Sub-replication and Replenishing Premium: Efficient Pricing of Multi-state Lookbacks

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Abstract. The direct valuation procedure of performing discounted expectation to obtain the prices of multi-state lookback options may lead to insurmountable complexity and numerical difficulties. The computation may require numerical differentiation of the joint distribution function of the extremum values, then followed by numerical integration over a semi-infinite domain. In this paper, we illustrate the use of an alternative approach that significantly simplifies the calculations of multi-state lookback option prices. The financial intuition behind the new approach involves the choice of a sub-replicating portfolio and the adoption of the corresponding replenishing strategy to achieve the subsequent full replication of the derivative. The replenishing premium is obtained by performing the integration of an appropriate distribution function over the range of asset price within which under replication occurs. The sub-replication and replenishment procedures may be utilized as hedging strategies for the lookback options. The pricing and hedging properties of multi-state lookback options are also discussed.

Keywords: option pricing, multi-state lookback options, replication strategy

1. Introduction

Lookback options provide the opportunity for the holders to realize attractive gains in the event of substantial price movement of the underlying assets during the life of the option. To capture the price volatility of an asset, an investor may be interested to purchase a lookback option on the spread between the maximum and minimum prices of the underlying asset over a given time period. This option has come to be known as the *lookback spread option*. Also, one may structure lookback options on two underlying assets. The *semi-double lookback options* are options whose terminal payoff depends on the extreme value of one asset price and the terminal value of another asset price. If the terminal payoff of a lookback option depends on the extreme values of both asset prices, then the option is called a *full double lookback option*. All these types of lookback options can be collectively called *two-state lookback options* (He *et al.*, 1998). More exotic forms of

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lookback payoffs, like the hot dog option, bounded cliquet lookback, etc., are discussed in Babsiri and Noel's paper (1998).

The pricing of lookback options poses interesting mathematical challenges. The analytic price formulas for one-asset lookback options have been systematically derived by Goldman *et al.* (1979), and Conze and Viswanathan (1991). For two-state lookback options, the analytic expressions of the joint probability density functions of the extreme values and terminal values of the prices of the underlying assets have been obtained by He *et al.* (1998) and Babsiri and Noel (1998). These probability density functions are used in the valuation of the lookback option prices via numerical integration of the discounted expectation integrals or Monte Carlo simulation.

A more careful examination of He *et al.*'s formulation for two-state lookback option models reviews that their computational procedures involve double numerical differentiation of the joint distribution function to obtain the joint density function of the extremum values. This is then followed by double numerical integration over infinite and/or semi-infinite intervals to evaluate the discounted expectation of the terminal payoff. It is well known that numerical differentiation is a highly unstable procedure and numerical integration over an infinite interval commonly faces with difficulties of treating the tailed region.

In this paper, we illustrate how to obtain the price formulas for European style multi-state lookback options where the final analytic forms involve only single integration of a probability distribution function over a finite interval. The complexity of numerical valuation of these price formulas is then significantly reduced. The resulted simplicity of the price formulas stems from an elegant financial intuition in the hedging process of the lookback options. Instead of following the usual approach of evaluating the discounted risk neutral expectation of the terminal payoff, we choose a *sub-replicating portfolio* for the lookback option, then followed by the adoption of the corresponding *replenishing strategy* to achieve the full replication of the option. The value of the lookback option is given by the sum of the value of the sub-replicating portfolio and the replenishing premium (the expected cost of implementing the replenishing strategy). The required replenishment depends on the probability of under replication, which is directly related to the probability distributions of the state variables in the option model. Our pricing approach is particularly suited for pricing multi-state lookback options since the joint probability distributions (rather than density functions) of extremum values are available in succinct analytic forms. Also, the pricing approach is independent of the model of the asset price process. The choice of the sub-replicating portfolio is not unique, and an ingenious choice dictates accordingly an efficient hedging strategy for the lookback option.

The paper is organized as follows. In Section 2, we discuss the concepts of sub-replication and replenishing premium and illustrate the use of the technique to the pricing of European vanilla options. We then apply the methodology to derive the put-call parity relations of the one-asset European floating strike and fixed strike lookback options. The analogy between the replenishing of sub-replication and the hedging by the rollover strategy (Garman, 1992) is highlighted. The price formulas of discretely monitored floating strike lookback call options are also dervied. In Section 3, we derive the price formulas of the one-asset and two-asset lookback spread options. The succinct representation of the price formula naturally reveals the financial intuition behind the derivation procedure. Our pricing methodology is applied further to the pricing of options on the extreme value of one asset and the terminal values of several assets in Section 4. In Section 5, we discuss the pricing and hedging properties of one-state and two-state lookback options. We show that the straddle provides the closest replication of the floating strike lookback call option. Also, it is observed that the gamma exposure of the one-asset lookback spread option stays nearly constant at varying level of asset price. The advantages of our pricing formulation over other formulations with regard to computational efficiencies are discussed. The paper is ended with conclusive remarks in the last section. In the Appendix, we list different probability distribution functions (under the assumption of lognormal process for the asset prices) that occur in the price formulas of various lookback options derived in the paper.

2. Concepts of sub-replication and replenishing premium

The innovative concept of *riskless hedging* initiates the development of the option pricing theory. Black and Scholes (1973) showed that the risk of an option can be hedged by combining the option with an appropriate amount of the underlying asset to form a riskless portfolio. In order to avoid arbitrage, the riskless portfolio should earn the riskless interest rate. Alternatively, Merton (1973) showed that the option can be replicated by a portfolio of the underlying asset and the riskless bond. Assuming frictionless market and no premature termination of the option contract, suppose the option's payoff matches with that of the replicating portfolio at maturity, then the value of the option is equal to the value of the replicating portfolio at all times throughout the life of the option. Harrison and Kreps (1979) showed that the replication based price of any contingent claim can be obtained by calculating the discounted expected value of its terminal payoff under the risk neutral probability. The concepts of *replicated contingent claims, absence of arbitrage* and *risk neutral valuation* form the cornerstones of the modern option pricing theory.

In the literature, the price formulas of lookback options were derived by calculating the discounted risk neutral expectation of the terminal payoff. In the coming subsections, we illustrate a new approach of developing pricing formulas of derivatives through a careful examination of the pricing of the vanilla options and the one-asset floating strike and fixed strike lookback options.

2.1. Alternative perspective on the pricing of vanilla options

Consider a European call option with the strike price K, whose terminal payoff is given by $\max(S_T - K, 0)$. Here, S_T denotes the asset price at option's maturity T. From the payoff structure of the call, it is intuitive to compare the call with a portfolio which consists of long holding of one unit of the underlying asset and short selling of a riskless bond with par value K and same maturity T. The terminal payoff of the above portfolio is $S_T - K$, so the portfolio gives only a partial replication of the terminal payoff of the European call. This is because the portfolio and the call have the same terminal payoff only when the call expires in-the-money or at-the-money, corresponding to $S_T \geq K$. The terminal value of the portfolio falls below that of the call option when $S_T < K$, that is, the call expires out-of-the-money. A partial replicating portfolio whose terminal value always stays equal or below the terminal value of the derivative to be replicated is said to be a *sub-replicating portfolio*. We normally choose a sub-replicating portfolio whose value is readily obtainable. The pricing of the call option then amounts to the determination of the additional premium for acquiring extra assets on top of the sub-replication that are required to achieve the full replication of the call. This additional premium is termed the *replenishing premium*.

The loss incurred to the writer of the call at maturity when the sub-replicating portfolio is employed to hedge the option's risk is given by the difference in the terminal payoffs of the call and the sub-replicating portfolio. This difference equals $K - S_T$ if $S_T < K$, and zero if otherwise. The writer is required to use additional assets to protect against the above loss scenario. In this case, we observe that the instrument required to replenish the mis-replication is simply the put option with strike K and same maturity T. This comes no surprise since this is just the manifestation of the put-call parity relation. The replenishing premium is the value of the put.

For the purpose of enhancing analytic tractability in the derivation procedure, it is preferable that we write the replenishing premium in an integral form that involves the probability distribution function rather than the probability density function. As usual, option valuation is taken to be performed in the risk neutral world. Let t denote the current time and write $\tau = T - t$. Consider

$$\begin{aligned}\text{put value} &= e^{-r\tau} E\left[(K - S_T)\mathbf{1}_{\{S_T \le K\}}\right] \\ &= e^{-r\tau} \int_\Omega (K - S_T)\mathbf{1}_{\{S_T(\omega)\le K\}}\, dP(\omega), \end{aligned} \tag{1}$$

where r is the constant riskless interest rate, E is the risk neutral expectation operator, $\mathbf{1}_{\{S_T\le K\}}$ is the indicator function for the event $\{S_T \le K\}$ and $dP(\omega)$ is the risk neutral probability measure over the domain set Ω for the random variable S_T. Applying the relation

$$(K - S_T)\mathbf{1}_{\{S_T(\omega)\le K\}} = \int_0^K \mathbf{1}_{\{S_T(\omega)\le \xi\}}\, d\xi, \tag{2}$$

we obtain

$$\begin{aligned}\text{put value} &= e^{-r\tau} \int_\Omega \int_0^K \mathbf{1}_{\{S_T(\omega)\le \xi\}}\, d\xi dP(\omega) \\ &= e^{-r\tau} \int_0^K \int_\Omega \mathbf{1}_{\{S_T(\omega)\le \xi\}}\, dP(\omega) d\xi \quad \text{(by Fubini's theorem)} \\ &= e^{-r\tau} \int_0^K P(S_T \le \xi)\, d\xi. \end{aligned} \tag{3}$$

It may be instructive to provide the following financial interpretation for the above formula. First, we divide the interval $[0, K]$ into n subintervals, each of equal width $\Delta\xi$ so that $n\Delta\xi = K$. The put can be decomposed into the sum of n portfolios, the j^{th} portfolio consists of long holding a put with strike $j\Delta\xi$ and short selling a put with strike $(j-1)\Delta\xi, j = 1, 2, \cdots, n$, where all puts have the same maturity date T. To the leading order in $\Delta\xi$, the present value of the j^{th} portfolio is $e^{-r\tau}\{(j\Delta\xi - S_T) - [(j-1)\Delta\xi - S_T]\}P(S_T \le \xi_j), \xi_j = j\Delta\xi$. Taking the limit $n \to \infty$ so that $\Delta\xi \to 0$, we obtain

$$\text{put value} = e^{-r\tau} \lim_{n\to\infty} \sum_{j=1}^n P(S_T \le \xi_j)\Delta\xi = e^{-r\tau} \int_0^K P(S_T \le \xi)\, d\xi. \tag{4}$$

These n portfolios can be visualized as the appropriate replenishment to the sub-replicating portfolio in order that the writer of the call option is immunized from possible loss at the maturity of the option. To refine the argument, we examine the role of each of these n portfolios. With the addition of the n^{th} portfolio [long a put with strike K and short a put with strike $(K-\Delta\xi)$] into the sub-replicating portfolio, the writer faces a loss only when S_T falls below $K-\Delta\xi$. Deductively, the protection over the interval $[(j-1)\Delta\xi, j\Delta\xi]$ in the out-of-the-money region of the call is secured with the addition of the j^{th} portfolio. One then proceeds one by one from the n^{th} portfolio down to the 1^{st} portfolio so that the protection over the whole interval $[0,K]$ is achieved. With the acquisition of all these replenishing portfolios, the writer of the call option is immunized from any possible loss at option's maturity even the call expires out-of-the-money. The cost of acquiring all these n portfolios is called the replenishing premium, and its value is given by the integral in Eq. (3).

A robust approach in developing pricing formulas of derivatives then emerges. The value of an option is given by the sum of the value of the sub-replicating portfolio and the replenishing premium. The pricing of an option amounts to an ingenious choice of the sub-replicating portfolio and the construction of the appropriate replenishing strategy.

The choice of the sub-replicating portfolio is not unique. Suppose the writer of the call option chooses the sub-replicating portfolio to be the null (empty) portfolio, then the replenishment is obtained by taking the collection of infinitely many portfolios, where the j^{th} portfolio consists of long holding of a call with strike $K+j\Delta\xi$ and short selling of a call with strike $K+(j+1)\Delta\xi, j = 1, 2, \cdots$. The present value of the replenishing premium is given by

$$\begin{aligned}\text{replenishing premium} &= e^{-r\tau}\sum_{j=1}^{\infty} P(S_T > K + j\Delta\xi)\Delta\xi \\ &= e^{-r\tau}\int_K^{\infty} P(S_T > \xi)\ d\xi. \end{aligned} \tag{5}$$

Since the sub-replicating portfolio has been chosen to be the null portfolio, the call value is then equal to the replenishing premium as defined in Eq. (5).

In the above formulations, it is not necessary to restrict the random asset price process to the usual lognormal process. Provided that $P(S_T \le \xi)$ or $P(S_T > \xi)$ for the specified asset price process is given, the integration of the distribution function in Eq. (4) or Eq. (5) can be evaluated accordingly. As a remark, our sub-replication and replenishing premium approach can be related to the static replication approach as advocated by Carr and his co-authors (say, Carr and Picron (1999)], though the financial arguments employed in the two approaches are quite different. Their static replication approach leads to pricing formulas that involve integration of call and put prices, while our approach results in integration of probability distributions.

2.2 One-asset floating strike and fixing strike lookbacks

In this subsection, we would like to demonstrate the robustness of the sub-replication and replenishment approach by pricing the European style one-asset floating strike and fixed strike lookback options under continuous and discrete monitoring of the extremum value of the asset price process. Our derivation procedure will be seen to be more direct, intuitive and simple compared to earlier methods reported in the literature (Conze and Viswanathan, 1991). In particular, we obtain the put-call parity relation of continuously monitored floating strike and fixed strike lookback options. Also, we illustrate that the rollover strategy of hedging lookback options can be interpreted as replenishment of sub-replication. The experience gained in the one-asset pricing models will be beneficial to the development of efficient pricing procedures for the multi-state lookback options.

2.2.1. Put-call parity relations of continuously monitored floating strike and fixed strike lookback options

We let $[T_0, T]$ be the continuously monitored period for the minimum value of the asset price process. It is assumed that the current time t is within the monitoring period so that $T_0 < t < T$, and that the period of monitoring ends with the maturity of the lookback call option. Let S_u denote the asset price at time $u, T_0 \leq u \leq T$. Let $\underline{S}[t_1, t_2]$ denote the realized minimum value of the asset price over the period $[t_1, t_2]$. The terminal payoff of the continuously monitored floating strike lookback call option is given by

$$c_{f\ell}(S_T, T) = S_T - \underline{S}[T_0, T]. \tag{6}$$

Note that the realized minimum value of S_u from the earlier time T_0 to the current time t (denoted by $\underline{S}[T_0, t]$) is already known. It is seen that

$$\underline{S}[T_0, T] = \min(\underline{S}[T_0, t], \underline{S}[t, T]). \tag{7}$$

Here, $\underline{S}[t, T]$ is a stochastic state variable with dependence on $S_u, u \in [t, T]$.

First, it seems natural to choose the sub-replicating instrument to be a forward with the same maturity and delivery price $\underline{S}[T_0, t]$. The terminal payoff of the sub-replicating instrument is below that of the forward only when $\underline{S}[t, T] < \underline{S}[T_0, t]$; otherwise, their terminal payoffs are equal. Here, $\underline{S}[t, T]$ is the random variable that determines the occurrence of under replication. Following similar argument as in Eq. (4), except that S_T is now replaced by $\underline{S}[t, T]$, the present value of the required replenishing premium to compensate for the occurrence of under replication is given by

$$\text{replenishing premium} = e^{-r\tau} \int_0^{\underline{S}[T_0, t]} P(\underline{S}[t, T] \leq \xi) \, d\xi. \tag{8}$$

The replenishing strategy is to purchase a series of portfolios so as to secure protection in the interval where $\underline{S}[t, T] \leq \underline{S}[T_0, t]$. Let $p_{fix}(S, t; K)$ denote the value of a fixed strike lookback put with strike K, whose terminal payoff is $\max(K - \underline{S}[T_0, T], 0)$. One then visualizes that the replenishing premium is simply $p_{fix}(S, t; \underline{S}[T_0, t])$, where the strike price of the fixed strike lookback put is taken to be $\underline{S}[T_0, t]$. The present value of the continuously monitored European floating strike lookback call option is given by the sum of the sub-replicating portfolio and the replenishing premium. This gives the following put-call parity relation for lookback options:

$$\begin{aligned} c_{f\ell}(S, t; \underline{S}[T_0, t]) &= S - e^{-r\tau}\underline{S}[T_0, t] + e^{-r\tau} \int_0^{\underline{S}[T_0, t]} P(\underline{S}[t, T] \leq \xi) \, d\xi \\ &= S - e^{-r\tau}\underline{S}[T_0, t] + p_{fix}(S, t; \underline{S}[T_0, t]), \end{aligned} \tag{9}$$

where S is the current asset price and $S - e^{-r\tau}\underline{S}[T_0, t]$ is the present value of the forward with delivery price $\underline{S}[T_0, t]$ and maturity date T. The probability distribution $P(\underline{S}[t, T] \leq \xi)$ is given by the distribution function for the restricted asset price process with the down barrier ξ over the interval $[t, T]$. For lognormally distributed asset price process, the corresponding distribution can be obtained using Eq. (A.1b) in Appendix.

Next, we would like to relate a fixed strike lookback call (whose strike price has been fixed) with a floating strike lookback put and a forward. Let $\overline{S}[t_1, t_2]$ denote the realized maximum value of the asset price over the period $[t_1, t_2]$. The terminal payoff of the continuously monitored fixed strike lookback call option is given by

$$c_{fix}(S_T, T) = (\overline{S}[T_0, T] - K)^+ \tag{10}$$

where K is the strike price and x^+ signifies $\begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$. When $\overline{S}[T_0, T] \geq K$, the fixed strike lookback call expires at-the-money or in-the-money. The terminal payoff can be expressed as $(\overline{S}[T_0, T] - S_T) + (S_T - K)$, which is the sum of the terminal payoffs of a floating strike put and a forward with delivery price K. It then becomes natural to choose the sub-replicating portfolio to be the sum of the European floating strike lookback put and the forward.

Since $\overline{S}[T_0, T] \geq \overline{S}[T_0, t]$, the fixed strike lookback call is guaranteed to expire in-the-money if it is currently in-the-money. Therefore, when $\overline{S}[T_0, t] \geq K$, the sub-replicating portfolio is guaranteed to be a full replication. On the other hand, when $\overline{S}[T_0, t] < K$ (the fixed strike lookback call is currently out-of-the-money), the sub-replicating portfolio would expire with payoff below that of the fixed strike lookback call when $\overline{S}[T_0, T] < K$. When $\overline{S}[T_0, t] < K$, the replenishing premium required to compensate for under replication is given by

$$\begin{aligned} \text{replenishing premium} &= e^{-r\tau} \int_0^K P(\max(\overline{S}[T_0, t], \overline{S}[t, T]) \leq \xi) \; d\xi \\ &= e^{-r\tau} \int_0^K P(\overline{S}[t, T] \leq \xi) \; d\xi - e^{-r\tau} \int_0^{\overline{S}[T_0, t]} P(\overline{S}[t, T] \leq \xi) \; d\xi. \end{aligned} \tag{11}$$

Let $p_{f\ell}(S, t; L)$ denote the current value of the continuously monitored European floating strike lookback put option, where L denotes the current realized maximum asset value. By following similar argument as in Eq. (9), one can deduce that

$$\begin{aligned} p_{f\ell}(S, t; L) &= e^{-r\tau} \int_0^L P(\overline{S}[t, T] > \xi) \; d\xi + [e^{-r\tau} L - S] \\ &= e^{-r\tau} \int_0^L P(\overline{S}[t, T] \leq \xi) \; d\xi - S. \end{aligned} \tag{12}$$

Accordingly, the replenishing premium in Eq. (11) can be expressed as $p_{f\ell}(S, t; K) - p_{f\ell}(S, t; \overline{S}[T_0, t])$.

In summary, the current value of the continuously monitored European fixed strike lookback call option can be expressed by the following put-call parity relation that combines both cases of $\overline{S}[T_0, t] \geq K$ and $\overline{S}[T_0, t] < K$:

$$c_{fix}(S, t; K) = p_{f\ell}(S, t; \max(\overline{S}[T_0, t], K)) + S - Ke^{-r\tau}. \tag{13}$$

We actually obtain a more direct insight into the put-call parity relation if we express the terminal payoff defined in Eq. (10) as

$$c_{fix}(S_T, T) = [\max(\overline{S}[T_0, T], K) - S_T] + S_T - K, \tag{14}$$

and treat $\max(\overline{S}[T_0, T], K)$ as the modified stochastic lookback variable defining the terminal payoff.

2.2.2. Rollover strategy and replenishment

Garman (1992) proposed the hedging of the floating strike lookback call by adopting the rollover strategy. At any time, we hold a European vanilla call with the strike price equals the current realized minimum asset value. In order to replicate the payoff of the floating strike lookback call at expiry, whenever a new realized minimum value of the asset price is established at a later time, one should sell the original call option and buy a new call with the same maturity date but with a strike price equal to the newly established minimum value. Since the call with a lower strike is always more expensive, an extra cost is required to adopt the rollover strategy. The sum of these expected costs of replacement is termed the *strike bonus premium*. We may write

$$c_{f\ell}(S, t; \underline{S}[T_0, t]) = c(S, t; \underline{S}[T_0, t]) + \text{strike bonus premium}, \tag{15}$$

where $c(S, t; \underline{S}[T_0, t])$ denotes the current value of the European vanilla call option with the same maturity date T and strike price $\underline{S}[T_0, t]$. One may interpret that if the sub-replicating instrument is taken to be this European call, then the strike bonus premium is simply the corresponding replenishing premium. The rollover strategy may be visualized as replenishment of sub-replication.

With the replacement of the forward by the call as the sub-replicating instrument, by virtue of the put-call parity relation for European vanilla options, the new replenishing premium is the old one minus the value of the European put with the same strike and maturity. By Eq. (8), we deduce that

$$\begin{aligned} \text{strike bonus premium} &= p_{fix}(S, t; \underline{S}[T_0, t]) - p(S, t; \underline{S}[T_0, t]) \\ &= e^{-r\tau} \int_0^{\underline{S}[T_0, t]} [P(\underline{S}[t, T] \le \xi) - P(S_T \le \xi)] \, d\xi \\ &= e^{-r\tau} \int_0^{\underline{S}[T_0, t]} P(\underline{S}[t, T] \le \xi < S_T) \, d\xi, \end{aligned} \tag{16}$$

where $p(S, t; \underline{S}[T_0, t])$ is the value of a European vanilla put struck at $\underline{S}[T_0, t]$. Fortunately, the stochastic state variable $\underline{S}[t, T]$ observes the property $\underline{S}[t, T] \le S_T$ so that the above simplication of the difference of the two distribution functions into single distribution function is feasible. Now, we obtain an alternative price formula for the continuously monitored floating strike lookback call as follows:

$$c_{f\ell}(S, t; \underline{S}[T_0, t]) = c(S, t; \underline{S}[T_0, t]) + e^{-r\tau} \int_0^{\underline{S}[T_0, t]} P(\underline{S}[t, T] \le \xi < S_T) \, d\xi. \tag{17}$$

To understand via financial intuition why the new replenishing premium involves the integration of the probability distribution $P(\underline{S}[t, T] \le \xi < S_T)$ over the interval $[0, \underline{S}[T_0, t]]$, we observe that under replication occurs only when $\underline{S}[t, T] < \min(S_T, \underline{S}[T_0, t])$. To immunize the loss due to under replication over the infinitesimal interval $[(j-1)\Delta\xi, j\Delta\xi]$, the present value of the corresponding replenishing portfolio is $e^{-r\tau} P(\underline{S}[t, T] \le \xi < \min(S_T, \underline{S}[T_0, t]) \Delta\xi$. The total replenishing premium is then given by

$$e^{-r\tau} \int_0^{\infty} P(\underline{S}[t, T] \le \xi < \min(S_T, \underline{S}[T_0, t]) \, d\xi = e^{-r\tau} \int_0^{\underline{S}[T_0, t]} P(\underline{S}[t, T] \le \xi < S_T) \, d\xi, \tag{18}$$

which agrees with that given in Eq. (16).

2.2.3. Discretely monitored floating strike lookback call options

Suppose the monitoring of the minimum value of the asset price takes place only at discrete instants $t_j, j = 1, 2, \cdots, n$, where t_n is on or before the maturity date of the lookback call option. Suppose the current time is taken to be within $[t_k, t_{k+1})$. The terminal payoff of the discretely monitored floating strike lookback call option is given by

$$c_{f\ell}^{dis}(S_T, T) = S_T - \min(S_{t_1}, S_{t_2}, \cdots, S_{t_n}). \tag{19}$$

We use the notation $\underline{S}[i,j]$ to denote $\min(S_{t_i}, S_{t_{i+1}}, \cdots, S_{t_j}), j > i$. At the current time, $\underline{S}[1,k] = \min(S_{t_1}, S_{t_2}, \cdots, S_{t_k})$ is already known. Similar to the continuously monitored case, we choose the sub-replicating instrument to be a forward with the same maturity date T and delivery price $\underline{S}[1,k]$.

If $\underline{S}[1,k] \le \underline{S}[k+1,n]$, then the forward expires with the same payoff as that of the discretely monitored European floating strike lookback call; otherwise, the sub-replicating forward expires with payoff below that of the lookback call. Similar to the continuously monitored case, the replenishing premium required to compensate for under replication is given by

$$\text{replenishing premium} \;=\; e^{-r\tau} \int_0^{\underline{S}[1,k]} P(\underline{S}[k+1,n] \le \xi)\; d\xi. \tag{20}$$

The present value of the discretely monitored European floating strike lookback call option is then given by

$$c_{f\ell}^{dis}(S, t; \underline{S}[1,k]) = S - e^{-r\tau}\underline{S}[1,k] + e^{-r\tau} \int_0^{\underline{S}[1,k]} P(\underline{S}[k+1,n] \le \xi)\; d\xi. \tag{21}$$

The distribution function $P(\underline{S}[k+1,n] \le \xi)$ can be expressed as

$$P(\underline{S}[k+1,n] \le \xi) = \sum_{j=k+1}^{n} E\mathbf{1}_{\{S_{t_j} \le \xi, S_{t_j}/S_{t_i} \le 1 \text{ for all } i \ne j, k+1 \le i \le n\}}, \tag{22}$$

where the indicator function in the jth term corresponds to the event that S_{t_j} is taken be the minima among $S_{t_{k+1}}, \cdots, S_{t_n}$; and j runs from $k+1$ to n. Suppose the asset price follows the lognormal process, then S_{t_j} and $S_{t_j}/S_{t_i}, i \ne j, k+1 \le i \le n$, are all lognormally distributed. In this case, the expectation values in Eq. (22) can be expressed in terms of multi-variate cumulative normal distribution functions [see Heynen and Kat (1995) for hints of such calculations].

3. Lookback spread options

In this section, we would like to apply the technique of sub-replication and replenishment to derive the price formulas of the one-asset and two-asset European lookback spread options. Traders may use the lookback spread options to hedge an existing position that is sensitive to price volatility or to bet on price volatility.

3.1. One-asset lookback spread option

The terminal payoff of an one-asset lookback spread option is given by

$$c_{sp}(S_T, T; K) = (\overline{S}[T_0, T] - \underline{S}[T_0, T] - K)^+. \tag{23}$$

From the above terminal payoff structure, a convenient choice of the sub-replicating portfolio would consist of long holding of one unit of European lookback call and one unit of lookback put, both of floating strike, and short holding of a riskless bond of par value K, all of them have the same maturity as that of the lookback spread option. The terminal payoff of the sub-replicating portfolio is $\overline{S}[T_0, T] - \underline{S}[T_0, T] - K$. It is observed that

$$\begin{aligned}\overline{S}[T_0, T] - \underline{S}[T_0, T] - K &= \max(\overline{S}[T_0, t], \overline{S}[T_0, T]) - \min(\underline{S}[T_0, t], \underline{S}[t, T]) - K \\ &\geq \overline{S}[T_0, t] - \underline{S}[T_0, t] - K,\end{aligned} \tag{24}$$

so the lookback spread option is guaranteed to expire in-the-money if it is currently in-the-money. In this case, the sub-replication is a full replication since the terminal payoffs of the sub-replicating portfolio and the lookback spread option are equal. However, if the lookback spread option is currently out-of-the-money, the terminal payoff of the sub-replicating portfolio would be less than that of the lookback spread option if the lookback spread option expires out-of-the-money, that is,

$$\max(\overline{S}[T_0, t], \overline{S}[t, T]) - \min(\underline{S}[T_0, t], \underline{S}[t, T]) - K < 0. \tag{25}$$

Suppose we treat $\max(\overline{S}[T_0, t], \overline{S}[t, T])$ as the stochastic state variable that determines full or under replication and $\min(\underline{S}[T_0, t], \underline{S}[t, T]) + K$ as the effective strike price, the required replenishing premium is then given by

$$\begin{aligned}&\text{replenishing premium} \\ &= e^{-r\tau} \int_0^\infty P(\max(\overline{S}[T_0, t], \overline{S}[t, T]) < \xi \leq \min(\underline{S}[T_0, t], \underline{S}[t, T]) + K)\, d\xi.\end{aligned} \tag{26a}$$

When $\xi \notin (\overline{S}[T_0, t], \underline{S}[T_0, t] + K]$, we observe that $P(\max(\overline{S}[T_0, t], \overline{S}[t, T]) < \xi \leq \min(\underline{S}[T_0, t], \underline{S}[t, T])$ $+K) = 0$. On the other hand, when $\xi \in (\overline{S}[T_0, t], \underline{S}[T_0, t] + K]$, we have

$$\max(\overline{S}[T_0, t], \underline{S}[t, T]) < \xi \quad \Leftrightarrow \quad \overline{S}[t, T] < \xi$$

and

$$\min(\underline{S}[T_0, t], \underline{S}[t, T]) + K \quad \Leftrightarrow \quad \xi \leq \underline{S}[t, T] + K.$$

Hence, the integral in Eq. (26a) can be simplfied as

$$\text{replenishing premium} \;=\; e^{-r\tau} \int_{\overline{S}[T_0, t]}^{\underline{S}[T_0, t] + K} P(\overline{S}[t, T] < \xi \leq \underline{S}[t, T] + K)\, d\xi. \tag{26b}$$

In summary, the current value of the one-asset European lookback spread option is given by

(i) $\overline{S}[T_0,t] - \underline{S}[T_0,t] - K \geq 0$

$$c_{sp}(S,t;\overline{S}[T_0,t],\underline{S}[T_0,t]) = c_{f\ell}(S,t;\underline{S}[T_0,t]) + p_{f\ell}(S,t;\overline{S}[T_0,t]) - Ke^{-r\tau}; \qquad (27a)$$

(ii) $\overline{S}[T_0,t] - \underline{S}[T_0,t] - K < 0$

$$\begin{aligned} c_{sp}(S,t;\overline{S}[T_0,t],\underline{S}[T_0,t]) &= c_{f\ell}(S,t;\underline{S}[T_0,t]) + p_{f\ell}(S,t;\overline{S}[T_0,t]) - Ke^{-r\tau} \\ &\quad + e^{-r\tau}\int_{\overline{S}[T_0,t]}^{\underline{S}[T_0,t]+K} P(\overline{S}[t,T] < \xi \leq \underline{S}[t,T] + K)\, d\xi. \qquad (27b) \end{aligned}$$

The distribution function $P(\overline{S}[t,T] < \xi \leq \underline{S}[t,T] + K)$ under the assumption of the lognormal asset price process is given in Appendix [see Eq. $(A.1e)$]. In general, the integral in Eq. $(27b)$ cannot be expressed in closed form. Since the distribution functions involving stochastic lookback variables have much simpler analytic forms than the corresponding density functions, the representation in Eq. $(27b)$ can be considered to be the most succinct form of the price formula for the lookback spread option. The numerical valuation of the price formula can be performed effectively using simple numerical quadrature.

3.2. Two-asset lookback spread option

Let $S_{1,u}$ and $S_{2,u}$ denote the price process of asset 1 and asset 2, respectively. Similarly, we write $\overline{S}_1[t_1,t_2]$ and $\underline{S}_2[t_1,t_2]$ as the realized maximum value of $S_{1,u}$ and realized minimum value of $S_{2,u}$ over the period $[t_1,t_2]$, respectively. The terminal payoff of a two-asset lookback spread option is given by

$$c_{sp}(S_{1,T},S_{2,T},T;K) = (\overline{S}_1[T_0,T] - \underline{S}_2[T_0,T] - K)^+, \qquad (28)$$

where K is the strike price. Since we can express $\overline{S}_1[T_0,T] - \underline{S}_2[T_0,T] - K$ as $(\overline{S}_1[T_0,T] - S_{1,T}) + (S_{2,T} - \underline{S}_2[T_0,T]) + S_{1,T} - S_{2,T} - K$, a natural choice of the sub-replicating portfolio would consist of long holding of one European floating strike lookback put on asset 1, one European floating strike lookback call on asset 2, one unit of asset 1 and short holding of one unit of asset 2 and a riskless bond of par value K. All instruments in the portfolio have the same maturity as that of the two-asset lookback spread option.

Similar to the one-asset counterpart, the two-asset lookback spread option is guaranteed to expire in-the-money if it is currently in-the-money; and the sub-replicating portfolio will expire with a terminal payoff below that of the lookback spread option if the lookback spread option expires out-of-the-money. Following the same argument as used in the derivation of Eqs. $(27a,b)$, the current value of the two-asset European lookback spread option is given by

(i) $\overline{S}_1[T_0,t] - \underline{S}_2[T_0,t] - K \geq 0$

$$\begin{aligned} c_{sp}(S_1,S_2,t;\overline{S}_1[T_0,t],\underline{S}_2[T_0,t]) &= p_{f\ell}(S_1,t;\overline{S}_1[T_0,t]) + c_{f\ell}(S_2,t;\underline{S}_2[T_0,t]) \\ &\quad + S_1 - S_2 - Ke^{-r\tau}; \qquad (29a) \end{aligned}$$

(ii) $\overline{S}_1[T_0,t] - \underline{S}_2[T_0,t] - K < 0$

$$\begin{aligned} c_{sp}(S_1,S_2,t;\overline{S}_1[T_0,t],\underline{S}_2[T_0,t]) &= p_{f\ell}(S_1,t;\overline{S}_1[T_0,t]) + c_{f\ell}(S_2,t;S_2[T_0,t]) \\ &\quad + S_1 - S_2 - Ke^{-r\tau} \\ &\quad + e^{-r\tau}\int_{\overline{S}_1[T_0,t]}^{\underline{S}_2[T_0,t]+K} P(\overline{S}_1[t,T] < \xi \leq \underline{S}_2[t,T] + K)\, d\xi. \qquad (29b) \end{aligned}$$

The distribution function $P(\overline{S}_1[t,T] < \xi \leq \underline{S}_2[t,T] + K)$ under the assumption of the lognormal asset price processes is given by Eq. *(A.3a)*. Though the analytic expression for the distribution function involves the infinite summation of double integration of modified Bessel functions, it is still quite manageable under the current state of art of mathematical software. The numerical implementation of the price formula (29b) will be discussed when we examine the price sensitivities of the double lookback spread option in Section 5.2.

4. Semi-lookback options

The terminal payoff of a semi-lookback option depends on the extreme value of the price of one asset and the terminal values of the prices of other assets. We further illustrate the robustness of the sub-replication and replenishment approach by deriving the price formulas of two-asset and multi-asset semi-lookback options.

4.1. Two-asset semi-lookback option

Let $V^2_{semi}(S_1, S_2, t; \overline{S}_2[T_0, t])$ denote the value of the two-asset semi-lookback option whose terminal payoff is given by $(\overline{S}_2[T_0,T] - S_{1,T} - K)^+$. Suppose we write $\overline{S}_2[T_0,T] - S_{1,T} - K = (\overline{S}_2[T_0,T] - S_{2,T}) + S_{2,T} - S_{1,T} - K$, the sub-replicating portfolio is chosen to consist of long holding of one European floating strike lookback put and one unit of forward on asset 2, and short holding of one unit of asset 1, all instruments having the same maturity. The sub-replicating portfolio will expire with a terminal payoff below that of the two-asset semi-lookback option if

$$\max(\overline{S}_2[T_0,t], \overline{S}_2[t,T]) - S_{1,T} - K < 0. \tag{30}$$

Following similar argument as above, the required replenishing premium is given by

$$\begin{aligned} \text{replenishing premium} &= e^{-r\tau} \int_0^\infty P(\max(\overline{S}_2[T_0,t], \overline{S}_2[t,T]) < \xi \leq S_{1,T} + K)\ d\xi \\ &= e^{-r\tau} \int_{\overline{S}_2[T_0,t]}^\infty P(\overline{S}_2[t,T] < \xi \leq S_{1,T} + K)\ d\xi. \end{aligned} \tag{31}$$

The present value of the two-asset semi-lookback option is then given by

$$\begin{aligned} V^2_{semi}(S_1, S_2, t; \overline{S}_2[T_0,t]) &= p_{f\ell}(S_2, t; \overline{S}_2[T_0,t]) + S_2 - S_1 - Ke^{-r\tau} \\ &\quad + \ e^{-r\tau} \int_{\overline{S}_2[T_0,t]}^\infty P(\overline{S}_2[t,T] < \xi \leq S_{1,T} + K)\ d\xi. \end{aligned} \tag{32}$$

The distribution function $P(\overline{S}_2[t,T] \leq \xi < S_{1,T} + K)$ under the assumption of lognormal asset price processes is given by Eq. $(A.2d)$.

4.2. Multi-asset semi-lookback option

Let $V^n_{semi}(S_1, S_2, \cdots, S_n, t; \underline{S}_1[T_0, t])$ denote the value of the multi-asset semi-lookback option whose terminal payoff is given by $\max(\max(S_{2,T}, \cdots, S_{n,T}) - \underline{S}_1[T_0, T], 0)$. From the terminal payoff structure, the value of the sub-replicating portfolio is given by $c^{n-1}_{max}(S_2, \cdots, S_n, t) + c_{f\ell}(S_1, t; \underline{S}_1[T_0, t]) - S_1$, where $c^{n-1}_{max}(S_2, \cdots, S_n, t)$ denotes the value of the $(n-1)$-asset maximum call option with zero strike. Under replication at maturity by the sub-replicating portfolio occurs when

$$\max(S_{2,T}, \cdots, S_{n,T}) < \underline{S}_1[T_0, T] = \min(\underline{S}_1[T_0, t], \underline{S}_1[t, T]). \tag{33}$$

Following analogous procedure as above, the present value of the n-asset semi-lookback option is given by

$$\begin{aligned} & V^n_{semi}(S_1, S_2, \cdots, S_n, t; \underline{S}_1[T_0, t]) \\ = \; & c^{n-1}_{max}(S_2, \cdots, S_n, t) + c_{f\ell}(S_1, t; \underline{S}_1[T_0, t]) - S_1 \\ & + e^{-r\tau} \int_0^{\underline{S}_1[T_0,t]} P(\max(S_{2,T}, \cdots, S_{n,T}) < \xi \leq \underline{S}_1[t, T]) \; d\xi. \end{aligned} \tag{34}$$

5. Pricing and hedging properties of lookback options

In Sec. 2.2, we have observed the analogy between the replenishment of sub-replication and rollover strategy. The choice of the sub-replicating portfolio would dictate the procedure of replenishment and in turn the rollover strategy. In this section, we examine the pricing and hedging behaviors of the floating strike lookback call, one-asset lookback spread and two-asset lookback spread. Under certain scenarios, a lookback option may be completely replicated by other lookback options or even vanilla options. The delta and gamma exposure of the lookback spread options are seen to exhibit some interesting phenomena.

5.1. Floating strike lookback call options

Equations (9) and (17) demonstrate the two possible choices of sub-replicating instrument for the sub-replication of a floating strike lookback call. When the forward is chosen as the sub-replicating instrument, it is seen that the replenishment can be accomplished by a fixed strike lookback put struck at $\underline{S}[T_0, t]$. With regard to hedging strategy, the under replication risk can be hedged by the rollover strategy of replacing the forward by its counterpart whose delivery price is set at the minimum asset value newly realized. On the other hand, if the sub-replicating instrument is a vanilla call struck at $\underline{S}[T_0, t]$, the replenishment is achieved by replacing the vanilla call by another call of lower strike whenever a new minimum asset value is realized.

Actually, there exists a third replication strategy that uses a combination of a call and a put both struck at $\underline{S}[T_0, t]$ (this is called a straddle). The idea behind is to take the value of a vanilla put struck at $\underline{S}[T_0, t]$ from the strike bonus premium defined in Eq. (16) and see whether the straddle may achieve a closer replication of the floating strike lookback call. From Eq. (17), we deduce that the amount of mis-replication by the straddle is given by

$$\begin{aligned}
&c_{f\ell}(S,t;\underline{S}[T_0,t]) - [c(S,t;\underline{S}[T_0,t] + p(S,t;\underline{S}[T_0,t]) \\
&= e^{-r\tau}\int_0^{\underline{S}[T_0,t]} \{P(\underline{S}[t,T] \le \xi < S_T) - P(S_T \le \xi)\}\ d\xi. \qquad (35)
\end{aligned}$$

Note that the difference of the above two distribution functions in the integrand can be positive (sub-replication), negative (super-replication) or even zero (full replication). Interestingly, it can be shown that the mis-replication becomes zero when the asset price is assumed to follow the lognormal process and $\alpha = r - \dfrac{\sigma^2}{2} = 0$, where σ is the volatility. [Hint: use Eqs. (*A1.a,b*) in Appendix; an alternative proof is given by Carr and Chou (1997).] In this special case of asset price process, we are able to achieve the full replication of a floating strike lookback call by a straddle. In general when full replication is not achieved, the rollover strategy of hedging amounts to replacing the straddle with a new strike set at the newly realized minimum asset value.

In Table 1, we list the percentage of mis-replication (ratio of mis-replication amount to option value) using the straddle and the vanilla call at varying asset price level. The asset price process is assumed to follow the lognormal distribution, and the parameter values used in the calculations are: $\underline{S}[T_0,t] = 60, r = 3\%$ and $\tau = 1$. Even with non-zero value of α, we observe that the percentage of mis-replication using the straddle can be significantly smaller than that using the vanilla call. The straddle is super-replicating the floating strike lookback call when $\alpha > 0$ and sub-replicating when $\alpha < 0$. With the choice of a closer replication, one may perform less frequent rebalancing if the hedging error falls within certain tolerable level.

We also examine the delta of the replenishing premium corresponding to the three strategies that are adopted to replicate a floating strike lookback call option. The three replicating instruments are, namely, a forward with delivery price $\underline{S}[T_0,t]$, a vanilla call struck at $\underline{S}[T_0,t]$, a straddle with the call and put both struck at $\underline{S}[T_0,t]$. The asset price is taken to be lognormally distributed, and the parameter values used in the calculations are: $\underline{S}[T_0,t] = 60, r = 3\%, \sigma = 12\%$ and $\tau = 1$. We observe in Figure 1 that when the asset value S is more than 40% above $\underline{S}[T_0,t]$, the delta values become vanishingly small. From the perspective of the rollover strategy, when S is well above $\underline{S}[T_0,t]$, the chance of rollover of the replicating instrument by its counterpart of lower strike or delivery price becomes small. The straddle is seen to be the best replicating instrument since the variation of the delta of the replenishing premium is relatively small even at S close to $\underline{S}[T_0,t]$.

Since the delta of a forward is one, the delta of the floating strike lookback call is given by one plus the delta of the replenishing premium corresponding to the sub-replication by a forward. The delta of the floating strike lookback call stays nearly at the constant value of one when S is far from $\underline{S}[T_0,t]$ but assumes a small positive value at S close to $\underline{S}[T_0,t]$.

5.2. Lookback spread options

Consider the one-asset lookback spread option that is currently out-of-the-money, it cannot be fully replicated by a straddle of floating strike lookback options plus short holding of a bond. The replenishing premium is given by an integral that involves the distribution function $P(\overline{S}[t,T] < \xi \le \underline{S}[t,T] + K)$. In Figure 2, we plot the distribution function against ξ at varying level of asset price S. We assume the lognormal asset price process and the parameter values used in the calculations are: $\underline{S}[T_0,t] = 100, \overline{S}[T_0,t] = 115, K = 25, r = 4\%, \sigma = 12\%, \tau = 1$. When S assumes a value that is close neither to $\underline{S}[T_0, t]$ or $\overline{S}[T_0, t]$, the lookback spread would have a higher chance of remaining out-of-the-money. The replenishing premium then has a larger value, and accordingly, the integral of the distribution function over $\xi \in [115, 125]$ has a higher value. Consider the distribution function curve corresponding to $S = 100$, the value of the distribution function is very small at ξ close to 125. This is because the chance of $\underline{S}[t, T] + K$ realizing a value slightly below 125 is very low.

From Eqs. $(27a, b)$, Eqs. $(29a, b)$ and the put-call parity relations of lookback options, the sub-replicating portfolio for either the one-asset or two-asset lookback spread option is composed of forwards and fixed strike lookback options with delivery prices and strike prices set at the current realized maximum and minimum asset values. To hedge a lookback spread option, we adopt similar rollover strategy where we replace the replicating instruments with their counterparts whose delivery price or strike price is set at the newly realized extremum asset value. The rollover will continue until the lookback spread option becomes in-the-money.

In Figure 3, we plot the delta of the one-asset lookback spread option against asset value S at varying strike price K. We assume lognormal asset price process and the parameter values used in the calculations are: $\underline{S}[T_0, t] = 100, \overline{S}[T_0, t] = 115, r = 4\%, \sigma = 12\%, \tau = 1$. The option value of the one-asset lookback spread and its delta are calculated using Eqs. $(27a, b)$, where the integral can be evaluated effectively by numerical integration of the distribution function. Suppose the discounted expectation approach of option valuation is used (like that in He *et al.*'s paper), one has to perform double differentiation of the distribution function to obtain the density function, then followed by double integration of the product of the terminal payoff and the density function. Figure 3 reveals that the gamma exposure of the one-asset lookback spread stays almost constant at varying asset value. This is not surprising since the one-asset lookback spread can be quite well replicated by a straddle of floating strike lookback call and put. This inherent property of uniform gamma exposure at varying asset value may be desirable for traders to use lookback spread options to trade on volatility. From Figure 3, the lookback spread that is currently in-the-money has a higher gamma exposure compared to that of its out-of-the-money counterpart.

We also perform the calculations of the delta values, $\dfrac{\partial V}{\partial S_1}$ and $\dfrac{\partial V}{\partial S_2}$, of the two-asset lookback spread via numerical valuation of the price formulas given in Eqs. $(29a, b)$. The plots for $\dfrac{\partial V}{\partial S_1}$ against S_1 and $\dfrac{\partial V}{\partial S_2}$ against S_2 at varying strike price K are shown in Figures 4 and 5, respectively. We again assume lognormal asset price processes and the parameter values used in the calculations are: $\overline{S}_1[T_0, t] = 115, \underline{S}_2[T_0, t] = 100, r = 4\%, \sigma = 12\%, \rho = 0.4, \tau = 1, S_1 = 110, S_2 = 110$. In our numerical implementation procedure, we employed the Matlab software to evaluate the modified Bessel functions. Though the integral formula for $g_n(\theta)$ in Eq. (A.3c) involves integration over a semi-infinite interval, we performed numerical integration over a truncated interval of integration. We tested the numerical accuracy by varying the width of the integration interval. Our numerical experiment revealed that sufficient accuracy to 5 significant figures can be achieved with about 50 Gaussian quadrature nodes and integration interval of 15. Also, similar level of numerical accuracy can be achieved by taking about 10 terms in the infinite series in the evaluation of $F(r_0, \theta_0, t)$ in Eq. (A.3b). The advantage of the representation of the price formula with only single integration of the distribution function is obvious. In the discounted expectation approach, the double integration of the product of the terminal payoff and the density function is required. It is almost intractable to perform double differentiation of the distribution function $P(\overline{S}_1[t, T] < \xi \leq \underline{S}_2[t, T])$ [see Eq.

$(A.3b)$] to obtain the density function. The delta values are increasing functions of both S_1 and S_2, and the gamma exposure of the two-asset lookback spread exhibits less uniformity over varying asset prices compared to its one-asset counterpart.

6. Conclusion

We observe that the use of an elegant financial intuition of sub-replication and replenishment has significantly reduced the complexity in the analytic pricing procedures of multi-state lookback options. The resulting price formulas for lookback options are in general expressed as the sum of the prices of sub-replicating instruments plus an integral of a probability distribution function representing the replenishing premium. For common types of asset price process, like the lognormal process, the analytic forms of the probability distribution functions are readily available. Compared to the representation of option price as discounted expectation of the terminal payoff, our price formulas avoid the valuation complexity of performing differentiation of the distribution functions to obtain the density functions. The pricing simplicity of our approach is most profound in the lookback spread options. The valuation of our price formula requires only single integration of a distribution function, while the discounted expectation approach requires double differentiation followed by double integration of the same distribution function.

The choice of the sub-replicating portfolio and replenishing strategy also leads naturally to a viable hedging strategy of the lookback option. We hedge lookback options using the rollover of the sub-replicating instrument by its counterpart with the strike or delivery price set equal to the newly realized extremum asset value. This rollover strategy can be visualized as replenishment of sub-replication. In lookback spread options, we show that the sub-replicating portfolio may achieve full replication when the spread option is currently in-the-money; thus an exotic lookback option can be decomposed into a portfolio of simpler derivative instruments.

We also obtain the put-call parity relations of one-asset floating strike and fixed strike lookback options. Under some special assumption of the asset price process, it is seen that a floating strike lookback call can be fully replicated by a straddle of vanilla call and put both struck at the current realized minimum asset value. For an one-asset lookback spread option, when it is currently in-the-money, it can be replicated fully by a straddle of floating strike lookback options plus shorting holding of a bond. We observe that the lookback spread options have almost constant gamma exposure at varying level of asset price. It is envisioned that traders may use lookback spread options to trade on volatility, like using straddles of vanilla options.

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Appendix

Under the assumption of the lognormal distribution for the asset price processes, we list the probability distributions that occur in the lookback option price formulas derived in the paper. We let $\sigma_i, i = 1, 2$, denote the volatility of the asset price process S_i In the risk neutral world, the dynamics of S_i is given by

$$\frac{dS_i}{S_i} = r dt + \sigma_i dZ_i, \qquad i = 1, 2,$$

where r is the riskless interest rate, dZ_i is the Wiener process and $dZ_1 dZ_2 = \rho dt$. Here, ρ is the correlation coefficient between dZ_1 and dZ_2. We write $X_i(t) = \ln S_i(t)$ so that

$$X_i(t) = \alpha_i t + \sigma_i Z_i(t), \quad i = 1, 2,$$

is a Brownian motion with drift rate α_i, where $\alpha_i = r - \dfrac{\sigma_i^2}{2}$. Further, we define

$$\underline{X}_i(t) = \min_{0\le u\le t} X_i(u) \qquad \text{and} \qquad \overline{X}_i(t) = \max_{0\le u\le t} X_i(u).$$

1. Probability distributions involving single asset

For notational simplicity, we drop the subscript for $X_1(t), \overline{X}_1(t), \underline{X}_1(t), \sigma_1$ and α_1.

$$\begin{aligned} P(\underline{X}(t) \ge \underline{x}, X(t) \ge x) &= G(x, \underline{x}, t; \alpha) \\ &= N\left(\frac{-x+\alpha t}{\sigma\sqrt{t}}\right) - e^{\frac{2\alpha \underline{x}}{\sigma^2}} N\left(\frac{-\underline{x}+2x+\alpha t}{\sigma\sqrt{t}}\right) \end{aligned} \quad (A.1a)$$

$$P(\underline{X}(t) \ge x) = G(x, x, t; \alpha) \quad (A.1b)$$

$$P(\overline{X}(t) \le \overline{x}, X(t) \le x) = G(-x, -\overline{x}, t; -\alpha) \quad (A.1c)$$

$$P(\overline{X}(t) \le x) = G(-x, -x, t; -\alpha) \quad (A.1d)$$

$$\begin{aligned} P(\underline{X} \ge x, \overline{X} \le y) = \sum_{n=-\infty}^{\infty} e^{[2n\alpha(y-x)]/\sigma^2} \Bigg\{ & \Bigg[N\left(\frac{y-\alpha t-2n(y-x)}{\sigma\sqrt{t}}\right) \\ & - N\left(\frac{x-\alpha t-2n(y-x)}{\sigma_1\sqrt{t}}\right)\Bigg] \\ & - e^{2\alpha x/\sigma^2}\Bigg[N\left(\frac{y-\alpha t-2n(y-x)-2x}{\sigma\sqrt{t}}\right) \\ & - N\left(\frac{x-\alpha t-2n(y-x)-2x}{\sigma\sqrt{t}}\right)\Bigg]\Bigg\} \end{aligned} \quad (A.1e)$$

2. Probability distributions involving two assets

(a) Semi-lookback options (He *et al.*, 1998)

$$\begin{aligned} & P(\underline{X}_1(t) \ge \underline{x}_1, X_1(t) \ge x_1, X_2(t) \le x_2) \\ &= G_{semi}(x_1, x_2, \underline{x}_1, t; \alpha_1, \alpha_2, \rho) \\ &= N_2\left(\frac{-x_1+\alpha_1 t}{\sigma_1\sqrt{t}}, \frac{x_2-\alpha_2 t}{\sigma_2\sqrt{t}}; -\rho\right) - e^{\frac{2\alpha_1 \underline{x}_1}{\sigma_1^2}} N_2\left(\frac{-x_1+\underline{x}_1+\alpha_1 t}{\sigma_1\sqrt{t}}, \frac{x_2-\alpha_2 t}{\sigma_2\sqrt{t}}; -\rho\right) \end{aligned} \quad (A.2a)$$

$$P(\underline{X}_1(t) \ge \underline{x}_1, X_2(t) \le x_2) = G_{semi}(\underline{x}_1, x_2, \underline{x}_1, t; \alpha_1, \alpha_2, \rho) \quad (A.2b)$$

$$P(\overline{X}_1(t) \le \overline{x}_1, X_1(t) \le x_1, X_2(t) \le x_2) = G_{semi}(-x_1, x_2, -\overline{x}_1, t; -\alpha_1, \alpha_2, -\rho) \quad (A.2c)$$

$$P(\overline{X}_1(t) \le \overline{x}_1, X_2(t) \le x_2) = G_{semi}(-\overline{x}_1, x_2, -\overline{x}_1, t; -\alpha_1, \alpha_2, -\rho) \quad (A.2d)$$

(b) Two-asset lookback options (Zhou, 2001)

$$\begin{aligned} P(\overline{X}_1(t) \le x_1, \overline{X}_2(t) \le x_2) &= G_2(x_1, x_2, t, \alpha_1, \alpha_2, \rho) \\ &= e^{a_1 x_1 + a_2 x_2 + bt} F(r_0, \theta_0, t) \end{aligned} \quad (A.3a)$$

where

$$F(r_0,\theta_0,t) = \frac{2}{\alpha t} e^{\frac{-r_0^2}{2t}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \int_0^{\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right) g_n(\theta)\ d\theta \tag{A.3b}$$

$$g_n(\theta) = \int_0^{\infty} r e^{\frac{-r^2}{2t}} e^{d_1 r \sin(\theta-\alpha) - d_2 r \cos(\theta-\alpha)} I_{\frac{n\pi}{\alpha}}\left(\frac{r r_0}{t}\right) dr. \tag{A.3c}$$

The other parameters are given by

$$\alpha = \begin{cases} \tan^{-1}\left(-\dfrac{\sqrt{1-\rho^2}}{\rho}\right), & \text{if } \rho < 0 \\ \pi + \tan^{-1}\left(-\dfrac{\sqrt{1-\rho^2}}{\rho}\right), & \text{otherwise} \end{cases}$$

$$\theta_0 = \begin{cases} \tan^{-1}\left(-\dfrac{Z_2\sqrt{1-\rho^2}}{Z_1 - Z_2\rho}\right), & \text{if } -\dfrac{Z_2\sqrt{1-\rho^2}}{Z_1 - Z_2\rho} > 0 \\ \pi + \tan^{-1}\left(-\dfrac{Z_2\sqrt{1-\rho^2}}{Z_1 - Z_2\rho}\right), & \text{otherwise} \end{cases}$$

$$r_0 = \frac{Z_2}{\sin\theta_0}, \quad Z_1 = \frac{x_1}{\sigma_1}, \quad Z_2 = \frac{x_2}{\sigma_2},$$

$$a_1 = \frac{\alpha_1\sigma_2 - \rho\alpha_2\sigma_1}{(1-\rho^2)\sigma_1^2\sigma_2}, \quad a_2 = \frac{\alpha_2\sigma_1 - \rho\alpha_1\sigma_2}{(1-\rho^2)\sigma_1\sigma_2^2},$$

$$b = -\alpha_1 a_1 - \alpha_2 a_2 + \frac{1}{2}\sigma_1^2 a_1^2 + \rho\sigma_1\sigma_2 a_1 a_2 + \frac{1}{2}\sigma_2^2 a_2^2,$$

$$d_1 = a_1\sigma_1 + \rho a_2\sigma_2, \quad d_2 = a_2\sigma_2\sqrt{1-\rho^2}. \tag{A.3d}$$

$$P(\overline{X}_1(t) \le x_1, \underline{X}_2(t) \ge x_2) = G_2(x_1, -x_2, t; \alpha_1, -\alpha_2, -\rho) \tag{A.3e}$$

$$P(\underline{X}_1(t) \ge x_1, \underline{X}_2(t) \ge x_2) = G_2(-x_1, -x_2, t; -\alpha_1, -\alpha_2, -\rho) \tag{A.3f}$$

asset value	percentage of mis-replication by a straddle		percentage of mis-replication by a vanilla call	
	$\alpha = 0.02$	$\alpha = -0.02$	$\alpha = 0.02$	$\alpha = -0.02$
60	4.9220	–6.4519	40.7589	46.7740
65	2.0786	–2.5823	19.6542	24.8320
70	0.7871	–1.8045	8.3746	10.7401
75	0.2874	–0.7944	3.3961	5.0424
80	0.1044	–0.2731	1.3554	1.4447
85	0.0381	–0.0884	0.5385	0.8351
90	0.0140	–0.0831	0.2137	0.4648
95	0.0052	–0.0734	0.0847	0.0846
100	0.0019	–0.0670	0.0336	0.0585

Table 1 A floating strike lookback call can be partially replicated either by a vanilla call or a straddle, all options are struck at $\underline{S}[T_0, t]$. The entries show the percentage of mis-replication (ratio of mis-replication amount to option value) by the vanilla call and the straddle at varying level of asset value and α. The other parameter values used in the calculations are: $\underline{S}[T_0, t] = 60, r = 3\%$ and $\tau = 1$.

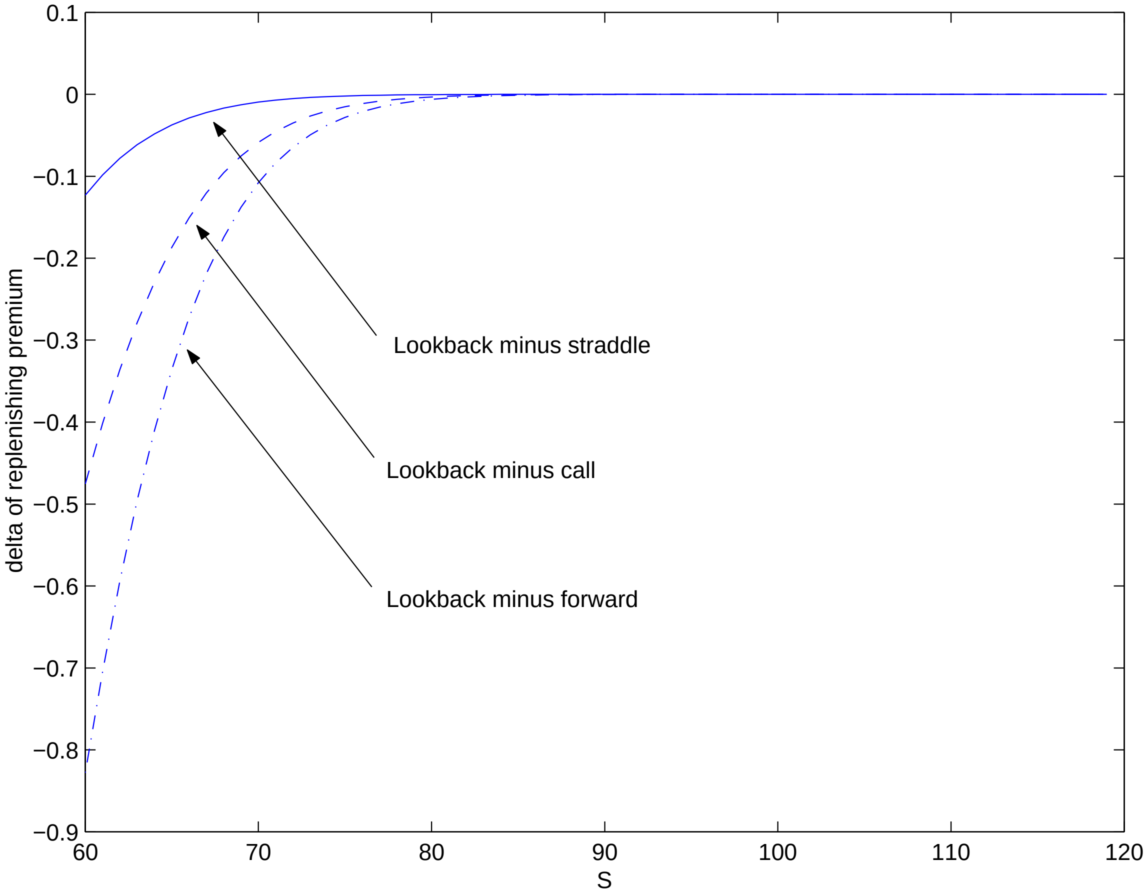

Figure 1 The curves show the plotting of the delta of the replenishing premium against the asset value S, corresponding to different replication strategies adopted in the replication of a floating strike lookback call option. The three replicating instruments are (i) a forward with delivery price $\underline{S}[T_0, t]$, (ii) a vanilla call with strike $\underline{S}[T_0, t]$ and (iii) a straddle with both call and put struck at $\underline{S}[T_0, t]$, where $\underline{S}[T_0, t]$ is the current realized minimum asset value. The parameter values used in the calculations are: $\underline{S}[T_0, t] = 60, r = 3\%, \sigma = 12\%, \tau = 1$. The delta becomes vanishingly small when S is more than 40% above $\underline{S}[T_0, t]$. However, the delta experiences some drastic variation at S close to $\underline{S}[T_0, t]$.

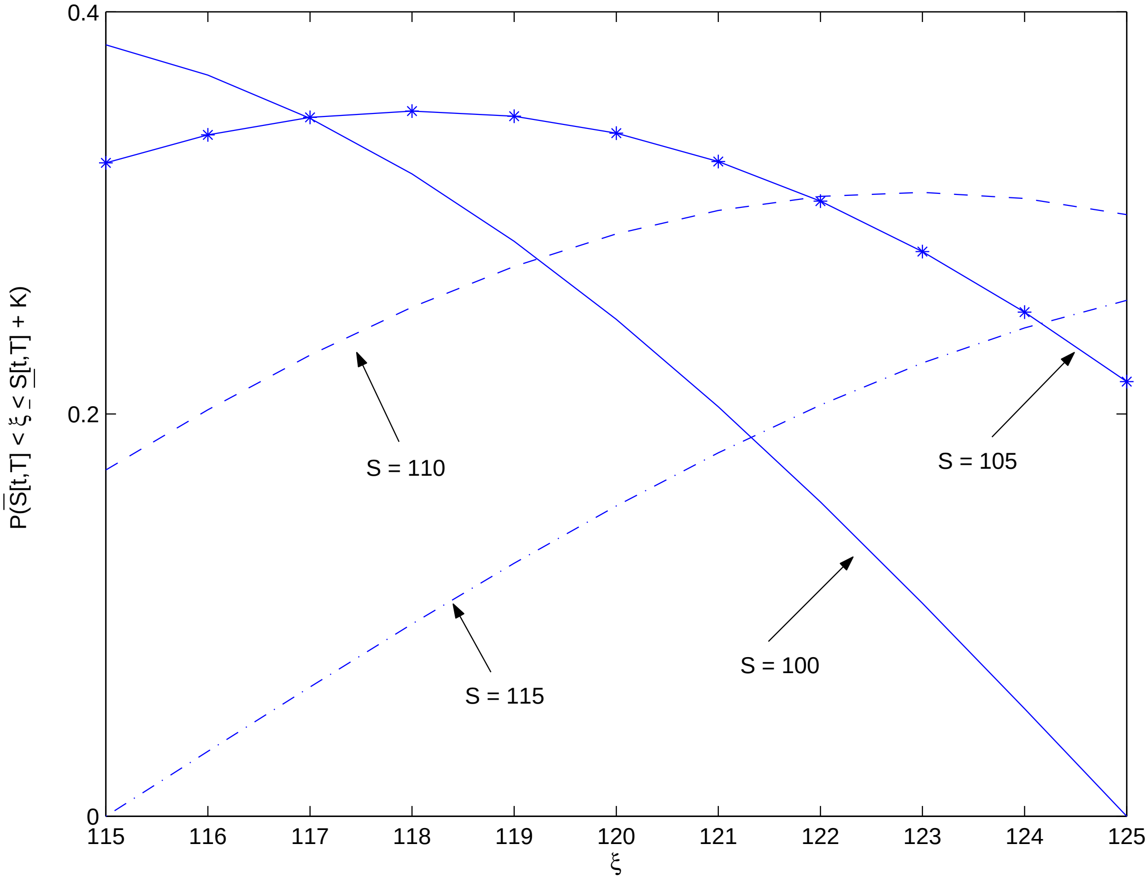

Figure 2 The curves show the plotting of the distribution function $P(\overline{S}[t,T] < \xi \leq \underline{S}[t,T] + K)$ against ξ for varying current asset value S. The parameter values used in the calculations are: $\underline{S}[T_0,t] = 100, \overline{S}[T_0,t] = 115, K = 25, r = 4\%, \sigma = 12\%, \tau = 1$. The replenishing premium is given by the discounted factor times the area under the distribution curve between $\overline{S}[T_0,t] = 115$ and $\underline{S}[T_0,t] + K = 125$.

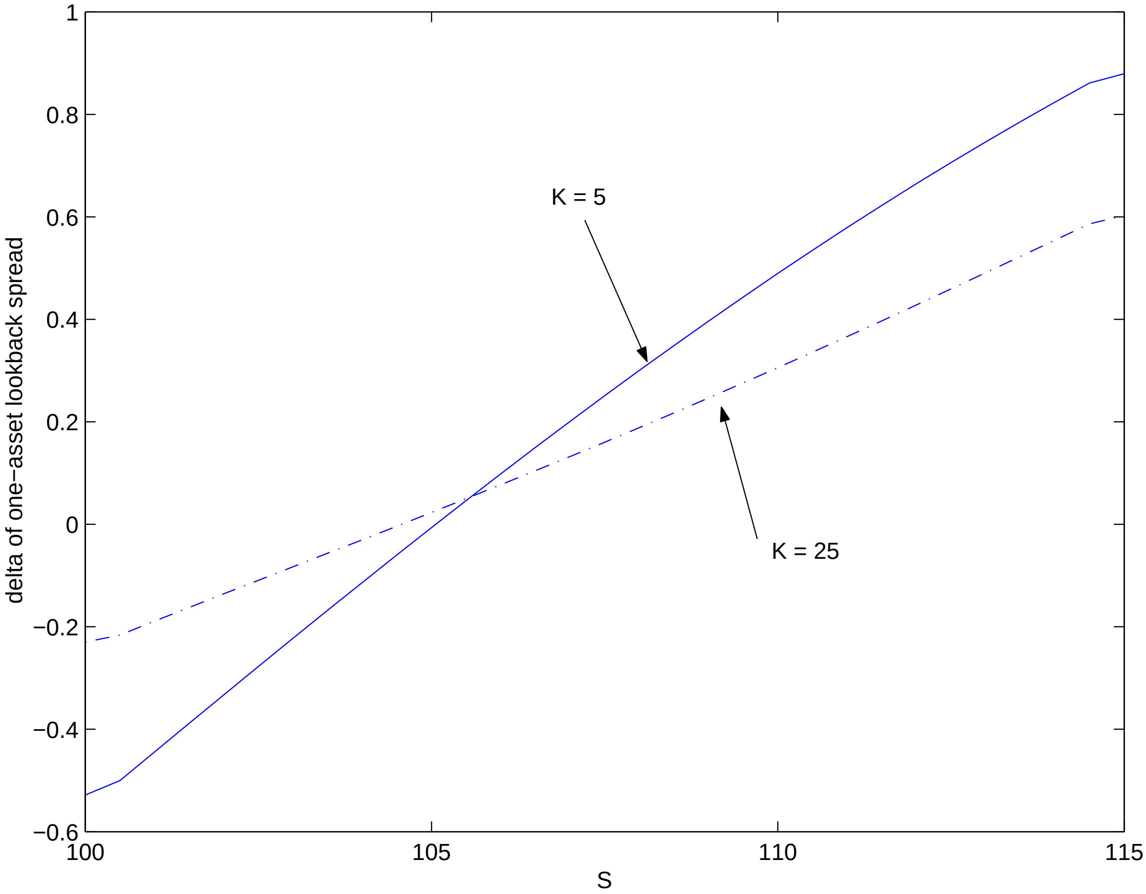

Figure 3 The curves show the plotting of the delta of the one-asset lookback spread option against asset value S at varying strike price K. The parameter values used in the calculations are: $\underline{S}[T_0, t] = 100, \overline{S}[T_0, t] = 115, r = 4\%, \sigma = 12\%, \tau = 1$. The gamma exposure of the lookback spread stays almost constant at varying level of the asset value. The lookback spread that is currently in-the-money ($K = 5$) has a higher gamma exposure compared to that of its out-of-the-money counterpart ($K = 25$).

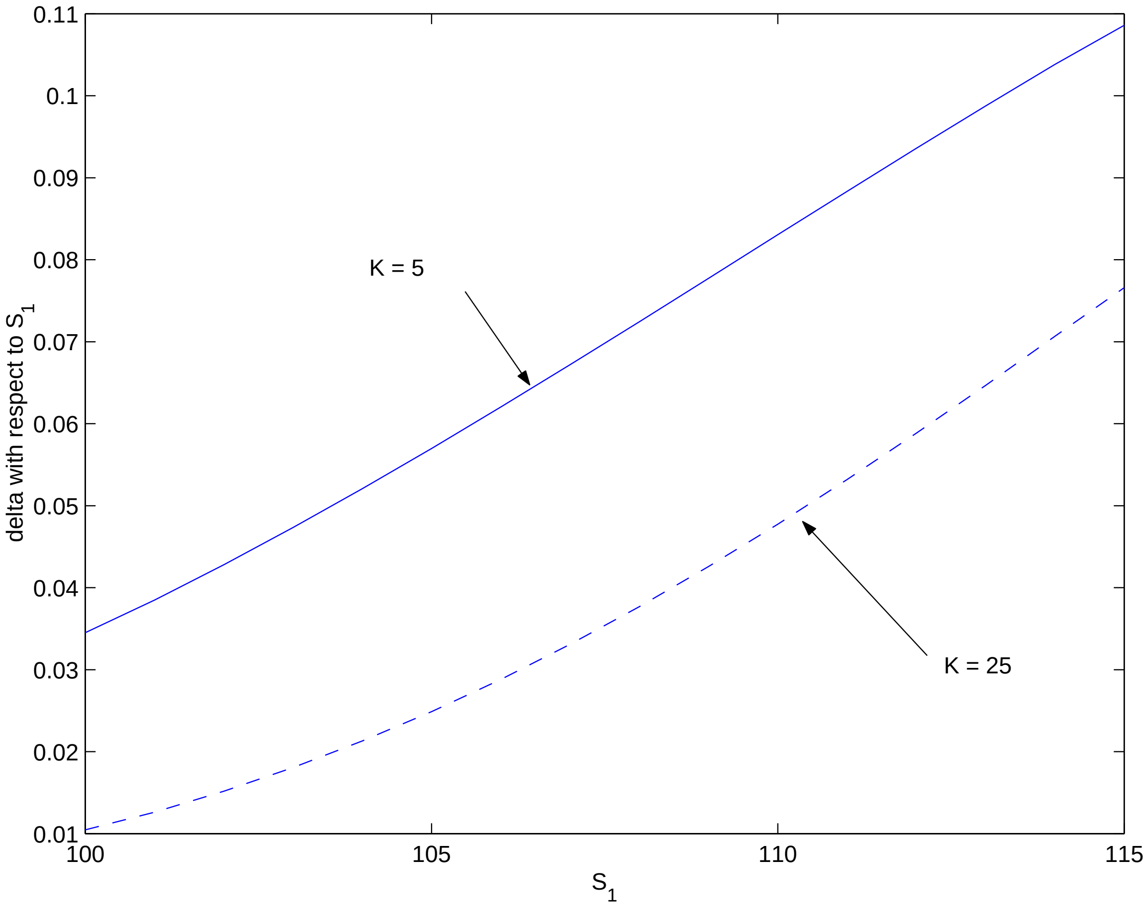

Figure 4 The curves show the plotting of the delta with respect to asset 1 of the two-asset lookback spread option against S_1 at varying strike price K. The parameter values used in the calculations are: $\overline{S}_1[T_0, t] = 115, \underline{S}_2[T_0, t] = 100, r = 4\%, \sigma = 12\%, \rho = 0.4, \tau = 1, S_2 = 110$. The delta with respect to S_1 is seen to be an increasing functioin of S_1 and a decreasing function of the strike price K.

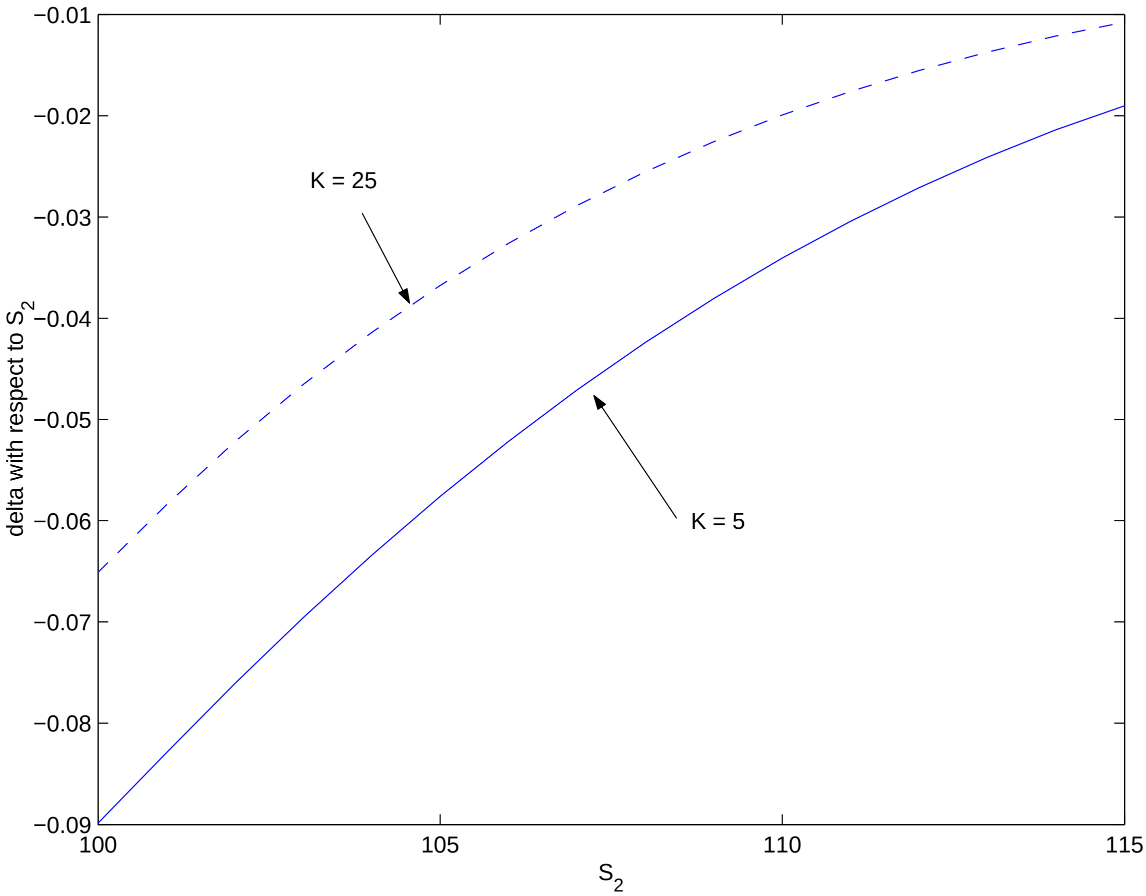

Figure 5 The curves show the plotting of the delta with respect to asset 2 of the two-asset lookback spread option against S_2 at varying strike price K. We take $S_1 = 110$ while other parameter values are the same as those in Figure 4. The delta with respect to S_2 is seen to be an increasing function of S_2 and its absolute value is a decreasing function of the strike price K.