical valuation of the option value. In Section 3, we present the results of the numerical experiments that were designed to reveal some of the behaviors of the pricing functions. In particular, we explore the characterization of the optimal exercise policies of the reload options and the impact of the length of the vesting period on the price functions. Conclusive remarks are presented in the last section.

2 Formulation of pricing models

First, we formulate the pricing model that computes the market value of a reload option subject to time vesting requirement. We adopt the risk neutral pricing approach under the Black-Scholes framework. In our pricing model, we suppress the warrant's dilution effects and employees' incentive effects on the stock price. Also, we asume the stock to be non-dividend paying. Under the risk neutral measure, the stochastic process of the stock price S is assumed to follow the Geometric Brownian process with constant volatility σ and drift rate same as the riskless interest rate r.

Let $V_n(S, t, u; X)$ denote the price function of the employee option granted with n reload rights (n can be infinite), where t is the calendar time, u denotes the remaining time required to satisfy the time vesting requirement and X is the strike price. For convenience, we take the original strike price X of the reload option to be unity. Also, we let δ denote the length of the time vesting period. Obviously, we have $0 \leq u \leq \delta$; and the (weak) path dependence nature of the time vesting feature is exhibited by the additional state variable u. Note that once u decreases to the value 0, the holder can exercise the reload at any time he wishes. Right after an exercise of the reload right, u is reassigned the value δ. For the new reload option, its strike price is set to be the prevailing stock price, the maturity date is the same as that of the original option and the number of reload rights is reduced by one.

The price function $V_n(S, t, u; X)$ observes the linear homogeneity property when the strike price equals the stock price, that is

$$V_n(S, t, u; S) = SV_n(1, t, u; 1). \tag{1}$$

Suppose there are n reload rights outstanding, $n \geq 1$, the payoff to the holder upon the exercise of the reload right is given by the sum of the call payoff $S - 1$ and $\frac{1}{S}$ units of new options reloaded (the strike price X is set to be unity). By observing the linear homogeneity property [see Eq. (1)], the payoff upon exercise can be expressed as

$$S - 1 + \frac{1}{S} V_{n-1}(S, t, \delta; S) = S - 1 + V_{n-1}(1, t, \delta; 1). \tag{2}$$

In our subsequent exposition, for the sake of simplicity, we suppress the strike price X in the price function V_n. Unless otherwise specified, the value of the strike price is taken to be unity.

For the complete specification of the reload policy subject to time vesting requirement, it is necessary to specify clearly the reload policy during the last δ-period before maturity. When the reload right is exercised during the last δ-period, we assume that there will be no new option received by the employee (since the vesting period of the new option reloaded would go beyond the maturity date). Hence, the employee reload option essentially becomes an American call option during the last δ-period of its life [see Eq. (6)]. Furthermore, suppose the holder exercises the last reload right at time earlier than δ-period before maturity, the new option received is an American call option but still subject to time vesting constraint. Only when the time vesting period has lapsed does the reload option become the usual American call option. If we take $n = 0$ to represent the status of "no further reload right", and when $u = 0$, we have

$$V_0(S, t, 0) = C_A(S, t), \tag{3}$$

where $C_A(S, t)$ is the price function of an American call with the same strike and maturity.

2.1 Path dependence nature of time vesting

The price function $V_n(S, t, u)$ apparently contains an extra vesting time variable u, similar to the excursion time variable in Parisian options. The path dependence nature of the time vesting feature is reflected by the property that u is set equal δ at the grant date and reset to δ after each exercise of the reload right. Fortunately, there exists a simple calculation procedure to obtain the solution $V_n(S,t,u), 0 < u \leq \delta$, once the solution $V_n(S,t+u,0)$ is available. As there is no reload allowed within the time interval $[t,t+u)$, so the reload option behaves like a compound option. Over the interval $[t,t+u)$, the price function $V_n(S,t,u)$ satisfies the Black-Scholes equation, subject to "known" payoff $V_n(S,t+u,0)$ at the future time $t+u$. Let $\psi(S_{t+u}|S_t = S)$ denote the transition density function of the stock price at time $t+u$, conditional on the stock price at time t equals S. It is straightforward to deduce the following relation:

$$V_n(S,t,u) = \int_0^\infty e^{-ru}\psi(S_{t+u}|S_t = S)V_n(S_{t+u}, t+u, 0)\, dS_{t+u}. \qquad (4)$$

This is like an integral transform on $V_n(S,t+u,0)$ with the kernel function $e^{-ru}\psi(S_{t+u}|S_t)$. For notational convenience in later discussion, we define the integral transform operator F_u^S by

$$\begin{aligned} V_n(S,t,u) &= F_u^S[V_n(\widetilde{S}, t+u, 0)] \\ &= \int_0^\infty e^{-ru}\psi(\widetilde{S}|S)V_n(\widetilde{S}, t+u, 0)\, d\widetilde{S}, \end{aligned} \qquad (5)$$

where $S_{t+u} = \widetilde{S}$ and $S_t = S$. Accordingly, we may rewrite the exercise payoff in Eq. (2) as $S - 1 + F_\delta^S[V_{n-1}(\widetilde{S}, t+\delta, 0)]\Big|_{S=1}$. Now, the function $V_n(S,t,u), u > 0$, is related directly to $V_n(S,t,0)$ by the relation in Eq. (4). Next, we concentrate our discussion on the properties of the price function at $u = 0$.

2.2 Analytic properties of $V_n(S,t,0)$

Let t_0 denote the grant date of the reload option, then $V_n(S,t,0)$ is defined over the time period $[t_0 + \delta, T]$. Assume that at the current time, the time vesting requirement has been satisfied so that $u = 0$. Recall that during the last δ-period before maturity, the remaining reload rights held by the employee has zero value. Hence, a n-reload option is the same as an American call option. We have

$$V_n(S,t,0) = V_0(S,t,0) = C_A(S,t) \quad \text{for } n \geq 1 \text{ and } t \in (T-\delta, T]. \qquad (6)$$

Alternatively, it is seen that the holder of an employee option with n reload rights outstanding can exercise all n reloads only when the time to expiry is not less than $n\delta$. Deductively, we then have

$$V_n(S,t,0) = \begin{cases} V_0(S,t,0) & t \in (T-\delta, T] \\ V_1(S,t,0) & t \in (T-2\delta, T-\delta] \\ \vdots & \vdots \\ V_k(S,t,0) & t \in (T-(k+1)\delta, T-k\delta]. \end{cases} \tag{7}$$

Equivalently, we may write

$$V_n(S,t,0) = V_k(S,t,0) \quad \text{when} \quad t \in (T-(k+1)\delta, T-k\delta], \tag{8}$$

for $n \geq k, k = 0, 1, 2, \cdots$.

Due to the time vesting feature, one reload right will be forfeited automatically when the calendar time advances towards maturity by one δ-period. Note that relation (2.8) remains valid when $n \to \infty$, that is, the employee option has unlimited number of reload rights.

2.3 Procedures to solve for $V_n(S, t_0, \delta)$

Given the grant date t_0, length of vesting period δ, maturity date T and number of allowable reloads n, we outline the procedures that solve for $V_n(S, t_0, \delta)$. From Eq. (2.5), we have $V_n(S, t_0, \delta) = F_\delta^S[V_n(\widetilde{S}, t_0 + \delta, 0)]$ so that it suffices to illustrate the procedures for solving $V_n(S, t_0 + \delta, 0)$. We consider two separate cases, depending on whether the actual maximum number of reloads allowable for a given time to expiry is less than n or same as n.

1. $t_0 + \delta > T - n\delta$

   In this case, the time to expiry of the option at time $t_0 + \delta$ is less than $n\delta$. Let $\widehat{k}$ denote the actual allowable maximum number of reloads. It is seen that $\widehat{k}$ is given by the largest integer less than $\dfrac{T-(t_0+\delta)}{\delta}$, thus $\widehat{k}$ is less than n when $t_0 + \delta > T - n\delta$. According to Eq. (7), we have $V_n(S, t_0 + \delta, 0) = V_{\widehat{k}}(S, t_0 + \delta, 0)$.

   To solve for $V_{\widehat{k}}(S, t_0 + \delta, 0)$, we first solve for $V_0(S, t, 0)$ over the last time interval $(T - \delta, T]$. This is an optimal stopping problem with exercise payoff $S - 1$. Next, we solve for $V_1(S, t, 0)$ over $(T - 2\delta, T - \delta]$. By continuity of the price function, we have $V_1(S, T - \delta, 0) =$

$V_0(S, T-\delta, 0)$. As part of the procedure, it is also necessary to obtain $V_0(S,t,\delta)$ over $(T-2\delta, T-\delta]$ using Eq. (5). This is because the exercise payoff associated with the one-reload option is $S-1+V_0(1,t,\delta)$. Deductively, we proceed the calculations backward in time over each successive time interval of width δ. Sequentially, we solve for $V_2(S,t,0)$ over $[T-3\delta, T-2\delta]$, $V_3(S,t,0)$ over $(T-4\delta, T-3\delta]$, etc., until we obtain $V_{\widehat{k}}(S, t_0+\delta, 0)$ over the last time interval $[t_0+\delta, T-\widehat{k}\delta]$.

2. $t_0 + \delta < T - n\delta$.

   At the first glance, we may proceed as above until we have solved for $V_j(S,t,0)$ over the successive time intervals $(T-(j+1)\delta, T-j\delta], j = 1, 2, \cdots, n-1$. Over the time interval $[t_0+\delta, T-n\delta]$, we solve for $V_n(S,t,0)$ based on the exercise payoff $S-1+V_{n-1}(1,t,\delta)$. A slight complication is encountered here. In order to obtain $V_{n-1}(1,t,\delta)$ over $[t_0+\delta, T-n\delta]$, it is necessary to have the solution for $V_{n-1}(1,t,0)$ over $[t_0+2\delta, T-(n-1)\delta]$. Therefore, one has to solve for $V_{n-1}(S,t,0)$ over a wider time interval, not just limited to the δ-period time interval $(T-n\delta, T-(n-1)\delta]$. Proceeding backward successively to $V_{n-2}(S,t,0), \cdots, V_1(S,t,0)$, we first solve for $V_j(S,t,0), j = 1, 2, \cdots, n-1$, over the corresponding wider time interval $[t_0+(n-j+1)\delta, T-j\delta]$.

2.4 Construction of the numerical scheme

Once the analytic properties of the pricing model have been well understood, it becomes quite straightforward to construct the numerical scheme for valuation of the price function of the reload options. We use the standard binomial tree to simulate the stochastic movement of the stock price. The optimal stopping problem can be solved by the dynamic programming procedure of comparing the continuation value and the exercise payoff at each binomial node. Let T denote the maturity date of the reload option. Let m denote the number of time steps from option's maturity, j be the number of net upward jumps in stock price in the binomial random walk. Let $S_{j,m}$ denote the stock price at node (j,m) in the binomial tree and Δt denote the time step. Let $V_{j,m}^n$ and $\overline{V}_{j,m}^n$ denote the numerical approximation to $V_n(S_{j,m}, T-m\Delta t, 0)$ and $V_n(S_{j,m}, T-m\Delta t, \delta)$, respectively, at node (j,m). The exercise payoff at node (j,m) is given by $\overline{V}_{j_0,m}^{n-1} + S_{j,m} - 1$. Here, j_0 corresponds to the node index where the stock price equals the strike price (taken to be unity). The dynamic programming procedure in the binomial calculations is given by

$$V_{j,m}^{n}=\max(e^{-r\Delta t}(p_u V_{j+1,m-1}^{n}+p_d V_{j-1,m-1}^{n}),\overline{V}_{j_0,m}^{n-1}+S_{j,m}-1). \qquad (9)$$

Here, the probability parameters in the binomial scheme are given by (Cox *et al.*, 1976)

$$p_u=\frac{e^{r\Delta t}-\overline{d}}{\overline{u}-\overline{d}} \quad \text{and} \quad p_d=1-p_u, \qquad (10)$$

where $\overline{u}=1/\overline{d}=e^{\sigma\sqrt{\Delta t}}$.

To compute $V_{j,m}^{n}$, it is necessary to proceed stepwise, starting from the numerical calculations of $V_{j,m}^{0}, V_{j,m}^{1}, \cdots$ successively according to the procedures outlined in Sec. 2.3. When $n=0$, according to Eqs. (3) and (6), we should set $V_{j,m}^{0}$ equal the American call option value.

Numerical calculation of $\overline{V}_{j,m}^{n}$

Suppose the width of the time vesting period δ corresponds to d time steps, that is, $d\ \Delta t=\delta$. The numerical valuation of $\overline{V}_{j,m}^{n}$ from the known solution of $V_{j,m-d}^{n}$ amounts to the numerical valuation of the expectation integral defined in Eq. (5). To obtain $\overline{V}_{j,m}^{n}$, we perform the usual binomial calculations backward in time over d time steps from the $(m-d)^{\text{th}}$ time level to the m^{th} time level. These binomial calculations are performed with the exclusion of the dynamic programming procedure, and the available numerical solution $V_{j,m-d}^{n}$ is used as known terminal values in the backward induction calculations.

Numerical valuation of infinite-reload options

The numerical valuation of the market value of the infinite-reload option $V_{j,m}^{\infty}$ can be effectively organized as follow.

1. In the first d steps, $0<m\leq d$, we compute the American call option value $V_{j,m}^{0}$.

2. For the next d steps, $d<m\leq 2d$, we first compute $\overline{V}_{j_0,m}^{0}$ by performing usual backward binomial calculations (without the dynamic programming procedure) with known terminal values $V_{j,m-d}^{0}$, then subsequently compute $V_{j,m}^{1}$ using Eq. (9).

3. In general, in the $(k+1)^{\text{th}}$ subinterval where $kd < m \leq (k+1)d$, we compute $\overline{V}_{j,m}^{k-1}$ and $V_{j_0,m}^{k}$ in a similar sequential order as outlined in Step (2).

Compared to the numerical algorithm proposed by Dybvig and Loewenstein (2003), our algorithm does not have to append an extra index (or dimension) to track the path dependence nature of the time vesting requirement. The successful design of a more efficient numerical scheme is achieved through better understanding of the pricing properties of the reload options.

3 Numerical results

Through the numerical calculations of the reload option values, we would like to examine the following issues.

1. Comparison of numerical accuracy and rate of convergence of the Dybvig-Loewenstein algorithm with our algorithm.

2. Analysis of the sensitivity of the reload option price function with respect to volatility, length of vesting period and number of reload rights granted.

3. Exploration of optimal exercise policies of infinite-reload options.

Throughout our sample calculations, we assume that all reload options are subject to time vesting requirement and the underlying stock is non-dividend paying. When the stock is dividend paying, either as discrete dividends or continuous dividend yield, we can use the well known standard procedures to modify the binomial tree for the inclusion of the dividend effects into the stock price dynamics. Once the appropriate binomial tree has been available, all other procedures in the numerical algorithm outlined in Sec. 2.4 remain the same.

Comparison of numerical accuracy

First, we would like to compare numerical accuracy and rate of convergence of Dybvig-Loewenstein's algorithm with our algorithm in the numerical valuation of an infinite-reload option. The parameter values used in the pricing model of the infinite-reload option are: $r = 0.05, X = S = 1$ and $\tau = 10$.

Here, τ denotes the remaining life of the reload option. Dybvig and Loewenstein also computed the upper bound and lower bound to the option value. The upper bound is given by the value of the infinite-reload option that allows continuous exercise. The lower bound is the value of the infinite-reload option with pre-determined exercise instants, where the interval between successive exercise instants equals the length of the vesting period.

Table 1 lists numerical option values obtained using Dybvig-Loewenstein's algorithm and our algorithm with varying values of total number of time steps N, volatility σ and length of vesting period δ. We used the numerical results obtained by taking $10,000$ time steps in the binomial calculations as the asymptotic estimate of the option value at infinite number of time steps. Though the asymptotic option values obtained from both algorithms agree very well (within 0.05% accuracy), the results obtained from our algorithm invariably demonstrate much faster rate of convergence. Our algorithm gives numerical option values that achieve 0.1% numerical accuracy using only 200 time steps, while Dybvig-Loewenstein's algorithm normally requires $4,000$ time steps in order to achieve 0.5% numerical accuracy.

σ	δ	Dai-Kwok			
		$N = 200$	$N = 400$	$N = 800$	$N = 10,000$
0.3	0.5	0.6323	0.6326	0.6327	0.6328
0.4	0.5	0.7172	0.7176	0.7177	0.7178
0.3	1.0	0.6172	0.6175	0.6175	0.6176
0.4	1.0	0.7019	0.7022	0.7022	0.7023

σ	δ	Dybvig-Loewenstein			lower bound	upper bound
		$N = 2000$	$N = 4000$	$N = \infty$		
0.3	0.5	0.6296	0.6302	0.6326	0.6285	0.6676
0.4	0.5	0.7125	0.7140	0.7176	0.7126	0.7524
0.3	1.0	0.6140	0.6151	0.6178	0.6144	0.6676
0.4	1.0	0.6970	0.6985	0.7021	0.6966	0.7524

Table 1 Comparison of numerical option values obtained from Dybvig-Loewenstein's algorithm and our algorithm for valuation of infinite-reload options. The parameter values of the reload option are: $r = 0.05, X = S = 1$ and $\tau = 10$. Here, N denotes the total number of time steps used in the numerical calculations, and the option value at $N = \infty$ refers to the asymptotic option value at infinite number of time steps.

Pricing behaviors of reload options

We examine the sensitivity of the price of multi-reload options with respect to the number of reload rights granted, volatility σ and length of vesting period δ. In Table 2, we list the numerical option values of one-reload, two-reload, three-reload and infinite-reload options. Other parameter values used in the calculations are: $r = 0.05, X = S = 1, \tau = 10$, and we used $1,000$ time steps in the numerical calculations. The numerical results reveal that the option price function is an increasing function of number of reload rights and σ, and a decreasing function of δ. The option values are not highly sensitive to δ when the number of reload rights is small.

σ	δ	one-reload	two-reload	three-reload	infinite-reload
0.3	0.5	0.5710	0.5936	0.6068	0.6328
0.3	1.0	0.5703	0.5918	0.6038	0.6176
0.4	0.5	0.6542	0.6787	0.6926	0.7178
0.4	1.0	0.6534	0.6768	0.6891	0.7023

Table 2 The entries are the option values of one-reload, two-reload, three-reload and infinite-reload options with different values of volatility σ and length of time vesting period δ. The parameter values used in the calculations are: $r = 0.05, X = S = 1$ and $\tau = 10$.

We would like to examine in more details the sensitivity of the option value of infinite-reload options with respect to the length of the time vesting period δ. In Figure 1, we plot the price function (evaluated at $S = 1$ and $r = 0.05$) of a 10-year infinite-reload option against δ. The plots agree with the financial intuition that the price function of an infinite-reload option is monotonically decreasing with increasing length of the vesting period and has a higher value with increasing volatility level. With moderate stock price volatility of 30%, the 10-year infinite-reload option value can drop by more than 10% when δ increases from zero to 2 years.

In Figure 2, we plot the price functions of American call option (dashed curve), one-reload option (dot-dashed curve), two-reload option (dotted curve) and infinite-reload option (solid curve), against time to expiry τ, $\tau = T - t$. Other parameter values used in the calculations are: $r = 0.05, \sigma = 0.3, \delta = 0.5, S = 1.243$ and $X = 1$. One observes that the price function of the American call option and infinite-reload option can differ quite significantly even at moderate value of time to maturity τ. Also, the price function of the one-reload opiton is seen to exhibit a jump in value at $\tau = \delta$. This is because the exercise payoff of the one-reload option at $\tau = \delta^+$ is $S - 1 + V_0(1, T - \delta^+, \delta)$ and the payoff reduces to $S - 1$ at $\tau = \delta^-$. The jump in the exercise payoff across $\tau = \delta$ leads to a jump in the price function. Similar argument can be extended to the two-reload and infinite-reload options, so the price function exhibits jumps at $\tau = \delta$ and $\tau = 2\delta$ for the two-reload option and at $\tau = k\delta, k = 1, 2, \cdots$, for the infinite-reload option.

Optimal exercise policies of infinite-reload options

Besides the calculations of the option values, we would like to explore the optimal exercise policies of time vesting infinite-reload options. For usual American call option, the optimal exercise policy dictates that the call is never exercised prematurely when the underlying stock is non-dividend paying. When the stock pays a continuous dividend yield, the call option should be optimally exercised when the stock price rises beyond some critical level $S^*(\tau)$. For reload options, even when the stock is non-dividend paying, our calculations reveal that there exists critical stock price above which the reload option should be exercised optimally. Let $S_n^*(\tau)[S_\infty^*(\tau)]$ denote the critical stock price at time to expiry τ for a n-reload [infinite-reload] option. In our sample calculations, depending on the parameter values chosen in the infinite-reload option model, we found that $S_\infty^*(\tau)$ may exist for all times or only over certain time intervals. Some interesting patterns of behaviors of $S_\infty^*(\tau)$ are shown in Figures 3 and 4.

The plot of $S_\infty^*(\tau)$ against τ in Figure 3 is obtained by choosing the following set of parameter values in the time vesting infinite-reload option: $r = 0.08, \sigma = 0.1, \delta = 0.5$ and $X = 1$. For $\tau \in [0, 0.5]$, $S_\infty^*(\tau)$ does not exist indicating that the reload option is never optimally exercised. This is because the reload option becomes an American call option during its last δ-period before maturity; and any American call on non-dividend paying stock is never exercised. For $\tau \in [\delta, 2\delta) = [0.5, 1.0)$, the infinite-reload option has essentially one reload right outstanding. Our numerical calculations show that there exists an interval $[\tau_1^*, 2\delta)$ such that $S_\infty^*(\tau)$ does not exist. In the next δ-interval, there also exists some range of value of τ where $S_\infty^*(\tau)$ is not defined. Such phenomena are not too surprising since similar phenomena also occur in finite-reload options without time vesting [see Dai and Kwok (2004)]. Indeed, a necessary condition for the existence of $S_\infty^*(\tau)$ is that the temporal rate of change of the expectation of the discounted value of the payoff upon exercising the reload right is negative. The non-existence of $S_\infty^*(\tau)$ reflects the fact that the above condition is not satisfied over some time interval.

In another set of sample calculations with the following parameter values for the infinite-reload option model: $r = 0.1, \sigma = 0.3, \delta = 0.5$ and $X = 1$, the critical stock price $S^*_\infty(\tau)$ exists when $\tau > \delta$, but exhibits jumps at $\tau = k\delta, k = 2, 3, \cdots$. In Figure 4, we plot the critical stock price $S^*_n(\tau)$, for $n = 1, 2, 3$ and ∞, against τ. Since $V_\infty(S, t, 0) = V_k(S, t, 0)$ for $\tau \in [k\delta, (k + 1)\delta)$, we expect that

$$S^*_\infty(\tau) = S^*_k(\tau) \quad \text{for} \quad \tau \in [k\delta, (k + 1)\delta). \tag{11}$$

Figure 4 reveals that $S^*_\infty(\tau)$ consists of piecewise continuous curves over each δ-interval and it possesses the following jump properties:

$$S^*_\infty((k\delta)^+) = S^*_k((k\delta)^+) < S^*_{k-1}((k\delta)^-) = S^*_\infty((k\delta)^-), \quad k = 1, 2, \cdots \tag{12}$$

that is, $S^*_\infty(\tau)$ exhibits discrete drops in value across time instants corresponding to $\tau = k\delta, k = 1, 2, \cdots$. Such phenomena agree with the financial intuition that the option holder is willing to exercise the reload right at a lower critical stock price as τ moves across the critical time instant $k\tau$ since the genuine allowable number of reload rights decreases by one. Also, our calculations reveal that the amount of discrete jump of $S^*_\infty(\tau)$ diminishes as τ increases. Interestingly, it is apparent that $S^*_\infty(\tau)$ tends asymptotically to some finite value as $\tau \to \infty$.

Lastly, we would like to explore in more details how $S^*_\infty(\infty)$ depends on δ and σ. In Figure 5, we plot $S^*_\infty(\infty)$ against δ for varying values of σ. Other parameter values used in the calculations are the same as those for Figure 4. With a longer time vesting period δ, the holder becomes more conservative to use the reload rights; so $S^*_\infty(\infty)$ increases with increasing δ. Also, like most other call options with the early exercise right, $S^*_\infty(\infty)$ increases with increasing volatility level of the stock price.

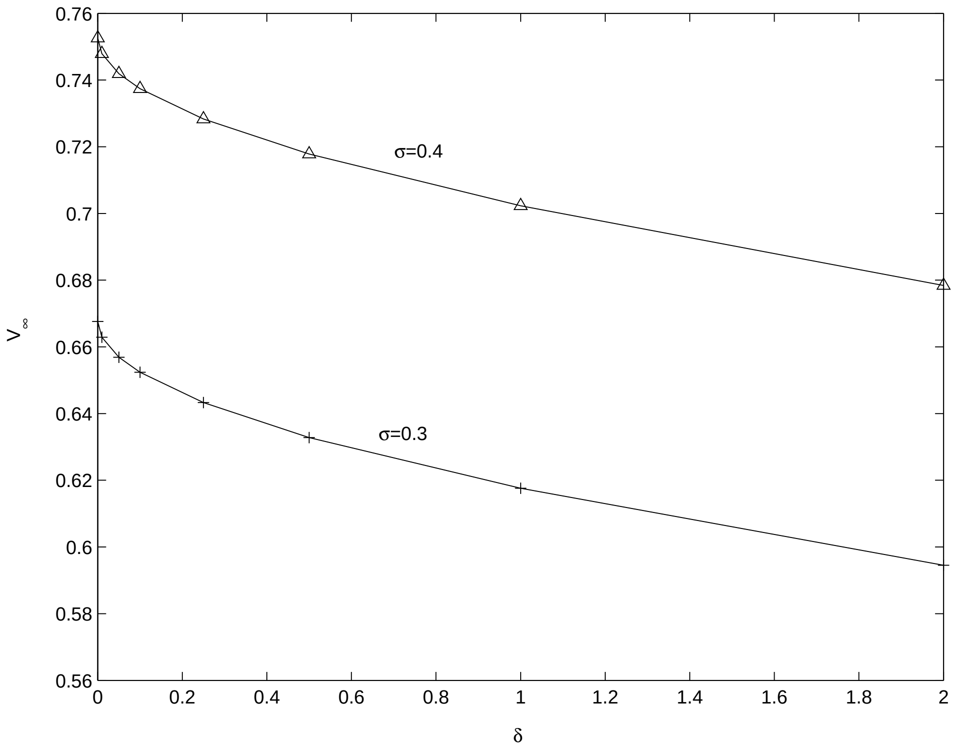

Figure 1 We plot the price function V_∞ of the infinite-reload option against the length of the vesting period δ. The parameter values used in the calculations are: $r = 0.05, S = 1, X = 1, \tau = 10$. We observe that the price function is monotonically decreasing with increasing δ and increasing with increasing volatility σ.

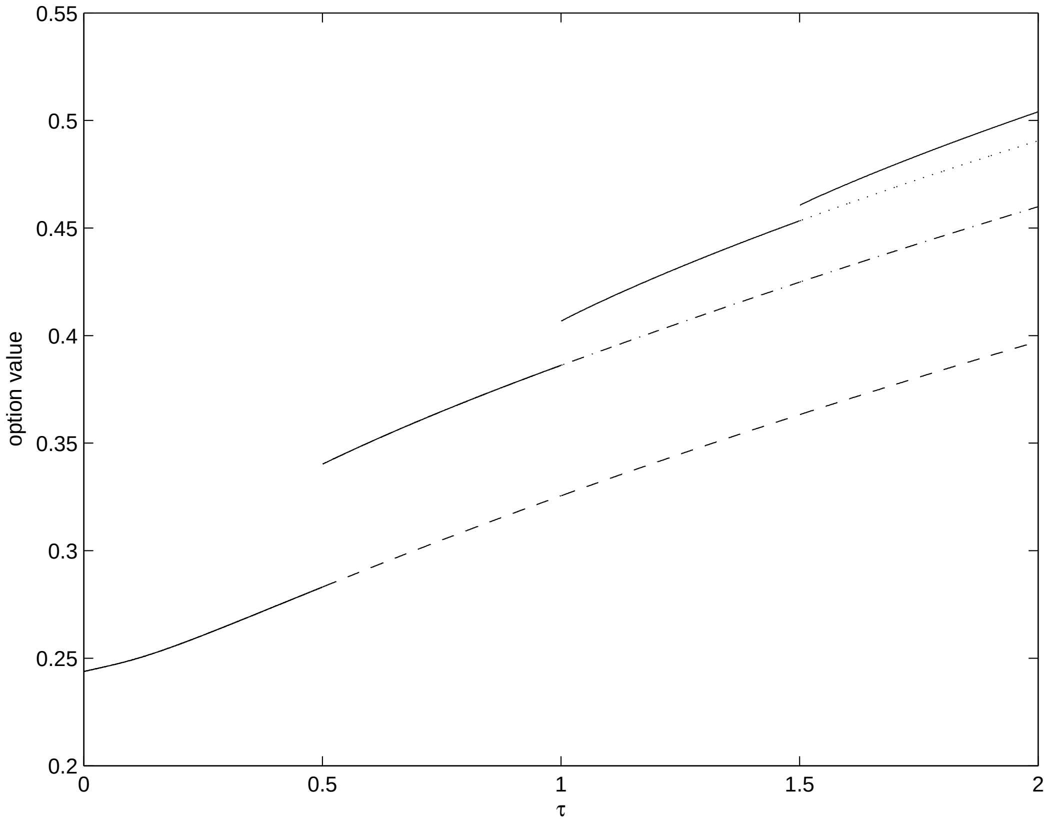

Figure 2 The price functions of American call option (dashed curve), one-reload option (dot-dashed curve), two-reload option (dotted curve) and infinite-reload option (solid curve) are plotted against time to expiry τ. The parameter values used in the calculations are: $r = 0.05, \sigma = 0.3, \delta = 0.5, S = 1.243$ and $X = 1$. The price function of the n-reload option has jumps at $\tau = \delta, 2\delta, \cdots, n\delta$, where $n = 1, 2, \cdots, \infty$.

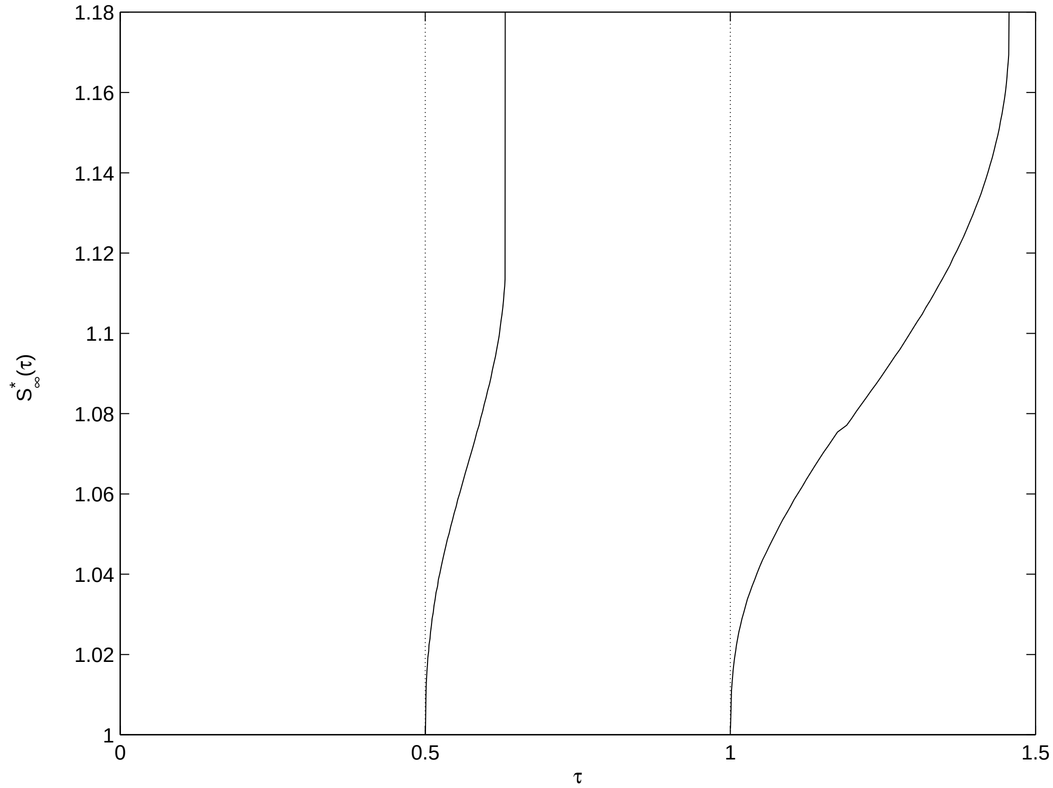

Figure 3 The critical stock price $S^*_\infty(\tau)$ of an infinite-reload option is plotted against time to expiry τ. The parameter values used in the calculations are: $r = 0.08, \sigma = 0.1, \delta = 0.5$ and $X = 1$. Within some time intervals, $S^*(\tau)$ is not defined, signifying that it is never optimal to exercise the reload feature during these time intervals.

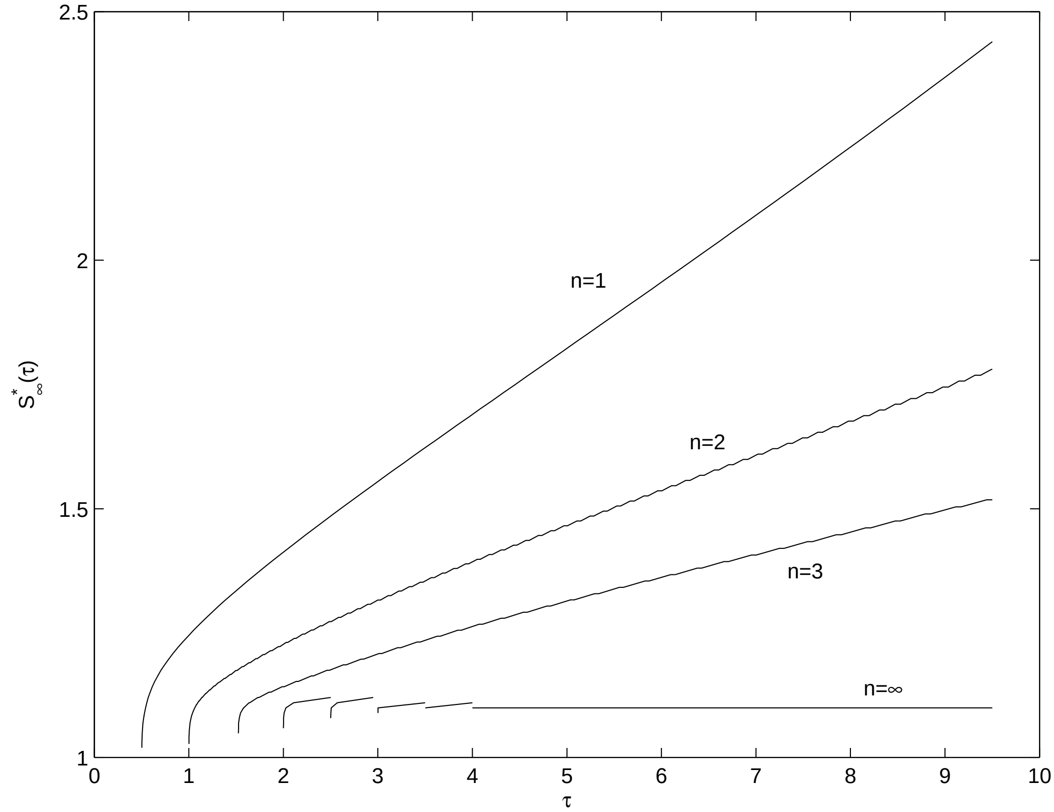

Figure 4 The critical stock price $S^*_\infty(\tau)$ for an infinite-reload option (labelled "$n = \infty$") is plotted against time to expiry τ. The plots of $S^*_n(\tau), n = 1, 2, 3,$ for the corresponding one-reload option (labelled "$n = 1$"), two-reload option (labelled "$n = 2$") and three-reload option (labelled "$n = 3$") against τ are also included for comparison. The parameter values used in the calculations are: $r = 0.05, \sigma = 0.3, \delta = 0.5$ and $X = 1$. Note that $S^*_n(\tau) = S^*_j(\tau)$ when $\tau \in [j\delta, (j+1)\delta), n \geq j, j = 0, 1, 2, \cdots$. Accordingly, $S^*_\infty(\tau)$ consists of piecewise continuous curves over each δ-interval.

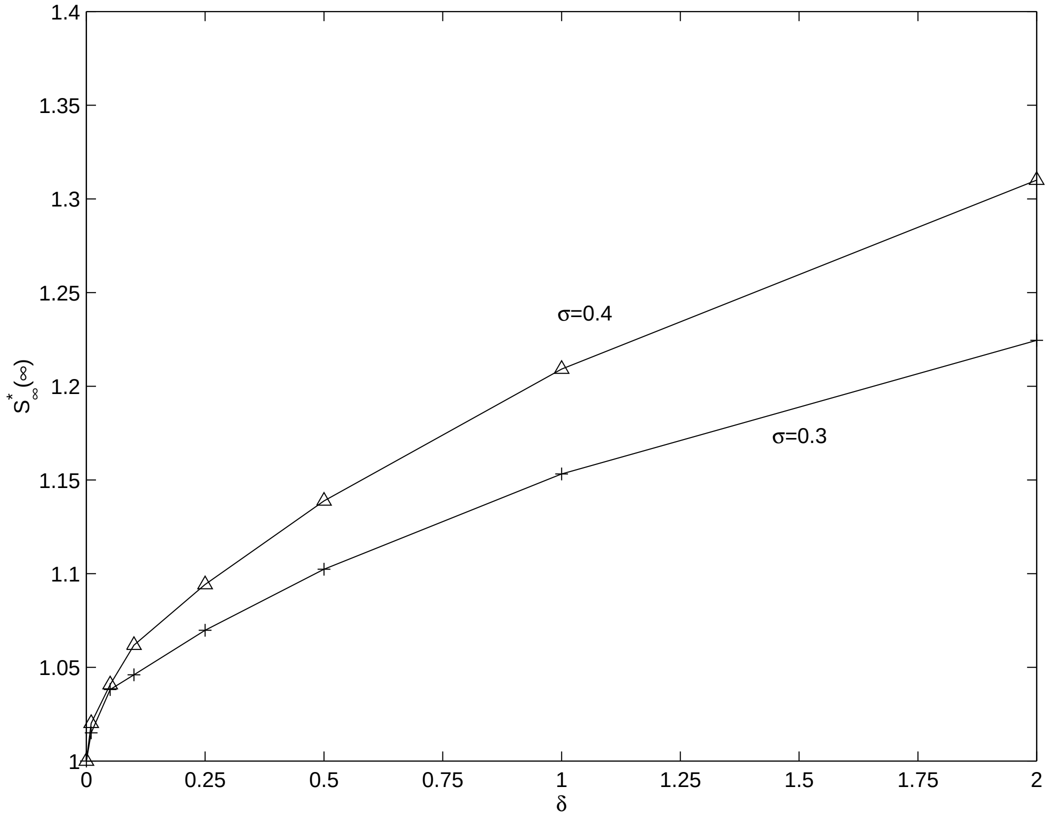

Figure 5 The critical stock price $S^*_\infty(\infty)$ of perpetual infinite-reload options is plotted against the time vesting period δ at varying levels of volatility σ.

4 Conclusion

In this paper, we analyze the impact of the time vesting requirement on the pricing behaviors of employee stock options with reload rights. An employee stock option with unlimited reload rights subject to time vesting requirement is essentially an option with finite reload rights, where the genuine allowable maximum number of reloads decreases as the calendar time is advancing towards maturity. Whenever the time to expiry decreases in value by an amount equals to the length of the vesting period, the price function of the infinite-reload option becomes the price function of a new finite-reload option with one less reload right. The price function of an infinite-reload option may exhibit a jump across those time instants corresponding to time to expiry which equals a multiple of the length of the vesting period.

By exploring the relevant analytic properties of the price functions, we have developed an efficient algorithm for numerical valuation of finite-reload and infinite-reload options. Since it becomes unnecessary to append an extra index to track the path dependence associated with the vesting time, the computational complexity of our algorithm has the same order as that of the reload option without the time vesting requirement.

We also have examined the optimal policies of exercising the reload for infinite-reload options subject to the time vesting requirement. Depending on the parameter values in the pricing model, there may exist some time interval during which it is never optimal to exercise the reload at any stock price level. For all reload options, the critical stock price increases with increasing length of the time vesting period and increasing volatility level of the stock price.

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