A Tale of Two Options:
Employee Reload Options and Shout Call Options

presented by

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Nature of employee stock options

as equity-linked compensation package

Attraction, retention and motivation of employees

- Non-transferable; may be exercised early (say, when the employee plans to leave) while an unconstrained investor ordinarily would sell the option.

- Constraints on short-selling the firm’s stock for hedging the risk of holding the option.

- Other embedded features, like reload and delayed vesting requirement.

Employees are inherently undiversified – physical as well as human capital invested disproportionately in their company.
Propensity of early exercise

Early exercise will be more pronounced when employees are more risk averse (portfolio diversification motives). The only way for employees to diversify portfolios that are dependent on firm’s fortune is to exercise. This would reduce the expected time until exercise and hence the option value.

Retention incentives

• *performance vesting feature*
  – exercisable only when the stock price rises above some threshold value

• *time vesting feature*
  – exercisable only after an initial period has lapsed

• *reload feature*

Combination of these features.
Importance of Value Estimation

- These options represent substantial claims against the employer.

Distinguish between

- cost to the employer
- value to the employees

Pricing methodologies

- Black-Scholes framework gives the market value with no constraints on selling, etc. Black-Scholes’ price gives the upper bound of the value to the holder.

- Utility maximization approach
  - Holder’s risk preference would enter into the model;
  - Use true probability distribution of stock price evolution.
How reloads work?

Example

\[ X_{original} = \$100, \quad S_\xi = \$150; \]

\[ \frac{\$100}{\$150} = \frac{2}{3} \text{ units of owned share tendered to pay as strike upon exercise.} \]

Under the reload provision, the holder will be granted \( \frac{2}{3} \) unit of new option, with the same expiration date as the original option and the strike price set at \( \$150 \).

Remark

The upper bound on a reload option is the underlying stock price, no matter how many reloads are possible and how long is the maturity.
Upper bound on the value of a reload option

- On the first exercise (at $S_1$), the holder receives $1 - \frac{K}{S_1}$ shares and $\frac{K}{S_1}$ new reload option with strike $S_1$.

- On the second exercise (at $S_2$), the employee nets an additional $\frac{K}{S_1} \left( 1 - \frac{S_1}{S_2} \right)$ shares for a total of $1 - \frac{K}{S_1} + \frac{K}{S_1} \left( 1 - \frac{S_1}{S_2} \right) = 1 - \frac{K}{S_2}$ shares and $\left( \frac{K}{S_1} \right) \left( \frac{S_1}{S_2} \right)$ new reload options with strike $S_2$.

- After the $i^{th}$ exercise (at $S_i$), the employee will hold $1 - \frac{K}{S_i}$ shares and $\frac{K}{S_i}$ new options with strike $S_i$.

Note that the value of the reload option is further reduced since the employee will not receive the early dividends.
Number of reloads allowed

- Norwest Corporation allows only one reload.

- First Bank System allows up to three reloads.

- First Chicago places no limit on the number of times an option may be reloaded.

* Some plans issue reload options for shares tendered to cover withholding tax on top of shares tendered to cover the strike price.

Benefits to employees

- The carried interest on the number of shares remain at one, independent of the number of reloads.

- It results in an increase in actual share ownership profit and voting rights on the profit shares.
Time vesting requirement

Allows unlimited reloads subject to a waiting period (most commonly 6 months) between reloads.
Performance vesting requirement

The reload provision is activated only when the stock price $S$ shoots beyond some threshold value $H$. 

![Graph showing performance vesting requirement]

- **Exercise boundary**
- **Stopping region**

stock price $X$ 

stock price $H$

time to expiry
Difficulties in valuation

The knock-in region intersects with the stopping region.
Valuations are needed for preparing accounting statements and tax returns

Non-feasibility in value estimation

For the reload feature, the Financial Accounting Standard Board (FASB) in 1995 recommended delaying estimation of the feature’s value:

The Board continues to believe that, ideally, the value of an option with a reload feature should be estimated at the grant date, taking into account all of its features. However, at this time, it is not feasible to do so. Accordingly, the Board concluded that the best way to account for an option with a reload feature is to treat both the initial grant and each subsequent grant of a reload option separately. (Statement of Financial Accounting Standards No. 123, ¶186, p.61)
### Panel A: Adoption Years for Option Plans with a Reload Feature

<table>
<thead>
<tr>
<th>Year</th>
<th>Number of Firms</th>
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<tr>
<td>Prior to 1990</td>
<td>12</td>
</tr>
<tr>
<td>1990</td>
<td>19</td>
</tr>
<tr>
<td>1991</td>
<td>30</td>
</tr>
<tr>
<td>1992</td>
<td>47</td>
</tr>
<tr>
<td>1993</td>
<td>37</td>
</tr>
<tr>
<td>1994</td>
<td>48</td>
</tr>
<tr>
<td>1995</td>
<td>24</td>
</tr>
<tr>
<td>Total</td>
<td>217</td>
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</table>

In US, 17% of new stock option plans in 1997 included some type of reload provision.
<table>
<thead>
<tr>
<th>Vesting Requirements</th>
<th>Count</th>
</tr>
</thead>
<tbody>
<tr>
<td>Six Months</td>
<td>34</td>
</tr>
<tr>
<td>One Year</td>
<td>15</td>
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<tr>
<td>Two Years</td>
<td>2</td>
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<tr>
<td>Performance Based</td>
<td>2</td>
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<td>Total</td>
<td>53</td>
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<td>Required Holding Period for Shares <em>Tendered</em> to Exercise an option</td>
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</tr>
<tr>
<td>---------------------------------------------------------------</td>
<td></td>
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<tr>
<td>Six Months</td>
<td>27</td>
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<tr>
<td>Other</td>
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<td>Total</td>
<td>30</td>
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</table>

<table>
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<tr>
<th>Required Holding Period for Shares <em>Received</em> from the Exercise of an Option</th>
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<tbody>
<tr>
<td>One Year</td>
</tr>
<tr>
<td>Other</td>
</tr>
<tr>
<td>Total</td>
</tr>
</tbody>
</table>
References


Justification of the Black-Scholes preference free pricing approach

- When the number of reloads is unlimited, Dybvig and Loewenstein showed by dominance argument that the holder should exercise whenever a new maxima of the stock price is realized. Such optimal policy is independent of the risk preferences of the holders.

- Though the utility value of the cashflows at exercise depends on the utility function of the risk averse holder, the exercise policy remains the same as that of a risk neutral holder.

- As the employer is aware of such optimal exercise policy adopted by the holder, and he has no constraint on hedging, so the market value of the options also represents the cost to the employer granting the reload option.
Objectives

1. Valuation of the reload feature under the Black-Scholes pricing framework.

2. Characterization of the optimal reload policies adopted by the holders
   - single-reload
   - multi-reload
   - infinite-reload.

3. Establish the relations of the reload options with the shout options and lookback options.

4. Time vesting requirement.
Shout call options

• Holder’s right to reset the strike price to the prevailing share price upon shouting. New terminal payoff becomes

$$\max(S_T - S_\xi, 0)$$

where $S_\xi$ is the share price at the shouting moment $\xi$.

• Let $c_{\text{shout}, n}(S, \tau; X)$ denote the price function of shout call with $n$ shouting rights outstanding. Upon shouting, the exercised payoff is

$$c_{\text{shout}, n-1}(S, \tau; S).$$
Single reload options

- Upon exercising, the employee receives one unit of stock and pays the strike $X$.

- Assumed that the employee uses $X/S$ units of owned stock for the strike payment so that

\[
\text{number of units of new stock received} = 1 - \frac{X}{S}.
\]

- The employee is assumed to keep these new stocks so that continuous dividend yields will be received.

- The employee receives $X/S$ units of new call option with strike price set at the prevailing stock price at the exercise moment and same maturity date $T$. 
Value of the reload option \[= S - X + \frac{X}{S} c(S, \tau; S, r, q).\]

By the linear homogeneity property of the call price function with respect to \(S\), we obtain

\[c(S, \tau; S, r, q) = S \hat{c}(\tau; r, q),\]

where

\[\hat{c}(\tau; r, q) = e^{-q\tau} N(\hat{d}_1) - e^{-r\tau} N(\hat{d}_2)\]

and

\[\hat{d}_1 = \frac{r - q + \frac{\sigma^2}{2}\sqrt{\tau}}{\sigma}\quad \text{and} \quad \hat{d}_2 = \frac{r - q - \frac{\sigma^2}{2}\sqrt{\tau}}{\sigma}.\]

Similarly,

\[p(S, \tau; S, r, q) = S \hat{p}(\tau; r, q)\]

where

\[\hat{p}(\tau; r, q) = e^{-r\tau} N(-\hat{d}_2) - e^{-q\tau} N(-\hat{d}_1).\]
Linear complementarity formulation for option pricing

The payoff function upon exercise of the single-reload option is $S - X + X\hat{c}(\tau; r, q)$.

$$\frac{\partial V_1}{\partial \tau} - \mathcal{L}_{r,q} V_1 \geq 0, \quad V_1(S, \tau) \geq S - X + X\hat{c}(\tau),$$

$$\left[ \frac{\partial V_1}{\partial \tau} - \mathcal{L}_{r,q} V_1 \right] \{V_1(S, \tau) - [S - X + X\hat{c}(\tau)]\} = 0,$$

$S \in (0, \infty), \tau \in (0, T],$

where the differential operator $\mathcal{L}_{r,q}$ is given by

$$\mathcal{L}_{r,q} = \frac{\sigma^2}{2} S^2 \frac{\partial^2}{\partial S^2} + (r - q) S \frac{\partial}{\partial S} - r, \quad r > 0 \quad \text{and} \quad q \geq 0.$$

$V_1(S, 0) = (S - X)^+.$
In the stopping region, $V_1 = S - X + X\hat{c}(\tau; r, q)$; we have

$$\left( \frac{\partial}{\partial\tau} - \mathcal{L}_{r,q} \right) [S - X + X\hat{c}(\tau; r, q)]$$

$$= qS - rX + X\frac{d\hat{c}}{d\tau}(\tau; r, q) + rX\hat{c}(\tau; r, q).$$

This quantity represents the delayed compensation cashflow rate and it must be positive in the stopping region.
Integral representation of the premium for the reload right

\[ V_1(S, \tau; X, r, q) = c(S, \tau; X, r, q) + R(S, \tau; X, r, q) \]

where \( R(S, \tau; X, r, q) \) is the reload premium. Let \( \psi(S_{\xi}; S) \) denote the transition density function of the stock price with \( S \) at the current time and \( S_{\xi} \) at \( \xi \) periods later.

\[
R(S, \tau; X, r, q) = \int_0^\tau e^{-r\xi} \int_{S^*(\tau-\xi)}^\infty \{ qS_{\xi} - rX + X[\hat{e}'(\tau - \xi) + rc(\tau - \xi)] \} \psi(S_{\xi}; S) \ dS_{\xi} d\xi
\]

\[
= \int_0^\tau \{ qSe^{-q\xi} N(d_{1,\xi}) - rX e^{-r\xi} N(d_{2,\xi}) + e^{-r\xi}X[\hat{e}'(\tau - \xi) + rc(\tau - \xi)]N(d_{2,\xi}) \} \ d\xi,
\]

where

\[
\psi(S_{\xi}; S) = \frac{1}{S_{\xi}\sigma\sqrt{2\pi\xi}} \exp \left( -\frac{\{ \ln S_{\xi} - \left[ \ln S + \left( r - q - \frac{\sigma^2}{2} \right) \xi \right] \}^2}{2\sigma^2\xi} \right),
\]

\[
d_{1,\xi} = \frac{\ln \frac{S}{S^*(\tau-\xi)} + \left( r - q + \frac{\sigma^2}{2} \right) \xi}{\sigma\sqrt{\xi}}, \quad d_{2,\xi} = d_{1,\xi} - \sigma\sqrt{\xi}.
\]
Integral equation

By setting $S = S^*(\tau)$, we obtain an integral equation to determine $S^*(\tau)$.

\[
c(S^*(\tau), \tau; X, r, q) + R(S^*(\tau), \tau; X, r, q) = S^*(\tau) - X + X\hat{c}(\tau; r, q).
\]

Solve for $S^*(\tau)$ at $\tau = j\Delta\tau, j = 1, 2, \cdots$ recursively, starting at $S^*(0^+)$. 

![Graph showing the integral equation and the solution process]
Zero dividend yield

The price function of a single-reload option can be expressed in terms of the price functions of a forward contract and a shout call option. By defining

\[ U_1(S, \tau; X, r, 0) = V_1(S, \tau; X, r, 0) - (S - Xe^{-r\tau}) \]

and observing the put-call parity

\[ X\hat{c}(\tau; r, 0) - (X - Xe^{-r\tau}) = X\hat{p}(\tau; r, 0), \]

one can show that \( U_1(S, \tau; X, r, 0) \) is governed by

\[ \frac{\partial U_1}{\partial \tau} - \mathcal{L}_{r,0} U_1 \geq 0, \quad U_1(S, \tau) \geq X\hat{p}(\tau; r, 0) \]

\[ \left[ \frac{\partial U_1}{\partial \tau} - \mathcal{L}_{r,0} U_1 \right] [U_1(S, \tau) - X\hat{p}(\tau; r, 0)] = 0, \quad S \in (0, \infty), \tau \in (0, T], \]

\[ U_1(S, 0) = (X - S)^+. \]
Suppose we use $S$ as the numeraire, and accordingly, we define

$$x = \frac{X}{S} \quad \text{and} \quad W_1(x, \tau) = \frac{1}{S} U_1(S, \tau).$$

By observing the put-call symmetry relation: $\hat{p}(\tau; r, 0) = \hat{c}(\tau; 0, r)$, then $W_1(x, \tau)$ observes the following linear complimentarity formulation

$$\frac{\partial W_1}{\partial \tau} - \mathcal{L}_{0,r} W_1 \geq 0, \quad W_1(x, \tau) \geq x \hat{c}(\tau; 0, r),$$

$$\left[ \frac{\partial W_1}{\partial \tau} - \mathcal{L}_{0,r} W_1 \right] \left[ W_1(x, \tau) - x \hat{c}(\tau; 0, r) \right] = 0, \quad x \in (0, \infty), \tau \in (0, T],$$

$$W_1(x, 0) = (x - 1)^+. $$
• The payoff upon exercise is $x \hat{c}(\tau; 0, r) = c(x, \tau; x_0, 0, r)$, which is a call option with strike price set at the prevailing stock price.

• Note that $W_1(x, \tau)$ is the price function of the one-shout call option with zero interest rate, unit strike price, dividend yield $r$ and strike price reset to the prevailing stock price upon shouting.

• In summary, we obtain

$$V_1(S, \tau; X, r, 0) = S - X e^{-r\tau} + c_{\text{shout}, 1}(X, \tau; S_0, 0, r),$$

where $c_{\text{shout}, 1}(S, \tau; X, r, q)$ is the price function of the one-shout call option.
Properties of the critical stock price $S^*_1(\tau; r, q)$

Let $S^*_1(\tau; r, q)$ denote the critical stock price of single-reload employee stock option that separates the stopping and continuation regions. The stopping region and $S^*_1(\tau)$ observe the following properties.

1. The stopping region is contained inside the region
   \[ \{(S, \tau) : S \geq X, \quad 0 \leq \tau \leq T \}. \]

2. At time close the expiry, $S^*_1(0^+; r, q) = X, q \geq 0, r > 0$. 
3. When the stock pays dividend at yield $q > 0$, the critical stock price at
infinite time to expiry is given by

$$S_1^*(\infty; r, q) = \frac{\mu_+}{\mu_+ - 1} X,$$

where

$$\mu_+ = \frac{- \left( r - q - \frac{\sigma^2}{2} \right) + \sqrt{\left( r - q - \frac{\sigma^2}{2} \right)^2 + 2\sigma^2 r}}{\sigma^2} > 1.$$

Both perpetual single-reload option and American option have the same critical
stock price. Why?
4. If the stock pays no dividend, then

(a) for \( r \leq \frac{\sigma^2}{2} \), \( S_1^*(\tau; r, 0) \) is defined for all \( \tau > 0 \) and \( S_1^*(\infty; r, 0) = \infty \);

(b) for \( r > \frac{\sigma^2}{2} \), \( S_1^*(\tau; r, 0) \) is defined only for \( 0 < \tau < \tau_1^* \), where \( \tau_1^* \) is the unique solution to the algebraic equation

\[-rN \left( -\frac{r + \frac{\sigma^2}{2}}{\sigma} \sqrt{\tau} \right) + \frac{\sigma}{2\sqrt{\tau}} n \left( -\frac{r + \frac{\sigma^2}{2}}{\sigma} \sqrt{\tau} \right) = 0.\]
Proof of Property (1)

It suffices to show that when $S < X$, the exercise payoff is always less than the value of the European call option.

Consider $D(S, \tau) = c(S, \tau; X, r, q) - [S - X + X \hat{c}(\tau; r, q)]$

$$D(S, \tau)\big|_{S=X} = 0 \quad \text{and} \quad \frac{\partial D}{\partial S} < 0.$$
**Proof of Property (2)**

Consider $\widehat{D}(S, \tau) = V_1(S, \tau; X, r, q) - [S - X + X\hat{c}(\tau; r, q)]$

Suppose $S_1^*(0^+) > X$ and for $S \in (X, S_1^*(0^+))$, we have

$$\widehat{D}(S, 0) = 0 \quad \text{and} \quad \frac{\partial \widehat{D}}{\partial \tau}(S, 0^+) < 0.$$ 

This leads to $\widehat{D}(S, \tau) < 0$ for $\tau \to 0^+$. A contradiction.
Multi-reload options

- When the stock pays no dividend, the price function of a $n$-reload option can be expressed as the sum of the price functions of a forward contract and a $n$-shout call option.

- From financial intuition, one would expect that the holder will exercise their reload right at a lower critical stock price when there are more reload rights outstanding.

- For $q > 0$, one can derive the recursive relation that relates the critical stock prices for perpetual multi-reload options.
Linear complementarity formulation

\[
\frac{\partial V_n}{\partial \tau} - \mathcal{L}_{r,q} V_n \geq 0, \quad V_n(S, \tau; X) \geq S - X + XV_{n-1}(1, \tau; 1)
\]

\[
\left(\frac{\partial V_n}{\partial \tau} - \mathcal{L}_{r,q} V_n\right) \{V_n(S, \tau; X) - [S - X + XV_{n-1}(1, \tau; 1)]\} = 0
\]

\[
S \in (0, \infty), \tau \in (0, T],
\]

\[
V_n(S, 0) = (S - X)^+.
\]

- The formulation is valid for \(n \geq 1\); and for notational convenience, we assume \(V_0(S, \tau; X, r, q)\) to be \(c(S, \tau; X, r, q)\).

- From financial intuition, it is obvious that \(V_{n+1}(S, \tau; X) > V_n(S, \tau; X)\) for all \(S > 0\) and \(\tau > 0\).
Relation with $n$-shout call option

When $q = 0$, we apply the transformation

$$W_n(x, \tau) = \frac{1}{S} [V_n(S, \tau; X, r, 0) - (S - X e^{-r\tau})] \quad \text{and} \quad x = \frac{X}{S}$$

to obtain

$$\frac{\partial W_n}{\partial \tau} - \mathcal{L}_{0,r} W_n \geq 0, \quad W_n(x, \tau) \geq x W_{n-1}(1, \tau),$$

$$\left[ \frac{\partial W_n}{\partial \tau} - \mathcal{L}_{0,r} W_n \right] [W_n(x, \tau) - x W_{n-1}(1, \tau)] = 0,$$

$$x \in (0, \infty), \quad \tau \in (0, T],$$

$$W_n(x, 0) = (x - 1)^+. 

Like the single-reload option, we can establish

$$V_n(S, \tau; X, r, 0) = S - X e^{-r\tau} + c_{\text{shout}, n}(X, \tau; S, 0, r),$$

where $c_{\text{shout}, n}(S, \tau; X, r, q)$ is the price function of a $n$-shout call option.
Properties of the critical stock price $S_n^*(\tau; r, q)$

- The holder should never exercise at $S < X$ and $S_n^*(\tau; r, q)$ starts at $X$ as $\tau \to 0^+$.

- Monotonic property: $S_{n+1}^*(\tau; r, q) < S_n^*(\tau; r, q)$, an obvious fact from financial intuition.

- When $q = 0$, the optimal exercise policy of a $n$-reload option can be related directly with that of the $n$-shout call counterpart.

- When $q > 0$, we obtain the recursive relation:

$$
\frac{S_n^*(\infty)}{X} = \frac{\mu_+}{\mu_+ - 1} - \frac{1}{\mu_+ - 1} \left[ \frac{S_{n-1}^*(\infty)}{X} \right]^{1-\mu_+}, \quad n > 1.
$$

- $S_n^*(\infty)$ is monotonically decreasing with respect to $n$ and $\lim_{n \to \infty} S_n^*(\infty) = X$. 
Value of $n$-reload employee option is bounded between an American call and one unit of stock.

1. At the first reload, we receive $1 - \frac{X}{S}$ shares and $\frac{X}{S_1}$ new options.

2. At the second reload, we receive

$$1 - \frac{X}{S_1} + \frac{X}{S_1} \left[ 1 - \frac{S_1}{S_2} \right] = 1 - \frac{X}{S_2}$$

share and $\frac{X}{S_2}$ new options, etc.

The value is further reduced because the employee will not receive the early dividends.
Employee stock options with infinite reloads

Consider the function

\[ F(S, \tau) = S e^{-q \tau} - X e^{-r \tau} + q \int_0^\tau (S e^{-q u} - X e^{-r u}) \, du, \]

which satisfies the equation

\[ \frac{\partial F}{\partial \tau} - L_{r,q} F = q (S - X) \]

\[ F(S, 0) = S - X. \]

We define the transformation

\[ U_\infty(S, \tau; X, r, q) = V_\infty(S, \tau; X, r, q) - F(S, \tau), \]

then \( U_\infty(S, \tau; X, r, q) \) satisfies linear complimentarity formulation

\[
\begin{align*}
\frac{\partial U_\infty}{\partial \tau} - L_{r,q} U_\infty & \geq q (X - S), \\
\left[ \frac{\partial U_\infty}{\partial \tau} - L_{r,q} U_\infty - q (X - S) \right] \left[ U_\infty(S, \tau; X) - X U_\infty(1, \tau; 1) \right] & = 0 \\
S & \in (0, \infty), \quad \tau \in (0, T], \\
U_\infty(S, 0) & = (X - S)^+. 
\end{align*}
\]
We define

\[ W_\infty(x, \tau) = \frac{1}{S} U_\infty(S, \tau) \quad \text{and} \quad x = \frac{X}{S}. \]

The corresponding governing equation for \( W_\infty(x, \tau) \) can be expressed as

\[
\frac{\partial W_\infty}{\partial \tau} - \mathcal{L}_{q,r} W_\infty \geq q(x - 1) \quad W_\infty(x, \tau) \geq x W_\infty(1, \tau) \\
\left[ \frac{\partial W_\infty}{\partial \tau} - \mathcal{L}_{q,r} W_\infty - q(x - 1) \right] \left[ W_\infty(x, \tau) - x W_\infty(1, \tau) \right] = 0,
\]

\[ x \in (0, \infty), \quad \tau \in (0, T], \]

\[ W_\infty(x, 0) = (x - 1)^+. \]
Let $c_{float}(S, m, \tau; r, q)$ denote the price function of a floating strike lookback call option with terminal payoff $S - m$, where $m$ is the realized minimum value of the stock price over the life of the option. Interestingly, $W_\infty(x, \tau)$ is related to $c_{float}(S, m, \tau; r, q)$ through the following relation:

$$W_\infty(x, \tau) = c_{float}(X, \min(S, X), \tau; q, r)$$
$$+ q \int_0^\tau c_{float}(X, \min(S, X), u; q, r) \, du.$$  

$$V_\infty(S, \tau; X, r, q) = SW_\infty \left( \frac{X}{S}, \tau \right) + F(S, \tau)$$
$$= c_{float}(S, \min(S, X), \tau; q, r) + (Se^{-q\tau} - Xe^{-r\tau})$$
$$+ q \int_0^\tau [c_{float}(S, \min(S, X), u; q, r) + (Se^{-qu} - Xe^{-ru})] \, du.$$
Dybvig and Loewenstein (2003) presented the following price formula for an infinite-reload option:

\[
V_\infty(S, T; X, r, q) = e^{-rT} E_{S_T} \left[ S_T \left( 1 - \frac{X}{M_T} \right)^+ \right] \\
+ q \int_0^T e^{-ru} E_{S_u} \left[ S_u \left( 1 - \frac{X}{M_u} \right)^+ \right] \, du
\]

where the current time is taken to be the zeroth time and $M_u$ is the realized maximum value of the stock price over $(0, u)$. To prove the equivalence of their price formula with ours, it suffices to show that

\[
G_0 = e^{-rT} E_{S_T} \left[ S_T \left( 1 - \frac{X}{M_T} \right)^+ \right] \\
= c_{float}(X, \min(X, S_0), T; q, r) + S_0 e^{-qT} - X e^{-rT}.
\]

**Remark**

They assume reloads to be allowed at preset instants, then take the number of allowable instants to be infinite.
Proof of the price formula for infinite-reload option

We take the limit $n \to \infty$ and define the infinite-shout call by

$$c_{\text{shout}}^\infty(S, \tau; X, r, q) = \lim_{n \to \infty} c_{\text{shout}, n}(S, \tau; X, r, q).$$

The governing equation for $c_{\text{shout}}^\infty(S, \tau)$ is given by

$$\frac{\partial c_{\text{shout}}^\infty}{\partial \tau} - \mathcal{L}_{r, q} c_{\text{shout}}^\infty = 0, \quad S > S_{\text{shout}, \infty}^*(\tau), \quad \tau > 0,$$

$$c_{\text{shout}}^\infty(S_{\text{shout}, \infty}^*(\tau), \tau) = S_{\text{shout}, \infty}^*(\tau) c_{\text{shout}}^\infty(1, \tau; 1),$$

$$\frac{\partial c_{\text{shout}}^\infty}{\partial S}(S_{\text{shout}, \infty}^*(\tau), \tau) = c_{\text{shout}}^\infty(1, \tau; 1),$$

$$c_{\text{shout}}^\infty(S, 0) = (S - X)^+. $$
Exercise policy of infinite-shout calls

The infinite-shout call has a simple exercise policy: the holder shouts when the option becomes at-the-money. We then have $S_{\text{shout,}\infty}^*(\tau) = X$ for all $\tau > 0$.

- Since the exercise boundary is known, the pricing model is no longer a free boundary value problem.

Remark

Mathematically, the free boundary value problem associated with the $n$-reload option reduces to a problem with fixed domain as $n \to \infty$. However, the dimension of the problem increases by one.
Pricing model of lookback call option

We consider the following pricing model of a floating strike lookback call option, whose price function is represented by $c_{float}(S, m, \tau; r, q)$:

$$\frac{\partial c_{float}}{\partial \tau} - r S c_{float} + \frac{\partial^2 c_{float}}{\partial m^2} |_{S=m} = 0, \quad S > m, \tau > 0,$$

$$c_{float}(S, m, 0) = S - m.$$
We apply the transformations

$$\bar{c}_{\text{float}}(y, \tau) = \frac{c_{\text{float}}(S, m, \tau)}{m}, \quad y = \frac{S}{m},$$

so that the governing equations of the pricing model of the lookback option can be transformed into

$$\frac{\partial \bar{c}_{\text{float}}}{\partial \tau} - \mathcal{L}_{r, q} \bar{c}_{\text{float}} = 0, \quad y > 1, \tau > 0,$$

$$\left. \frac{\partial \bar{c}_{\text{float}}}{\partial y} \right|_{y=1} = \bar{c}_{\text{float}}(1, \tau),$$

$$\bar{c}_{\text{float}}(y, 0) = y - 1.$$
Comparing the pricing formulations of the infinite-shout call and the floating strike lookback call option, we then conclude that

(a) when $S > X$,

$$c^\infty_{\text{shout}}(S, \tau; X) = Xc^\infty_{\text{shout}} \left( \frac{S}{X}, \tau; 1 \right) = X\tau_{\text{float}} \left( \frac{S}{X}, \tau \right) = c_{\text{float}}(S, X, \tau).$$

(b) when $S \leq X$,

$$c^\infty_{\text{shout}}(S, \tau; X) = Sc^\infty_{\text{shout}}(1, \tau; 1) = S\tau_{\text{float}}(1, \tau) = c_{\text{float}}(S, S, \tau).$$

Combining the above results, we obtain

$$c^\infty_{\text{shout}}(S, \tau; X) = c_{\text{float}}(S, \min(X, S), \tau).$$
We rewrite the pricing formulation of $W_\infty(x, \tau)$ into the following alternative form:

$$\frac{\partial W_\infty}{\partial \tau} - \mathcal{L}_{q, r} W_\infty = q(x - 1), \quad x > 1, \tau > 0$$

$$\frac{\partial W_\infty}{\partial x} - W_\infty|_{x=1} = 0,$$

$$W_\infty(x, 0) = x - 1.$$

Comparing with the formulation of $\tau_{float}(y, \tau)$, we then deduce that

$$W_\infty(x, \tau) = c_{float}(S, \min(S, X), \tau; q, r)$$

$$+ q \int_0^\tau c_{float}(S, \min(S, X), u; q, r) \, du.$$ 

Lastly, the price function of the infinite-reload option is given by

$$V_\infty(S, \tau; X, r, q) = SW_\infty \left( \frac{X}{S}, \tau \right) + F(S, \tau)$$

$$= c_{float}(S, \min(S, X), \tau; q, r) + (Se^{-q\tau} - Xe^{-r\tau})$$

$$+ q \int_0^\tau [c_{float}(S, \min(S, X), u; q, r) + (Se^{-qu} - Xe^{-ru})] \, du.$$
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<th>$q$</th>
<th>$N = 500$</th>
<th>$N = 1000$</th>
<th>$N = 2000$</th>
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Option values for the infinite-reload options obtained using binomial calculations and analytic formulas.
• The calculations were performed to check the validity of the analytic price formulas for the infinite-reload options.

• The numerical data also illustrate the convergence of the binomial calculations with respect to the number of binomial steps, $N$.

• The effectiveness of the non-linear extrapolation to hasten the rate of convergence is also demonstrated.

\[
V_{extra} = \frac{V(500)V(2000) - V(1000)^2}{V(500) + V(2000) - 2V(1000)}.
\]

• Other parameter values used in the calculations are: $r = 0.04, \sigma = 0.3, \tau = 10$ and $X = 1$. 
Plots of critical stock price against time to expiry for employee options with one, two and three reloads \( (q = 0, r > \frac{\sigma^2}{2}) \).

The parameter values used in the calculations are: \( q = 0, r = 0.1, \sigma = 0.3 \) and \( X = 1 \). The critical stock price \( S^*_n(\tau; r, q) \) is defined only for \( \tau < \tau^*_n, n = 1, 2, 3 \). These threshold values on time to expiry are found to be: \( \tau^*_1 = 6.78, \tau^*_2 = 12.38 \) and \( \tau^*_3 = 17.86 \); and they observe the monotonic property: \( \tau^*_1 < \tau^*_2 < \tau^*_3 \).
Plots of critical stock price against time to expiry for employee options with one, two and three reloads \( (q = 0, r \leq \frac{\sigma^2}{2}) \).

The parameter values used in the calculations are: \( q = 0, r = 0.04, \sigma = 0.3 \) and \( X = 1 \). Apparently, the critical stock price is defined for all values of time to expiry and there is no asymptotic value for the critical stock price at infinite time to expiry.
Plots of critical stock price against time to expiry for employee options with one, two and three reloads ($q > 0$).

The parameter values used in the calculations are: $q = 0.03, r = 0.04, \sigma = 0.3$ and $X = 1$. The asymptotic values for the critical stock price at infinite time to expiry are found to be: $S_1^*(\infty) = 3.45, S_2^*(\infty) = 1.99$ and $S_3^*(\infty) = 1.59$. 
Plots of option value against dividend yield for employee options with one, two, three and infinite reloads.

The parameter values used in the calculations are: $r = 0.04, \sigma = 0.3, \tau = 10, S = X = 1$. The price functions are seen to be decreasing functions of the dividend yield.
Plots of option value against interest rate for employee options with one, two, three and infinite reloads.

The parameter values used in the calculations are: $q = 0.03, \sigma = 0.3, \tau = 10, S = X = 1$. The price functions are seen to be monotonic increasing with respect to interest rate.
Plots of option value against volatility for employee options with one, two, three and infinite reloads.

The parameter values used in the calculations are: $q = 0.03, r = 0.04, \tau = 1.0, S = X = 1$. The price functions always increase in value with increasing volatility.
Time vesting requirement

- The first reload can be exercised only after an initial vesting period (say, $\delta$-period).

- The new reload options received after each exercise are subject to the same vesting period.

Let $V_n(S, t, u; \delta, X)$ denote the price function of the employee option with $n$ reloads, where $u$ is the remaining time required to satisfy the time vesting requirement. Obviously, we have $0 \leq u \leq \delta$.

It suffices to consider the solution of $V_n(S, t, 0)$ since

$$V_n(S, t, u) = e^{-ru} \int_0^\infty \psi(S_{t+u}|S_t = S)V_n(S_{t+u}, t + u, 0) \, dS_{t+u},$$

as there is no reload allowed inside the time interval $[t, t + u)$. 
Analytic properties of $V_n(S, t, 0)$

Let $t_0$ denote the grant date of the reload option, then $V_n(S, t, 0)$ is defined only within $[t_0 + \delta, T]$.

An employee option with $k$ reload rights outstanding can have the exercise of all $k$ reloads only when $\tau \geq (k - 1)\delta$. Set

$$V_n(S, t, 0) = V_k(S, t, 0) \quad \text{for} \quad n \geq k \quad \text{when} \quad t \in [T - k\delta, T].$$

In more details,

$$V_n(S, t, 0) = V_1(S, t, 0) \quad \text{for} \quad t \in [T - t, T], \quad n \geq 1,$$

$$V_n(S, t, 0) = V_2(S, t, 0) \quad \text{for} \quad t \in [T - 2t, T - t), \quad n \geq 2, \text{etc.}$$

**Continuity of price functions**

$$V_{k+1}(S, t, 0) = V_k(S, t, 0) \quad \text{at} \quad t = T - k\delta, \quad k = 1, 2, \ldots.$$
Infinite-reload options

\[
\begin{aligned}
V_\infty (S, t, 0) &= V_3 (S, t, 0) \\
V_\infty (S, t, 0) &= V_2 (S, t, 0) \\
V_\infty (S, t, 0) &= V_1 (S, t, 0)
\end{aligned}
\]

for \(T-3\delta \leq t \leq T-2\delta\)  
for \(T-2\delta \leq t \leq T-\delta\)  
for \(T-\delta \leq t \leq T\)

\[T-3\delta \quad T-2\delta \quad T-\delta \quad T\]

\[\longrightarrow \quad \text{calendar time}\]

Analytic behaviors of the price function \(V_\infty (S, t, 0)\) of the infinite-reload option over successive time intervals.
Two-reload options

\[ \text{solve } V_2(S, t, 0) \text{ over } [t_0 + \delta, T-\delta] \]

\[ V_2(S, t, \delta) = F_\delta[V_2(S, t + \delta, 0)] \]

for \( t \leq T-2\delta \)

\[ V_2(S, t, \delta) = F_\delta[V_1(S, t + \delta, 0)] \]

for \( T-2\delta \leq t \leq T-\delta \)

\[ V_1(S, t, 0) \text{ over } [t_0 + 2\delta, T] \]

\[ V_2(S, t, \delta) = C_E(S, t) \]

for \( T-\delta \leq t \leq T \)

---

Summary of the solution procedures for pricing the two-reload option at the grant date \( t_0 \), where \( t_0 < T - 3\delta \).
We plot the price function $V_{\infty}(1, 0, \delta)$ of the infinite-reload option against the length of the vesting period $\delta$. The parameter values used in the calculations are: $r = 0.05, q = 0, X = 1, \tau = 10$. We observe that $V_{\infty}(1, 0, \delta)$ is monotonically decreasing with increasing $\delta$ and increasing with increasing volatility $\sigma$. 
The price functions of one-reload option (dashed curve), two-reload option (dot-dashed curve), three-reload option (dotted curve) and infinite-reload option (solid curve) are plotted against time to expiry $\tau$. The parameter values used in the calculations are: $r = 0.05, q = 0, \sigma = 0.4, \delta = 0.5, S = 1$ and $X = 1$. 
The critical stock price $S^*_n(\tau)$ for an infinite-reload option (labelled “$n = \infty$”) is plotted against time to expiry $\tau$. The plots of $S^*_n(\tau), n = 1, 2, 3$, for an one-reload option (labelled “$n = 1$”), two-reload option (labelled “$n = 2$”) and three-reload option (labelled “$n = 3$”) against $\tau$ are also included for comparison. The parameter values used in the calculations are: $r = 0.1, q = 0, \sigma = 0.3, \delta = 0.5$ and $X = 1$. The curves $S^*_1(\tau), S^*_2(\tau), S^*_3(\tau)$ and $S^*_\infty(\tau)$ overlap with each other over $\tau \in [0, 0.5)$. At $\tau = k\delta, k = 1, 2, 3, \cdots$, $S^*_\infty(\tau)$ is continuous but its $\tau$-derivative $S^*'_{\infty}(\tau)$ is not continuous.
Numerical results

Multi-reload options with time vesting

\[ T = 5, r = 0.05, q = 0, S = X = 1 \]

<table>
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<tr>
<th>Volatility</th>
<th>( \delta )</th>
<th>one-reload</th>
<th>two-reload</th>
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<td>0.4058</td>
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## Unlimited reload options with time vesting

### six-month time vesting

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<th>Lower bound</th>
<th>Dybvig et al.</th>
<th>Dai-Kwok</th>
<th>$n = 200$</th>
<th>$n = 400$</th>
<th>$n = \infty$</th>
<th>Upper bound</th>
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<tr>
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<td>0.7204</td>
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### one-year time vesting

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<th>Dai-Kwok</th>
<th>$n = 200$</th>
<th>$n = 400$</th>
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Numerical results for different $\delta$

Parameters: $T = 10, \sigma = 0.4, r = 0.05, X = S = 1$

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<td>0.6572</td>
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