# Timer Options: expiry floats with realized variance 

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## Agenda

1. Product nature and uses of timer options

- Barrier options in the volatility space: knock-out depends on the discrete realized variance hitting the preset variance budget

2. Perpetual timer options

- Black-Scholes type representation
- Joint density function of functionals of instantaneous invariance

3. Analytic price formulas of finite maturity timer options (twodimensional Fourier integrals) under 3/2-model of stochastic volatility

- Decomposition into a portfolio of timerlets
- Joint characteristic function of log-asset price and integrated variance


## Variance budget

The investor specifies a maximum bound $T$ on the option life and a target volatility $\sigma_{0}$ to define a variance budget

$$
B=\sigma_{0}^{2} T
$$

Let $t_{i}, i=0,1,2, \ldots, N$, be the monitoring dates. Let $\tau_{B}$ be the random first hitting time in the tenor of monitoring dates at which the discrete realized variance exceeds the variance budget $B$, namely,

$$
\tau_{B}=\min \left\{j \left\lvert\, \sum_{i=1}^{j}\left(\ln \frac{S_{t_{i}}}{S_{t_{i-1}}}\right)^{2} \geq B\right.\right\} \Delta .
$$

Here, $\Delta$ is the uniform time interval between consecutive monitoring dates.

Termination date of a finite-maturity timer option $=\min \left(\tau_{B}, T\right)$, where $T$ is the preset mandated expiration date.


Knocked out at $\tau_{B}$ since the variance budget has been breached. This occurs earlier than $T$.

## Uses of timer options

- Portfolio managers can use the timer put options on an index to hedge sudden market drops (with uncertainty in timing). Conpensations from the timer put payoff are received earlier (due to increased volatility) after the incidence of market drop. In the bullish market where the stock price increases, the realized volatility decreases thus giving longer life of the time put.
- The implied volatility is often higher than the realized volatility. If one feels the implied volatility in the market is too high currently, then one can capture the volatility risk premium by longing a timer call and shorting a vanilla call (higher price due to higher level of implied volatility). The volatility target is set below the current implied volatility and the volatility risk premium is captured by the difference in values of the two call options.

Analytic pricing formula of discretely monitored finite-maturity timer options under stochastic volatility models

Define the continuous integrated variance to be $I_{t}=\int_{0}^{t} v_{s} d s$. We use $I_{t}$ as a proxy of the discrete realized variance for the monitoring of the first hitting time. We define $\tau_{B}$ to be

$$
\tau_{B}=\min \left\{j \mid I_{t_{j}} \geq B\right\} \Delta
$$

This approximation does not introduce a noticeable error for daily monitored timer options. Note that

$$
\begin{aligned}
C_{0}\left(X_{0}, I_{0}, V_{0}\right)= & \mathbb{E}_{0}\left[e^{-r\left(T \wedge \tau_{B}\right)} \max \left(S_{T \wedge \tau_{B}}-K, 0\right)\right] \\
= & \mathbb{E}_{0}\left[e^{-r T} \max \left(S_{T}-K, 0\right) 1_{\left\{\tau_{B}>T\right\}}\right. \\
& \left.+e^{-r \tau_{B}} \max \left(S_{\tau_{B}}-K, 0\right) 1_{\left\{\tau_{B} \leq T\right\}}\right]
\end{aligned}
$$

where $K$ is the strike price and $r$ is the constant interest rate.

## Decomposition into a portfolio of timerlets

1. No knock-out occurs prior to $T:\left\{\tau_{B}>T\right\}=\left\{I_{T}<B\right\}$

$$
\text { Terminal payoff }=\max \left(S_{T}-K, 0\right) \mathbf{1}_{\left\{I_{T}<B\right\}}
$$

2. Knock-out occurs at $t_{j+1}$ :

$$
\left\{\tau_{B}=t_{j+1}\right\}=\left\{I_{j+1} \geq B, I_{j}<B\right\}=\left\{I_{t_{j}}<B\right\} \backslash\left\{I_{t_{j+1}}<B\right\}
$$

That is, there is no knock-out by $t_{j}$ but rule out "no knock-out by $t_{j+1}{ }^{\prime \prime}$.

Summing $j=0,1, \ldots, N-1$, we have

$$
\left\{\tau_{B}=T\right\}=\bigcup_{j=0}^{N-1}\left\{\tau_{B}=t_{j+1}\right\}=\bigcup_{j=0}^{N-1}\left\{I_{t_{j}}<B\right\} \backslash\left\{I_{t_{j+1}}<B\right\}
$$

A finite-maturity discrete timer call option can be decomposed into a portfolio of timerlets:

$$
\begin{aligned}
& C_{0}=\mathbb{E}_{0}\left[e^{-r T} \max \left(S_{T}-K, 0\right) 1_{\left\{I_{T}<B\right\}}\right] \\
&+\mathbb{E}_{0}\left[\sum_{j=0}^{N-1} e^{-r t_{j+1}}\right.\left(\max \left(S_{t_{j+1}}-K, 0\right) 1_{\left\{I_{t_{j}}<B\right\}}\right. \\
&\left.\left.\quad-\max \left(S_{t_{j+1}}-K, 0\right) 1_{\left\{I_{t_{j+1}}<B\right\}}\right)\right] .
\end{aligned}
$$

The challenge is the modeling of the joint processes of $\left\{S_{t_{j+1}}, I_{t_{j}}\right\}$ (two state variables at two different time levels $t_{j}$ and $t_{j+1}$ ) and $\left\{S_{t_{j+1}}, I_{t_{j+1}}\right\}$.

Under a stochastic volatility model, the $\log$ return $\log \frac{S_{t_{j+1}}}{S_{t_{j}}}$ has dependence on the stochastic process of the instantaneous variance.

## Stochastic volatility model

Consider the stochastic volatility model specified as follows:

$$
\begin{aligned}
\frac{\mathrm{d} S_{t}}{S_{t}} & =(r-q) \mathrm{d} t+\sqrt{v_{t}}\left(\rho \mathrm{~d} W_{t}+\sqrt{1-\rho^{2}} \mathrm{~d} W_{t}^{v}\right) \\
\mathrm{d} v_{t} & =\alpha\left(v_{t}\right) \mathrm{d} t+\beta\left(v_{t}\right) \mathrm{d} W_{t}^{v}
\end{aligned}
$$

where $\rho$ is the correlation coefficient between the asset price process $S_{t}$ and instantaneous variance process $v_{t}, W_{t}$ and $W_{t}^{v}$ are two independent Brownian motions. The drift function $\alpha\left(v_{t}\right)$ and the volatility function $\beta\left(v_{t}\right)$ are measurable functions with respect to the natural filtration generated by the two correlated Brownian motions.

Usually, $\beta\left(v_{t}\right)$ assumes the form of a power function. The common choices of the power are $3 / 2$ and $1 / 2$.

- For the 3/2-model, we choose

$$
\alpha\left(v_{t}\right)=v_{t}\left(\theta_{t}-\kappa v_{t}\right) \quad \text { and } \quad \beta\left(v_{t}\right)=\epsilon v_{t}^{3 / 2}
$$

- For the Heston 1/2-model, we choose

$$
\alpha\left(v_{t}\right)=\lambda\left(\theta_{t}-v_{t}\right) \quad \text { and } \quad \beta\left(v_{t}\right)=\eta v_{t}^{1 / 2}
$$

## Analytic evaluation in the Fourier domain

Write $\log$ asset price $X_{t}=\ln S_{t}$ and integrated variance $I_{t}=\int_{0}^{t} v_{s} \mathrm{~d} s$, where $I_{t}$ is used as a proxy for the discrete realized variance used in the knock-out condition in the timer option.

Pricing of the timerlets involves the joint process of $S_{t}$ and $I_{t}$ (may or may not be at the same time point).

Let $x$ stands for $\ln S_{t_{j+1}}$ and $y$ stands for $I_{t_{j}}$ or $I_{t_{j+1}}$. The Fourier transform $\widehat{F}(\omega, \eta)$ of the terminal payoff $\left(S_{t_{j+1}}-K, 0\right) 1_{\left\{I_{t_{j}}<B\right\}}$ and $\left(S_{t_{j+1}}-K, 0\right) 1_{\left\{I_{t_{j+1}}<B\right\}}$ admit the same analytic representation

$$
\widehat{F}(\omega, \eta)=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\mathrm{i} \omega x-\mathrm{i} \eta y}\left(e^{x}-K\right)^{+} 1_{\{y<B\}} \mathrm{d} x \mathrm{~d} y=\frac{K^{1-\mathrm{i} \omega} e^{-\mathrm{i} \eta B}}{\left(\mathrm{i} \omega+\omega^{2}\right) \mathrm{i} \eta}
$$

We take $\omega=\omega_{R}+i \omega_{I}$ and $\eta=\eta_{R}+i \eta_{I}$, where the damping factors are chosen such that $\omega_{I}<-1$ and $\eta_{I}<0$, to ensure the existence of the two-dimensional Fourier transform.

## Fourier integral representation of the price formula

By the Parseval Theorem, the finite-maturity discrete timer option price admits the following analytic formula in terms of a twodimensional Fourier integral:

$$
\begin{aligned}
C_{0}= & \frac{1}{4 \pi^{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-r T} \widehat{F}(\omega, \eta) \mathbb{E}_{0}\left[e^{\mathrm{i} \omega X_{t_{N}}+\mathrm{i} \eta I_{t_{N}}}\right] \mathrm{d} \omega_{R} \mathrm{~d} \eta_{R} \\
& +\sum_{j=0}^{N-1} \frac{1}{4 \pi^{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-r t_{j+1}} \\
& \left\{\widehat{F}(\omega, \eta) \mathbb{E}_{0}\left[e^{\mathrm{i} \omega X_{t_{j+1}}+\mathrm{i} \eta I_{t}}\right]-\widehat{F}(\omega, \eta) \mathbb{E}_{0}\left[e^{\mathrm{i} \omega X_{t_{j+1}}+\mathrm{i} \eta I_{t}}\right]\right\} \mathrm{d} \omega_{R} \mathrm{~d} \eta_{R} .
\end{aligned}
$$

The challenging tasks involve the determination of the conditional characteristic functions:

$$
E_{0}\left[e^{i \omega X_{t_{j+1}}+i \eta I_{t}+1}\right] \quad \text { and } \quad E_{0}\left[e^{i \omega X_{t_{j+1}}+i \eta I_{t_{j}}}\right]
$$

under the relevant stochastic volatility model.

## Partial Fourier transform of the triple joint density function

Let $G\left(t, x, y, v ; t^{\prime}, x^{\prime}, y^{\prime}, v^{\prime}\right)$ be the joint transition density of the triple ( $X, I, V$ ) from state $(x, y, v)$ at time $t$ to state $\left(x^{\prime}, y^{\prime}, v^{\prime}\right)$ at a later time $t^{\prime}$. The joint transition density $G$ satisfies the following threedimensional Kolmogorov backward equation:

$$
\begin{aligned}
-\frac{\partial G}{\partial t}= & \left(r-q-\frac{v}{2}\right) \frac{\partial G}{\partial x}+\frac{v}{2} \frac{\partial^{2} G}{\partial x^{2}}+v \frac{\partial G}{\partial y}+\alpha(v) \frac{\partial G}{\partial v} \\
& +\frac{\beta(v)^{2}}{2} \frac{\partial^{2} G}{\partial v^{2}}+\rho \sqrt{v} \beta(v) \frac{\partial^{2} G}{\partial x \partial v}
\end{aligned}
$$

with the terminal condition:

$$
G\left(t^{\prime}, x, y, v ; t^{\prime}, x^{\prime}, y^{\prime}, v^{\prime}\right)=\delta\left(x-x^{\prime}\right) \delta\left(y-y^{\prime}\right) \delta\left(v-v^{\prime}\right)
$$

where $\delta(\cdot)$ is the Dirac delta function.

We define the generalized partial Fourier transform of $G$ by $G$ as follows:

$$
\breve{G}\left(t, x, y, v ; t^{\prime}, \omega, \eta, v^{\prime}\right)=\int_{-\infty}^{\infty} \int_{0}^{\infty} e^{\mathrm{i} \omega x^{\prime}+\mathrm{i} \eta y^{\prime}} G\left(t, x, y, v ; t^{\prime}, x^{\prime}, y^{\prime}, v^{\prime}\right) \mathrm{d} y^{\prime} \mathrm{d} x^{\prime}
$$

where the transform variables $\omega$ and $\eta$ are complex variables.

The partial transform $\breve{G}$ solves the three-dimensional Kolmogorov equation with the terminal condition:

$$
\breve{G}\left(t^{\prime}, x, y, v ; t^{\prime}, \omega, \eta, v^{\prime}\right)=e^{\mathrm{i} \omega x+\mathrm{i} \eta y} \delta\left(v-v^{\prime}\right)
$$

The conditional characteristic function of $\left(X_{t}, I_{t}\right)$ conditional on ( $X_{t^{\prime}}, I_{t^{\prime}}$ ) can be obtained by integrating $\breve{G}\left(t, x, y, v ; t^{\prime}, \omega, \eta, v^{\prime}\right)$ with respect to $v^{\prime}$ from 0 to $\infty$.

Note that $G$ admits the following solution form:

$$
\breve{G}\left(t, x, y, v ; t^{\prime}, \omega, \eta, v^{\prime}\right)=e^{\mathrm{i} \omega x+\mathrm{i} \eta y} g\left(t, v ; t^{\prime}, \omega, \eta, v^{\prime}\right)
$$

where $g$ satisfies the following one-dimensional partial differential equation:

$$
\begin{aligned}
-\frac{\partial g}{\partial t}= & {\left[\mathrm{i} \omega\left(r-q-\frac{v}{2}\right)-\omega^{2} \frac{v}{2}+\mathrm{i} \eta v\right] g } \\
& +[\alpha(v)+\mathrm{i} \omega \rho \sqrt{v} \beta(v)] \frac{\partial g}{\partial v}+\frac{\beta(v)^{2}}{2} \frac{\partial^{2} g}{\partial v^{2}}
\end{aligned}
$$

with the terminal condition:

$$
g\left(t^{\prime}, v ; t^{\prime}, \omega, \eta, v^{\prime}\right)=\delta\left(v-v^{\prime}\right)
$$

The double generalized Fourier transform on the log-asset and integrated variance pair reduces the three-dimensional governing equation to a one-dimensional equation.

## Conditional characteristic functions

The conditional characteristic function of $\left(X_{t_{j}}, I_{t_{j}}\right)$ is found by integrating $\breve{G}\left(t_{0}, X_{t_{0}}, I_{t_{0}}, v_{0} ; t_{j}, X_{t_{j}}, I_{t_{j}}, v^{\prime}\right)$ with respect to $v^{\prime}$ from 0 to $\infty$

$$
\begin{aligned}
\mathbb{E}_{0}\left[e^{\mathrm{i} \omega X_{t_{j}}+\mathrm{i} \eta I_{t_{j}}}\right] & =\int_{0}^{\infty} \breve{G}\left(t_{0}, X_{t_{0}}, I_{t_{0}}, v_{0} ; t_{j}, X_{t_{j}}, I_{t_{j}}, v^{\prime}\right) \mathrm{d} v^{\prime} \\
& =e^{\mathrm{i} \omega X_{0}+\mathrm{i} \eta I_{0}} \int_{0}^{\infty} g\left(t_{0}, v_{0} ; t_{j}, \omega, \eta, v^{\prime}\right) \mathrm{d} v^{\prime} \\
& =e^{\mathrm{i} \omega X_{0}+\mathrm{i} \eta I_{0}} h\left(t_{0}, v_{0} ; t_{j}, \omega, \eta\right)
\end{aligned}
$$

The expectation calculation

$$
\mathbb{E}_{0}\left[e^{-r t_{j+1}} \max \left(S_{t_{j+1}}-K, 0\right) \boldsymbol{1}_{\left\{I_{\left.t_{j}<B\right\}}\right.}\right]
$$

requires the joint conditional characteristic function of $I_{t}$ at $t_{j}$ and $S_{t}$ at $t_{j+1}$.

## Iterated expectation

Working backward in time from $t_{j+1}$ to $t_{j}$, we compute $E_{t_{j}}\left[e^{\mathrm{i} \omega X_{t_{j+1}}}\right]$; and from $t_{j}$ to $t_{0}$, we compute $E_{0}\left[e^{i \omega X_{t_{j}}+i \eta I_{t}}\right]$. This is done by setting $\eta=0$ in $h\left(t_{j}, v^{\prime} ; t_{j+1}, \omega, \eta\right)$ and integrating $g\left(t_{0}, v_{0} ; t_{j}, \omega, \eta, v^{\prime}\right)$ $h\left(t_{j}, v^{\prime} ; t_{j+1}, \omega, 0\right)$ over $v^{\prime}$ from 0 to $\infty$.


By the two-step expectation calculation, we obtain

$$
\begin{aligned}
& \mathbb{E}_{0}\left[e^{\mathrm{i} \omega X_{t_{j+1}}+\mathrm{i} \eta I_{t_{j}}}\right] \\
= & e^{\mathrm{i} \omega X_{0}+\mathrm{i} \eta I_{0}} \int_{0}^{\infty} g\left(t_{0}, v_{0} ; t_{j}, \omega, \eta, v^{\prime}\right) h\left(t_{j}, v^{\prime} ; t_{j+1}, \omega, 0\right) \mathrm{d} v^{\prime}
\end{aligned}
$$

Here, $v^{\prime}$ is the dummy variable for the instantaneous variance $v_{t_{j}}$.

Analytic expressions for $g$ and $h$ under the 3/2-model

For the 3/2-model, we manage to obtain

$$
\begin{aligned}
& g\left(t, v ; t^{\prime}, v^{\prime}\right) \\
= & e^{a\left(t^{\prime}-t\right)} \frac{A_{t}}{C_{t}} \exp \left(-\frac{A_{t} v+v^{\prime}}{C_{t} v v^{\prime}}\right)\left(v^{\prime}\right)^{-2}\left(\frac{A_{t} v}{v^{\prime}}\right)^{\frac{1}{2}+\frac{\tilde{\tilde{\kappa}}}{\varepsilon^{2}}} I_{2 c}\left(\frac{2}{C_{t}} \sqrt{\frac{A_{t}}{v v^{\prime}}}\right),
\end{aligned}
$$

where $I_{2 c}$ is the modified Bessel function of order $2 c$,

$$
\begin{gathered}
a=\mathrm{i} \omega(r-q), \quad \tilde{\kappa}=\kappa-\mathrm{i} \omega \rho \varepsilon, \quad A_{t}=e^{\int_{t}^{t^{\prime}} \theta_{s} \mathrm{~d} s} \\
C_{t}=\frac{\varepsilon^{2}}{2} \int_{t}^{t^{\prime}} e^{\int_{t}^{s} \theta_{s^{\prime}} \mathrm{d} s^{\prime}} \mathrm{d} s, \quad c=\sqrt{\left(\frac{1}{2}+\frac{\tilde{\kappa}}{\varepsilon^{2}}\right)^{2}+\frac{\mathrm{i} \omega+\omega^{2}-2 \mathrm{i} \eta}{\varepsilon^{2}}} \\
h\left(t, v ; t^{\prime}, \omega, \eta\right)=\int_{0}^{\infty} g\left(t, v_{t} ; t^{\prime}, \omega, \eta, v^{\prime}\right) d v^{\prime} \\
\quad=e^{a\left(t^{\prime}-t\right)} \frac{\Gamma(\widetilde{\beta}-\tilde{\alpha})}{\Gamma(\widetilde{\beta})}\left(\frac{1}{C_{t} v}\right)^{\tilde{\alpha}} M\left(\tilde{\alpha}, \widetilde{\beta},-\frac{1}{C_{t} v}\right),
\end{gathered}
$$

where $\tilde{\alpha}=-\frac{1}{2}-\frac{\tilde{\kappa}}{\varepsilon^{2}}+c, \widetilde{\beta}=1+2 c$, $\Gamma$ is the gamma function, $M$ is the confluent hypergeometric function of the first kind.

Proof of the formula for $g$
Write $\tilde{g}=g e^{a\left(t-t^{\prime}\right)}$ and define

$$
\tilde{g}\left(t, v ; t^{\prime}, \omega, \eta, v^{\prime}\right)=\frac{u^{\alpha}}{\left(u^{\prime}\right)^{\alpha-2}} f\left(t, u ; t^{\prime}, \omega, \eta, u^{\prime}\right)
$$

then the governing equation for $f$ becomes

$$
\begin{aligned}
-\frac{\partial f}{\partial t}= & \frac{\varepsilon^{2} u}{2} \frac{\partial^{2} f}{\partial u^{2}}+\left[\varepsilon^{2}(\alpha+1)+\tilde{\kappa}-\theta_{t} u\right] \frac{\partial f}{\partial u}-\alpha \theta_{t} f \\
& +\left[\frac{\varepsilon^{2}}{2}\left(\alpha^{2}+\alpha\right)+\tilde{\kappa} \alpha-\frac{i \omega+\omega^{2}}{2}+i \eta\right] \frac{f}{u}
\end{aligned}
$$

subject to

$$
f\left(t^{\prime}, u ; t^{\prime}, \omega, \eta, u^{\prime}\right)=\delta\left(u-u^{\prime}\right)
$$

We choose the free parameter $\alpha$ so that the coefficient of $f / u$ vanishes, thus eliminating the occurrence of complex numbers in the governing equation.

Apparently, we choose $\alpha=\alpha(\omega, \eta)$ such that

$$
\frac{\varepsilon^{2}}{2}\left(\alpha^{2}+\alpha\right)+\tilde{\kappa} \alpha-\frac{i \omega+\omega^{2}}{2}+i \eta=0
$$

so the governing equation of $f$ becomes

$$
-\frac{\partial f}{\partial t}=\frac{\varepsilon^{2} u}{2} \frac{\partial^{2} f}{\partial u^{2}}+\left[\varepsilon^{2}(\alpha+1)+\tilde{\kappa}-\theta_{t} u\right] \frac{\partial f}{\partial u}-\alpha \theta_{t} f
$$

where all the coefficients are affine in $u$.
It follows that $\alpha$ can take two values

$$
\alpha=-\left(\frac{1}{2}+\frac{\tilde{\kappa}}{\varepsilon^{2}}\right) \pm c
$$

where

$$
c(\omega, \eta)=\sqrt{\left(\frac{1}{2}+\frac{\tilde{\kappa}}{\varepsilon^{2}}\right)^{2}+\frac{\mathrm{i} \omega+\omega^{2}-2 \mathrm{i} \eta}{\varepsilon^{2}}}
$$

We choose the positive sign for $c$ in $\alpha$ due to the technical condition required for the existence of the Laplace transform of $f$ in the later procedure.

We write the Laplace transform of $f$ with respect to $u^{\prime}$ as follows

$$
\widehat{f}\left(t, u ; t^{\prime}, \omega, \eta, \xi\right)=\int_{0}^{\infty} e^{-\xi u^{\prime}} f\left(t, u ; t^{\prime}, \omega, \eta, u^{\prime}\right) \mathrm{d} u^{\prime}
$$

Then, $\hat{f}$ admits the following exponential affine solution:

$$
\widehat{f}\left(t, u ; t^{\prime}, \omega, \eta, \xi\right)=\exp (B(t, \xi) u+D(t, \xi))
$$

where $B(t, \xi)$ and $D(t, \xi)$ are parameter functions determined by the following Riccati system of ODEs:

$$
\begin{aligned}
-\frac{\partial B}{\partial t} & =\frac{\varepsilon^{2}}{2} B^{2}-\theta_{t} B \\
-\frac{\partial D}{\partial t} & =\left[\varepsilon^{2}(\alpha+1)+\tilde{\kappa}\right] B-\alpha \theta_{t}
\end{aligned}
$$

with boundary conditions $B\left(t^{\prime}, \xi\right)=-\xi$ and $D\left(t^{\prime}, \xi\right)=0$. It can be found that

$$
B(t, \xi)=-\frac{\xi}{A_{t}+C_{t} \xi}
$$

where

$$
A_{t}=e^{\int_{t}^{t^{\prime}}} \theta_{s} \mathrm{~d} s, \quad C_{t}=\frac{\varepsilon^{2}}{2} \int_{t}^{t^{\prime}} e^{\int_{t}^{s} \theta_{\tau} \mathrm{d} \tau} \mathrm{~d} s
$$

We obtain

$$
D(t, \xi)=-2\left[\alpha+1+\frac{\tilde{\kappa}}{\varepsilon^{2}}\right] \ln \left(A_{t}+C_{t} \xi\right)+\left[\alpha+2+\frac{2 \tilde{\kappa}}{\varepsilon^{2}}\right] \int_{t}^{t^{\prime}} \theta_{s} \mathrm{~d} s
$$

Next, we take the inverse Laplace transform of

$$
\widehat{f}\left(t, u ; t^{\prime}, \omega, \eta, \xi\right)=A_{t}^{\alpha+2+\frac{2 \tilde{\kappa}}{\varepsilon^{2}}} \exp \left(-\frac{\xi u}{A_{t}+C_{t} \xi}\right)\left(A_{t}+C_{t} \xi\right)^{-2 \alpha-2-\frac{2 \tilde{\kappa}}{\varepsilon^{2}}}
$$

to obtain

$$
\begin{aligned}
f\left(t, u ; t^{\prime}, \omega, \eta, u^{\prime}\right) & =\frac{A_{t}^{\frac{3}{2}+c+\frac{\tilde{\kappa}}{\varepsilon^{2}}}}{2 \pi \mathrm{i} C_{t}} \int_{\tau-\mathrm{i} \infty}^{\tau+\mathrm{i} \infty} e^{\frac{u^{\prime}\left(p-A_{t}\right)}{C_{t}}} p^{-2 c-1} e^{-\frac{u\left(p-A_{t}\right)}{C_{t} p}} \mathrm{~d} p \\
& =\frac{A_{t}^{\frac{3}{2}+c+\frac{\tilde{\mathfrak{F}}}{\varepsilon^{2}}} e^{-\frac{u+A_{t} u^{\prime}}{C_{t}}}}{2 \pi \mathrm{i} C_{t}} \int_{\tau-\mathrm{i} \infty}^{\tau+\mathrm{i} \infty} e^{\frac{u^{\prime} p}{C_{t}}} p^{-2 c-1} e^{\frac{u A_{t}}{C_{t} p}} \mathrm{~d} p \\
& =\frac{A_{t}^{\frac{3}{2}-c+\frac{\tilde{\mathcal{F}}}{\varepsilon^{2}}}}{C_{t}} e^{-\frac{u+A_{t} u^{\prime}}{C_{t}}}\left(\frac{A_{t} u^{\prime}}{u}\right)^{c} I_{2 c}\left(\frac{2}{C_{t}} \sqrt{A_{t} u u^{\prime}}\right)
\end{aligned}
$$

Expressed in terms of $v$ and $v^{\prime}, g$ is found to be
$g\left(t, v ; t^{\prime}, \omega, \eta, v^{\prime}\right)=e^{a\left(t^{\prime}-t\right)} \frac{A_{t}}{C_{t}} \exp \left(-\frac{A_{t} v+v^{\prime}}{C_{t} v v^{\prime}}\right) \frac{1}{\left(v^{\prime}\right)^{2}}\left(\frac{A_{t} v}{v^{\prime}}\right)^{\frac{1}{2}+\frac{\tilde{\kappa}}{\varepsilon^{2}}} I_{2 c}\left(\frac{2}{C_{t}} \sqrt{\frac{A_{t}}{v v^{\prime}}}\right)$.

Proof of the formula for $h$

$$
\begin{aligned}
h\left(t, v ; t^{\prime}, \omega, \eta\right) & =\int_{0}^{\infty} g\left(t, v ; t^{\prime}, \omega, \eta, v^{\prime}\right) \mathrm{d} v^{\prime} \\
& =e^{a\left(t^{\prime}-t\right) \frac{A_{t}}{C_{t}} \int_{0}^{\infty} e^{-\frac{u+A_{t} u^{\prime}}{C_{t}}}\left(\frac{A_{t} u^{\prime}}{u}\right)^{\frac{1}{2}+\frac{\tilde{\kappa}}{\varepsilon^{2}}} I_{2 c}\left(\frac{2}{C_{t}} \sqrt{A_{t} u u^{\prime}}\right) \mathrm{d} u^{\prime}} \\
& =\frac{e^{a\left(t^{\prime}-t\right)-\frac{u}{C_{t}}}}{C_{t} u^{\frac{1}{2}+\frac{\tilde{\kappa}}{\varepsilon^{2}}}} \int_{0}^{\infty} e^{-\frac{z}{C_{t}} z^{\frac{1}{2}+\frac{\tilde{\kappa}}{\varepsilon^{2}}} I_{2 c}\left(\frac{2 \sqrt{u z}}{C_{t}}\right) \mathrm{d} z}
\end{aligned}
$$

where $z=A_{t} u^{\prime}$. Thanks to the closed form formula for the inverse Laplace transform of $t^{\iota} I_{\varsigma}(\lambda \sqrt{t})$, we have

$$
\int_{0}^{\infty} e^{-s t} t^{\iota} I_{\varsigma}(\lambda \sqrt{t}) \mathrm{d} t=\frac{\Gamma(\phi)}{\Gamma(\psi)} \frac{X^{\varsigma / 2}}{s^{1+\iota}} M(\phi, \psi, X)
$$

where $\phi=1+\iota+\varsigma / 2, \psi=1+\varsigma, X=\frac{\lambda^{2}}{4 s}$ and $\mathfrak{R}(\phi, s)>0$, we obtain

$$
\begin{aligned}
h\left(t, v ; t^{\prime}, \omega, \eta\right) & =e^{a\left(t^{\prime}-t\right)-\frac{u}{C_{t}}} \frac{\Gamma(1-\alpha)}{\Gamma(2 c+1)}\left(\frac{u}{C_{t}}\right)^{\tilde{\alpha}} M\left(1-\alpha, 2 c+1, \frac{u}{C_{t}}\right) \\
& =e^{a\left(t^{\prime}-t\right) \frac{\Gamma(\widetilde{\beta}-\tilde{\alpha})}{\Gamma(\widetilde{\beta})}\left(\frac{u}{C_{t}}\right)^{\tilde{\alpha}} M\left(\tilde{\alpha}, \widetilde{\beta},-\frac{u}{C_{t}}\right)} \\
& =e^{a\left(t^{\prime}-t\right) \frac{\Gamma(\widetilde{\beta}-\tilde{\alpha})}{\Gamma(\widetilde{\beta})}\left(\frac{1}{C_{t} v}\right)^{\tilde{\alpha}} M\left(\tilde{\alpha}, \widetilde{\beta},-\frac{1}{C_{t} v}\right)} .
\end{aligned}
$$

Here,

$$
\tilde{\alpha}=-\frac{1}{2}-\frac{\tilde{\kappa}}{\varepsilon^{2}}+c, \quad \text { and } \quad \tilde{\beta}=1+2 c
$$

Note that the second equality follows from the identity:

$$
M(a, b, z)=e^{z} M(b-a, b,-z)
$$

Heston model

| $S_{0}$ | $T$ | $r$ | $q$ | $B$ | $N$ | $\lambda$ | $\eta$ | $\bar{v}$ | $v_{0}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 100 | 1.5 | 0.015 | 0 | 0.087 | 300 | 2 | 0.375 | 0.09 | 0.087 |

Parameter values in the Heston model and finite-maturity discrete timer options

| $K$ | $\rho$ | Hilbert | MC | RE(\%) |
| :---: | :---: | :---: | :---: | :---: |
| 90 | -0.5 | 17.6905 | 17.6927 | -0.0124 |
|  | 0 | 17.5517 | 17.5551 | -0.0194 |
|  | 0.5 | 17.4910 | 17.4882 | 0.0160 |
| 100 | -0.5 | 12.3996 | 12.4099 | -0.0830 |
|  | 0 | 12.2804 | 12.2909 | -0.0854 |
|  | 0.5 | 12.2647 | 12.2692 | -0.0367 |
| 110 | -0.5 | 8.4174 | 8.4313 | -0.1649 |
|  | 0 | 8.3503 | 8.3634 | 0.1566 |
|  | 0.5 | 8.3716 | 8.3774 | -0.0692 |

Comparison of the numerical results for the finite-maturity discrete timer call options for varying strike prices $K$ and correlation values $\rho$ obtained from the fast Hilbert transform algorithm with the benchmark results obtained using the Monte Carlo method (MC) under the Heston model.

3/2-model

| $S_{0}$ | $T$ | $r$ | $q$ | $B$ | $N$ | $\lambda$ | $\eta$ | $\bar{v}$ | $v_{0}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 100 | 1.5 | 0.015 | 0 | 0.087 | 200 | 22.84 | 8.56 | 0.218 | 0.087 |

Parameter values in the $3 / 2$ model and finite-maturity discrete timer options

| $K$ | $\rho$ | Hilbert | MC | RE(\%) |
| :---: | :---: | :---: | :---: | :---: |
| 90 | -0.5 | 17.7155 | 17.7383 | -0.1285 |
|  | 0 | 17.5778 | 17.5892 | -0.0648 |
|  | 0.5 | 17.4923 | 17.5016 | -0.0531 |
| 100 | -0.5 | 12.4366 | 12.4594 | -0.1830 |
|  | 0 | 12.3195 | 12.3328 | -0.1078 |
|  | 0.5 | 12.2759 | 12.2856 | -0.0790 |
| 110 | -0.5 | 8.4608 | 8.4802 | -0.2287 |
|  | 0 | 8.3951 | 8.4063 | -0.1332 |
|  | 0.5 | 8.3897 | 8.3962 | -0.0774 |

Comparison of the numerical results for finite-maturity discrete timer call options for varying strike prices $K$ and correlation values $\rho$ obtained from the fast Hilbert transform algorithm with the benchmark results obtained using the Monte Carlo method (MC) under the 3/2 stochastic volatility model.

Sensitivity analysis on volatility of variance $\eta$ and correlation coefficient $\rho$ under the Heston model

- The price function may not be a monotonically increasing function of $\eta$.
- When $\rho=-0.5$, the discrete timer call option price firstly increases and then decreases with increasing value of $\eta$.
- When $\rho=0.5$, the discrete timer call option price is a decreasing function of $\eta$.

Sensitivity analysis under the Heston model

| $\rho$ | $\eta$ | $K=90$ | $K=94$ | $K=98$ | $K=102$ | $K=106$ | $K=110$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 0.15 | 17.6571 | 15.3986 | 13.3621 | 11.5434 | 9.9315 | 8.5091 |
| -0.5 | 0.3 | 17.7028 | 15.4356 | 13.3888 | 11.5585 | 9.9342 | 8.4989 |
|  | 0.45 | 17.6654 | 15.3651 | 13.2840 | 11.4197 | 9.7630 | 8.2986 |
|  | 0.15 | 17.5859 | 15.3234 | 13.2845 | 11.4650 | 9.8537 | 8.4333 |
| 0.5 | 0.3 | 17.5453 | 15.2842 | 13.2475 | 11.4307 | 9.8226 | 8.4056 |
|  | 0.45 | 17.4522 | 15.1898 | 13.1569 | 11.3472 | 9.7483 | 8.3413 |

Comparison of the numerical values for finite-maturity discrete timer call option prices with varying values of strike prices, volatility of variance and correlation coefficient under the Heston model.


Plot of the finite-maturity discrete timer call option prices against variance budget $B$. The discrete timer call option price tends to that of the vanilla European call option (shown in the dashed line) when $B$ is sufficiently large.


Plot of the finite-maturity discrete timer call option prices against number of monitoring instants $N$. The dashed line represents the finite-maturity timer call option price under continuous monitoring.


Plot of the finite-maturity discrete timer call option price versus maturity (mandated) under two different values of the variance budget. The price sensitivity to maturity can be quite significant for shortlived timer options.

## Conclusion

- Assuming perpetuity, we manage to derive closed form price formula of a timer option in terms of a triple integral. This is done by performing integration of the Black-Scholes type formula with respect to the joint law of functionals of the instantaneous variance.
- By decomposing a timer option into a portfolio of timerlets, we manage to price a finite-maturity timer option based on the explicit representation of the joint characteristic function of log asset price and its integrated variance.
- Our numerical tests on pricing the finite-maturity discrete timer options under the Heston model and $3 / 2$ model demonstrate high level of numerical accuracy and robustness of the fast Hilbert algorithm for pricing options with exotic barrier feature.

