Pricing and hedging variable annuities products: Guaranteed Lifelong Withdrawal Benefits

presented by

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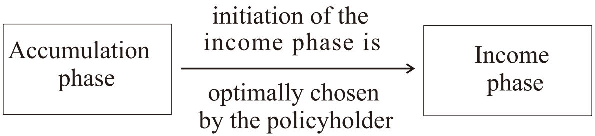
Outline

- Product nature of the Guaranteed Lifelong Withdrawal Benefit (GLWB) in variable annuities
  - Policy value and benefit base
  - Bonus (roll-up) provision and ratchet (step-up) provision
- Pricing formulation as dynamic control models
  - Withdrawals and initiation of income phase as controls
- Optimal dynamic withdrawal policies and initiation of the income phase
  - Bang-bang analysis: discrete set of decision choices
- Sensitivity analysis of pricing and hedging
  - Bonus rate on optimal withdrawal strategies
  - Suboptimal withdrawal strategies on value function
  - Contractual withdrawal rate on optimal initiation
  - Hedging strategies on profits and losses

Product nature of GLWB

- The policyholder pays an initial premium and possibly subsequent premiums during the accumulation phase to the issuer. The amount is then invested into the policyholder's choice of portfolio of mutual funds.

- Two phases
  - Accumulation phase: growth of the policy value and benefit base with equity participation (limited withdrawals may be allowed in some contracts).
  - Income phase: guaranteed annualized withdrawals regardless of the policy value until the death of the last surviving Covered Person.



Market successes of GLWB

Retirement protection

Policyholders maintain equity positions in their retirement assets, thus take advantage of the potential market upside growth, while getting lifelong annuities guaranteed.

$\xrightarrow{\text{up}}$ equity participation of market upside growth

$\xrightarrow{\text{down}}$ lifelong annuities guaranteed on the downside

Market size

In 2018, the sales of variable annuities in the US markets are around 100 billion dollars. The GLWB rider is structured in about half of the new variable annuities sales.

Policy fund account value

- This is the ongoing value of the investment account, subject to changes due to investment returns and withdrawal amounts, payment of the rider charges and increment in value due to additional purchases of funds after initiation of the contract.
- Upon the death of the last Covered Person, the remaining (positive) amount in the policy fund account will be paid to the beneficiary.

Benefit base (used as the notional for determining the contractual withdrawal amount)

- The benefit base is initially set to be the upfront payment. The benefit base may grow by virtue of the bonus (roll-up) provision in the accumulation phase and ratchet (step-up) provision in the income phase.
- Under the lifelong withdrawal guarantee, the policyholder is entitled to withdraw a fixed proportion of the benefit base periodically (say, annual withdrawals) after initiation of the income phase for life even when the policy fund account value has been fully depleted.

$$\begin{aligned} & \text{Lifelong guaranteed withdrawal amount} \\ = \; & \text{Lifelong withdrawal scheduled rate} \times \text{benefit base} \end{aligned}$$

Lifetime withdrawal rate

The contractual lifetime withdrawal rate is dependent on the age of the policyholder entering into the Income phase.

In some contracts, the jumps in the lifetime withdrawal rate occur in 3-year or 5-year time periods.

Age when withdrawals begin	59 and under	60-64	65-79	80^+
Lifetime withdrawal percentage	single: 3.5% spousal: 3%	single: 4.5% spousal: 4%	single: 5.5% spousal: 5%	single: 6.5% spousal: 6%

Below is another example from a GLWB contract.

Age	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65
Single life	3.5%	3.6%	3.7%	3.8%	3.9%	4.0%	4.1%	4.2%	4.3%	4.4%	4.5%	4.6%	4.7%	4.8%	4.9%	5.0%
joint life	2.8%	2.9%	3.0%	3.1%	3.2%	3.3%	3.4%	3.5%	3.6%	3.7%	3.8%	3.9%	4.0%	4.1%	4.2%	4.3%

Bonus provision

Let W_i and γ_i be the fund (policy) value and withdrawal amount at year i, respectively, η_b be the percentage of the benefit base charged on the policy fund value as the annual rider fee, $G(\tau_I)$ is the contractual withdrawal rate with dependence on the initiation year τ_I of the income phase.

- Suppose the policyholder chooses not to withdraw at year i, either in the accumulation or income phase, then the benefit base A_i is increased proportionally by the bonus rate b_i under the bonus provision, where

$$A_i^+ = A_i\,(1 + b_i) \quad \text{if } \gamma_i = 0.$$

- In the income phase, when $0 < \gamma_i \leq G(\tau_I)A_i$, then the benefit base would not be reduced and the withdrawal is not subject to penalty charge.

  When $\gamma_i > G(\tau_I)A_i$, then the benefit base decreases proportionally according to the amount of excess withdrawal. The ratio of decrease is given by

$$\frac{\text{excess withdrawal}}{\text{net fund value after withdrawal}} = \frac{\gamma_i - G(\tau_I)A_i}{W_i - \eta_b A_i - G(\tau_I)A_i}.$$

Ratchet provision

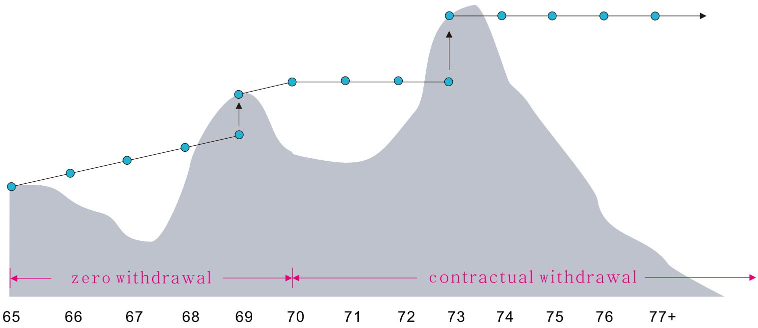
The jump condition on the benefit base arising from the ratchet provision on a ratchet date $i \in \mathcal{T}_e$ (preset dates that allow ratchet) is given by

$$
\begin{aligned}
& A_i^+ \\
= & \begin{cases} \max\left(A_i, \left((W_i - \eta_b A_i)^+ - \gamma_i\right)^+ \mathbf{1}_{\{i \in \mathcal{T}_e\}}\right) & \text{if } 0 < \gamma_i \leq GA_{i-} \\ \max\left(\frac{W_i - \eta_b A_i - \gamma_i}{W_i - \eta_b A_i - G(\tau_I) A_i} A_i, \left((W_i - \eta_b A_i)^+ - \gamma_i\right) \mathbf{1}_{\{i \in \mathcal{T}_e\}}\right) & \text{if } GA_{i-} < \gamma_i \leq W_i - \eta_b A_i. \end{cases}
\end{aligned}
$$

The value of A_i^+ right after time i have dependence on their values W_i and A_i right before time i and the withdrawal amount γ_i.

- When the withdrawal γ_i does not exceed GA_{i-}, A_{i^+} takes the maximum between A_i and the net fund value after paying the withdrawal amount and rider fee.
- When $\gamma_i > GA_i$, the effect of proportional decrease on A_i is taken into account.

Schematic plot to show the growth of the benefit base



- Under zero withdrawal during the accumulation phase, the benefit base increases by a proportional amount (bonus provision).
- During the income phase, the benefit base is increased to the policy fund value if the benefit base is below the policy fund value (ratchet provision).

Cashflows received by the policyholder

Let k_i be proportional penalty charge applied on the excess of withdrawal amount over the contractual withdrawal at year i.

- In the accumulation phase, the cash flow $f_i^A(\gamma_i; A_i)$ received by the policyholder as resulted from the withdrawal amount γ_i is given by

$$f_i^A(\gamma_i; A_i) = \begin{cases} \gamma_i & \text{if } -BA_i \leq \gamma_i \leq 0 \\ (1-k_i)\gamma_i & \text{if } 0 < \gamma_i \leq (W_i - \eta_b A_i)^+ \end{cases}.$$

  Here, B is the cap multiplier of the benefit base that fixes the upper bound of additional purchase (corresponding to $\gamma_i < 0$).

- In the income phase, the excess withdrawal beyond the contractual withdrawal amount $G(\tau_I)A_i$ is charged at proportional penalty rate k_i. The actual cash amount received by the policyholder as resulted from the withdrawal amount γ_i is given by

$$f_i^I(\gamma_i; A_i, G(\tau_I)) = \begin{cases} \gamma_i & \text{if } 0 \leq \gamma_i \leq G(\tau_I)A_i \\ G(\tau_I)A_i + (1-k_i)[\gamma_i - G(\tau_I)A_i] & \text{if } G(\tau_I)A_i < \gamma_i \leq W_i - \eta_b A_i \end{cases}.$$

Pricing formulation as dynamic control models

Let Γ denote the optimal withdrawal strategies as characterized by the vector $(\gamma_1, \gamma_2, \ldots, \gamma_{T-1})$, where γ_i is the annual withdrawal amount or additional purchase (considered as negative withdrawal) on the withdrawal date i and T is the perceived latest withdrawal date.

Let $\mathcal{E}$ be the admissible strategy set for the pair of control variables (Γ, τ_I), where τ_I is the optimal time for the initiation of the income phase.

Under Q-measure, the value function of the GLWB products is formally given by

$$
\begin{aligned}
V(W, A, v, 0) = \sup_{(\Gamma, \tau_I) \in \mathcal{E}} E_Q \Bigg[& \sum_{i=1}^{\tau_S \wedge (T-1)} e^{-ri} p_{i-1} q_{i-1} W_i + \sum_{i=1}^{(\tau_I - 1) \wedge \tau_S} e^{-ri} p_i f_i^A(\gamma_i; A_i) \\
& + \sum_{i=\tau_I}^{\tau_S \wedge (T-1)} e^{-ri} p_i f_i^I(\gamma_i; A_i, G(\tau_I)) + \mathbf{1}_{\{\tau_S > T-1\}} e^{-rT} p_{T-1} W_T \Bigg].
\end{aligned}
$$

Here, p_i is the survival probability up to year i and q_i is the death probability in $(i, i+1)$. The **optimal complete surrender time** τ_S is dictated by the optimal choice of the withdrawal amount γ_i, where

$$
\tau_S = \inf \{ i \in \mathcal{T} | \gamma_i = W_i - \eta_b A_i > 0 \}.
$$

Four sources of cashflows to the policyholder

- The first summation term represents the death payment weighted by the probability of mortality from the initiation date of the contract to the complete surrender time τ_S or $T-1$, whichever comes earlier.

- The second summation term gives the sum of discounted withdrawal cash flows from the initiation date of the contract to the last withdrawal date in the accumulation phase or the complete surrender time τ_S, whichever comes earlier.

- The third summation term gives the sum of discounted withdrawal cash flows from the initiation date of the income phase to the complete surrender time τ_S or $T-1$, whichever comes earlier.

- The last single term is the discounted cash flow received by the policyholder at the maximum remaining life T provided that complete surrender has never been adopted throughout the whole life of the policy.

Stochastic volatility model for the fund value process

The general formulation of the stochastic volatility model for the Q-dynamics of the underlying fund value process of W_t can be expressed as

$$dW_t = (r - \eta)\, W_t\, dt + \sqrt{v_t} W_t \left[\rho\, dB_t^{(1)} + \sqrt{1 - \rho^2}\, dB_t^{(2)} \right]$$

and

$$dv_t = \kappa v_t^a \left(\theta - v_t \right)\, dt + \epsilon v_t^b\, dB_t^{(1)},$$

for $a = \{0, 1\}$ and $b = \{1/2, 1, 3/2\}$.

Here, $B_t^{(1)}$ and $B_t^{(2)}$ are uncorrelated Q-Brownian motions, ρ is the correlation coefficient between W_t and v_t, ϵ is the volatility of variance, κ is the risk neutral speed of mean reversion, θ is the risk neutral long-term mean variance, and r is the riskless interest rate.

Analytic expressions for the characteristic function of the fund value process W_t are available for these choices of stochastic volatility models.

Bang-bang analysis

The design of the numerical algorithm would be much simplified if the choices of the optimal withdrawal amount γ_i are limited to a finite number of discrete values. The technical analysis relies on the convexity and monotonicity properties of the value function.

As part of the technical procedure, it is necessary to require the two-dimensional Markov process $\{(W_t, v_t)\}_t$ to observe the following mathematical properties:

Property 1 (Convexity preservation)

For any convex terminal payoff function $\Phi(W_T)$, the corresponding European price function as defined by

$$\phi(w, v) = e^{-r(T-t)} E\left[\Phi(W_T) \middle| W_t = w, v_t = v\right], \; t \leq T,$$

is also convex with respect to w.

Property 2 (Scaling)

For any positive K, the two stochastic processes $\{(W_t, v_t)\}_t$ and $\{(\frac{W_t}{K}, v_t)\}_t$ have the same distribution law given that their initial values are the same with each other almost surely.

The stochastic volatility models under $a = \{0, 1\}$ and $b = \{1/2, 1, 3/2\}$ satisfy these two properties.

By virtue of Property 2, the value functions $V^{(I)}$ and $V^{(A)}$ satisfy the following scaling properties for any positive scalar K:

$$V^{(I)}(KW, KA, v, t; G_0) = KV^{(I)}(W, A, v, t; G_0)$$
$$V^{(A)}(KW, KA, v, t) = KV^{(A)}(W, A, v, t).$$

- Since the benefit base A does not change within consecutive withdrawal dates, by virtue of the above scaling properties, we can achieve reduction in dimensionality of the pricing model by one by defining $\widetilde{W} = W/A$.
- The scaling properties are also crucial in establishing the bang-bang control analysis.

We write $\mathrm{GLWB}^{(\mathrm{A})}$ and $\mathrm{GLWB}^{(\mathrm{I})}$ to represent the value function of the GLWB rider in the accumulation phase and income phase, respectively. The main results on the bang-bang control strategies for $\mathrm{GLWB}^{(I)}$ and $\mathrm{GLWB}^{(A)}$ are summarized follows.

Theorem

Assume that $\{(W_t, v_t)\}_t$ satisfies both Properties 1 and 2, $GLWB^{(I)}$ and $GLWB^{(A)}$ observe the following optimal withdrawal strategy, respectively.

1. *On any withdrawal date i, the optimal withdrawal strategy γ_i within the income phase for $GLWB^{(I)}$ with a positive guaranteed rate G_0 is limited to (i) $\gamma_i = 0$; (ii) $\gamma_i = G_0 A_i$; or (iii) $\gamma_i = W_i - \eta_b A_i$.*

2. *On any withdrawal date i, the optimal strategy on this withdrawal date within the accumulation phase for $GLWB^{(A)}$ is either*

   (2a) *to initiate the income phase on this withdrawal date if $V_C^{(I)}(i) > V_C^{(A)}(i)$ and the subsequent optimal withdrawal strategy γ_i is limited to (i) $\gamma_i = 0$; (ii) $\gamma_i = G(i)A_i$; or (iii) $\gamma_i = W_i - \eta_b A_i$;*

   (2b) *or to remain in the accumulation phase on this withdrawal date if $V_C^{(I)}(i) \leq V_C^{(A)}(i)$ and the optimal withdrawal strategy γ_i is limited to (i) $\gamma_i = -BA_i$; (ii) $\gamma_i = 0$; or (iii) $\gamma_i = W_i - \eta_b A_i$.*

- When the policy is already in the income phase, the withdrawal policies are limited to zero withdrawal, withdrawal at the contractual rate or complete surrender.
  - Due to the penalty charge on excess withdrawal, the bang-bang analysis shows that it is not optimal to withdraw more than the scheduled withdrawal amount, except choosing complete surrender.
- When the policy is in the accumulation phase, the policyholder may choose to enter into the income phase or stay in the accumulation phase.
  - The subsequent optimal policies while staying in the accumulation phase are limited to maximum allowable purchase, zero withdrawal or complete surrender.

Remark

We obtain the above bang-bang strategies under discrete withdrawals of the GLWB rider. The knowledge of these optimal withdrawal strategy avoid the tedious procedure of searching for the optimal strategies from continuum of choices of withdrawal amounts. Note that such bang-bang results fail under the GMWB counterpart.

Dynamic programming procedure

The time-t value function of GLWB$^{(\mathrm{I})}$, denoted by $V^{(I)}(W, A, v, i; G_0)$, is seen to have dependence on the guaranteed withdrawal rate $G(\tau_I)$.

Since the contractual withdrawal rate depends on the optimal initiation time of the income phase τ_I, it is necessary to calculate a set of $V^{(I)}(W, A, v, t; G_0)$ with G_0 being set to be $G(i)$, $i = 1, 2, \cdots, T_a + 1$. Here, T_a be the last date on which the GLWB contract may stay in the accumulation phase.

Using the dynamic programming principle of backward induction, we compute $V^{(I)}(W, A, v, i; G_0)$ as follows:

$$
\begin{aligned}
&V^{(I)}(W, A, v, T; G_0) = p_{T-1} W_T, \\
&V^{(I)}(W, A, v, i; G_0) \\
= \;& p_{i-1} q_{i-1} W_i + \sup_{\gamma_i \in [0, \max(W_i - \eta_b A_i, G_0 A_i)]} \{ p_i f_i^I(\gamma_i; A_i, G_0) \\
&+ e^{-r} E_Q[V^{(I)}(W, A, v, i+1; G_0) | (W_{i^+}, A_{i^+}) = \boldsymbol{h}_i^I(W_i, A_i, \gamma_i; G_0), v_{i^+} = v_i] \},
\end{aligned}
$$

where $i = 1, 2, \cdots, T - 1$ and $G_0 = G_{n_k}$, $k = 1, \cdots, K$. Here, $\boldsymbol{h}_i^I$ is the jump function of the policy fund and benefit base associated with γ_i and G_0.

We let $V^{(A)}(W, A, v, t)$ be the time-t value function of GLWB$^{(\mathrm{A})}$.

For an event date, $1 \leq i \leq T_a - 1$, we have

$$V^{(A)}(W, A, v, i) = \; p_{i-1}q_{i-1}W_i + \max\{V_C^{(A)}(i), V_C^{(I)}(i)\},$$

where

$$\begin{aligned}
V_C^{(A)}(i) = & \sup_{\gamma_i \in [-BA_i, 0, (W_i - \eta_b A_i)^+]} \{p_i f_i^A(\gamma_i; A_i) \\
& + e^{-r} E_Q[V^{(A)}(W, A, v, i+1) | (W_{i^+}, A_{i^+}) = \boldsymbol{h}_i^A(W_i, A_i, \gamma_i), v_{i^+} = v]\}, \\
V_C^{(I)}(i) = & \sup_{\gamma_i \in [0, \max((W_i - \eta_b A_i)^+, G(i)A_i)]} \{p_i f_i^I(\gamma_i; A_i, G(i)) \\
& + e^{-r} E_Q[V^{(I)}(W, A, v, i+1; G(i)) | (W_{i^+}, A_{i^+}) = \boldsymbol{h}_i^I(W_i, A_i, \gamma_i; G(i)), v_{i^+} = v_i]\}.
\end{aligned}$$

Here, $V_C^{(A)}(i)$ corresponds to the value function under continuation that the policyholder chooses not to activate the income phase at year i.

Since the policyholder is entitled to choose to stay in the accumulation phase or activate the income phase in the next year $i+1$, so $\max\{V_C^{(A)}(i), V_C^{(I)}(i)\}$ is taken.

Maximizing the monetary value of the riders

- The price function and optimal strategies are formulated as a stochastic control model based on the assumption that the policyholder adopts optimal strategies that maximize the monetary value of the GLWB riders. The policyholder may follow what appears to be a sub-optimal strategy that does not maximize the monetary value of the embedded guarantees, say, due to tax considerations and liquidity needs.
- The cost of hedging under this assumption of maximizing monetary value serves as an important benchmark in the sense that it is the worse case scenario for the writer.

Numerical algorithms

1. Under the affine Heston stochastic volatility model, we performed the calculations using the fast Fourier transform method since the characteristic function of the fund value process is available in simple analytic form.

   "Optimal initiation of guaranteed lifelong withdrawal benefit with dynamic withdrawal," *SIAM Journal on Financial Mathematics*, 2017, vol.8, p.804-840 (with Y.T. Huang and P.P. Zeng).

2. For the 3/2-model, we employed the regression-based Monte Carlo simulation method in finding the optimal withdrawal strategies, similar to most optimal stopping models.

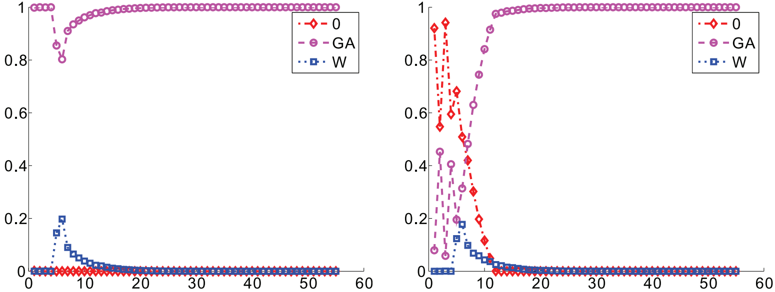
   "Regression-based Monte Carlo methods for stochastic control models: Variable annuities with lifelong guarantees," *Quantitative Finance*, 2016, vol.16, p.905-928 (with Y.T. Huang).

Numerical studies on the value functions and choices of optimal policies on contractual specifications, like bonus rate, penalty charge, contractual withdrawal rate (dependent on the age at which income phase is initiated)

Parameter	Value
Volatility, σ	0.20
Interest rate, r	0.04
Penalty for excess withdrawal, $k(t)$	$0 \leq t \leq 1 : 3\%$, $1 < t \leq 2 : 2\%$, $2 \leq t \leq 3 : 1\%$, $3 < t \leq 4 : 0\%$
Expiry time, T (years)	57
Initial payment, S_0	100
Mortality	DAV 2004R (65 year old male)
Mortality payments	At year end
Withdrawal rate, G	0.05 annual
Bonus (no withdrawal)	0.06 annual
Withdrawal strategy	Optimal
Withdrawal dates	yearly

Model and contract parameters.

Withdrawal strategies in the income phase – impact of bonus rate



(a) Bonus: 4%, Ratchet: Every two years (b) Bonus: 7%, Ratchet: Every two years

Plots of proportion of withdrawal policies adopted (measured by number of simulation paths)

- The zero withdrawal strategy is suboptimal when the bonus rate is low (4%), below the contractual withdrawal rate 5%. Surrender as the optimal choice occurs most likely on earlier withdrawal dates.

- When the bonus rate is increased to 7%, the policyholder chooses zero withdrawal on early withdrawal dates with almost certainty. However, this tendency decreases on later withdrawal dates since the advantage of building the benefit base at later time becomes less significant.

Impact on value function under suboptimal withdrawal strategies

Penalty for excess withdrawal, $k(t)$	$0 \le t \le 1$: 6% (10%), $1 < t \le 2$: 5% (9%), $2 \le t \le 3$: 4% (8%), $3 < t \le 4$: 3% (7%), $4 \le t \le 5$: 2% (6%), $5 < t \le 25$: 1% (5%), $25 < t \le T$: 0% (0%)

Table: Penalty charge settings "Penalty 1" and "Penalty 2".

Bonus rate	Penalty charge	Optimal Strategy	Suboptimal Strategy 1	Suboptimal Strategy 2
0	1	100.3315	100.2746	97.8960
0	2	98.9781	98.9669	97.8960
0.04	1	100.2770	100.2746	97.8960
0.04	2	98.9728	98.9669	97.8960
0.07	1	101.4455	100.2746	97.8960
0.07	2	99.8056	98.9669	97.8960
0.08	1	**103.0922**	**100.2746**	**97.8960**
0.08	2	101.6182	98.9669	97.8960

Table: Sensitivity analysis of the contractual features on the GLWB price.

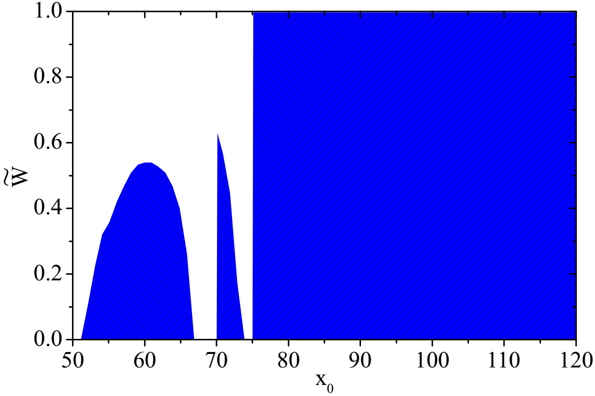
- "Suboptimal Strategy 1": only take two strategies on each withdrawal date: $\gamma = GA$ and $\gamma = W$.
- "Suboptimal Strategy 2": only takes $\gamma = GA$ until death.
- The additional flexibility of choosing $\gamma = 0$ adds little value when the bonus rate is low and/or the penalty charge is high.

Optimal initiation region with respect to the age x_0

age	contractual withdrawal rate
$65 \leq x_0 \leq 70$	0.05
$71 \leq x_0 \leq 75$	0.055
$76 \leq x_0 \leq 120$	0.06

$$r = 0.05 \quad \text{and} \quad \eta_b = 0.01, \quad b_i = 0.06$$

- When $\widetilde{W}$ falls below some threshold value, it may be optimal to initiate the income phase to receive the withdrawals.
- When the age is approaching the trigger age of moving to the next higher withdrawal rate, it is optimal to delay initiation even at very low $\widetilde{W}$ in order to enjoy the higher withdrawal rate.



Plot of the optimal initiation region (shaded) in the $\widetilde{W}$-x_0 plane under the impact of the contractual withdrawal rate $G_{x_0}(t)$. The contractual withdrawal rate would not increase beyond $x_0 = 76$, so it is optimal to initiate the income phase at any level of $\widetilde{W}$.

Three main sources of risk

1. Mortality risk
   - When the mortality risk is fully diversifiable, one may hedge mortality risk by selling independent policies to a group of policyholders with similar risk of death.
   - There may be systematic change in mortality risk affecting all of the population simultaneously, called the longevity risk.
2. Policyholder behavior risk
   - Deterministic assumption: inferred from historical statistics on decision making processes.
   - Optimal decision making or at least based on some market factors (like moneyness of the guarantee).

3. Financial risk

One may follow delta hedging, similar to hedging of options. The crucial difference is that the costs of these guarantees are not paid upfront, unlike initial option premiums.

- Fees are paid periodically as a percentage of the fund value throughout the life of the contract. The challenge is how to use the collected fees to match the payoff of the guarantees when they should be paid.
- The value of collected fees and the cost of hedging move in opposite direction. When the value of fee is low (low fund value), the value of embedded options is high.

Difficulties of using static hedging using put options

(i) Neglect path dependent feature of the fund value.

(ii) Costs of buying options may not match with the uncertain fees collected.

Hedging procedures

We let $B(t)$ be the money market account, $S(t)$ be the underlying fund, $W(t)$ be the policy fund value and $V(t)$ be the value of the GLWB.

- Between consecutive withdrawal dates, the value process W_t follows the same dynamic equation as that of $S(t)$ except for the proportional rider fee charged on the policy fund.
- On each withdrawal date, unlike the underlying fund S, W decreases by the withdrawal amount chosen by the policyholder.
- We use S as a tradable proxy to hedge the exposure of the GLWB on W.

We construct a portfolio that consists of the money market account, underlying fund and GLWB as follows:

$$\Pi(t) = \Delta_B(t)B(t) + \Delta_S(t)S(t) - V(t),$$

where Δ_B and Δ_S are the number of holding units of the money market account and the underlying fund, respectively. Also, we denote the number of holding units of the policy fund value by Δ_W.

By equating the dollar values of the underlying fund and policy fund value in the portfolio, we have $\Delta_S S = \Delta_W W$.

We impose the self-financing condition on $\Pi(t)$ with the initial value $\Pi(0)$ being zero. The value of $\Pi(t)$ may be interpreted as the profit and loss of the portfolio at time t.

We consider three hedging strategies: (i) non-active hedging; (ii) delta hedging; (iii) minimum variance hedging.

- Under non-active hedging, the insurance company puts the upfront premium paid by the investor of the GLWB into the money market account and does not hold any position in the underlying fund at any time, so that Δ_S is identically zero throughout all times. The fees collected are put into the money market account.
- For the delta hedging strategy, Δ_W is set to be $\frac{\partial V}{\partial W}$, so that Δ_S is equal to $\frac{W}{S}\frac{\partial V}{\partial W}$.

For the minimum variance hedging strategy, Δ_W is chosen to minimize the variance of the portfolio's instantaneous changes. One can show that Δ_W for the (local) minimum variance hedging under the 3/2-model is given by

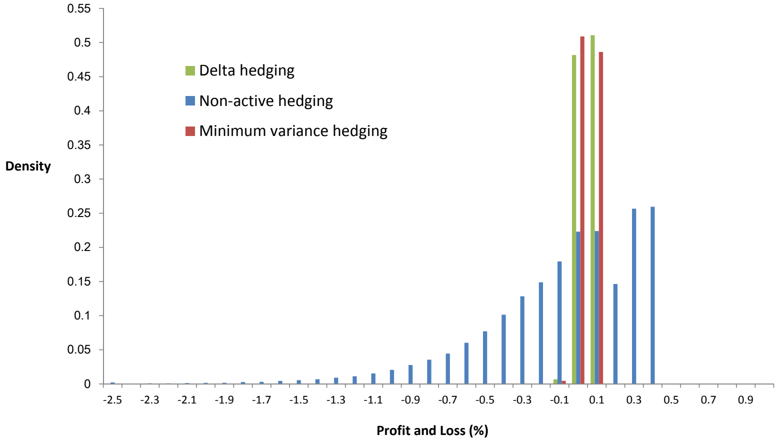
$$\Delta_W = \frac{\partial V}{\partial W} + \rho \frac{\epsilon v}{W} \frac{\partial V}{\partial v}.$$

Hence, we have

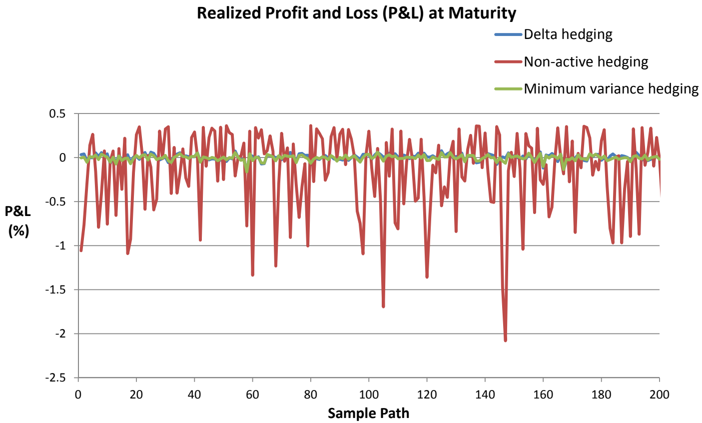
$$\Delta_S = \frac{W}{S}(\frac{\partial V}{\partial W} + \rho \frac{\epsilon v}{W} \frac{\partial V}{\partial v}).$$

Remark

Bernard and Kwak (2016) propose the net liability hedge where the collected fees are utilized right the way in the hedging strategy.



Histogram of profit and loss of the non-active hedging strategy, delta hedging strategy and minimum variance hedging strategy. The profit and loss is in the form of relative percentage of the initial payment. The number of simulation paths is 50,000 and the hedging frequency is monthly.



Realization of the profit and loss at maturity by following the non-active hedging strategy, delta hedging strategy and minimum variance hedging strategy with **200** sample paths. The profit and loss is in the form of relative percentage of the initial payment.

The variance of the profit and loss under the delta hedging and minimum variance hedging are seen to be much smaller than that under the non-active hedging. This indicates good efficiency of the delta hedging and minimum variance hedging.

The standard deviations of the profit and loss by following the delta hedging and minimum variance hedging stay almost at the same level. The monthly hedging procedure may be too infrequent for the minimum variance hedging to be effective in reducing the standard deviation of profit and loss.

There are other more sophisticated hedging strategies, such as the delta-gamma hedging that is used in complex structured derivatives. The success of employing these hedging strategies relies on accurate sensitivity estimation, which is a challenging topic itself.

Conclusion

- We present the optimal control models to compute the value functions of GLWB in both the accumulation and income phases.
- Effective pricing calculations can be performed by fast Fourier transform method or regression-based Monte Carlo simulation method. Computational efficiency of the numerical procedure is enhanced by the bang-bang analysis of the set of control policies on withdrawal strategies and initiation time of the income phase.
- We analyze the optimal withdrawal policies and optimal initiation policies under various contractual specifications and study the sensitivity analysis on the price of the GLWB by varying the embedded contractual features and the assumption on the policyholder's withdrawal behavior.
- We consider the hedging efficiencies and profit-loss profiles under various hedging strategies.