Saddlepoint Approximation Methods for Pricing

Financial Options on Discrete Realized Variance

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* This is a joint work with Wendong Zheng.
Overview

• The saddlepoint method derives approximation formulas for the density function and the cumulative distribution function of a random variable based on knowledge of its moment generating function (mgf). The method is applicable to a large class of Markov processes for which the mgf or characteristic function, but not the transition function, can be found in closed form.

• The mgf of the discrete realized variance of a stock price process is not available in closed form. We price options on discrete realized variance via small time asymptotic approximation to the moment generating function of the quadratic variation process and discretely sampled realized variance.
Complex moment generating functions

Let \( F_U(u) \) and \( f_U(u) = F'_U(u) \) denote the cumulative distribution function and continuous density function of a random variable \( U \). Define the complex moment generating function of \( U \) by

\[
M(z) = \int_{-\infty}^{\infty} e^{zu} f(u) \, du, \quad \text{where} \quad z = x + iy.
\]

(i) When \( y = 0 \), we recover the ordinary real moment generating function \( M(x) \).

(ii) When \( x = 0 \), \( M(iy) = \int_{-\infty}^{\infty} e^{iyu} f(u) \, du \) is the characteristic function.

We assume \( M(z) \) to be analytic in some open vertical strip \( G \) containing the imaginary axis. In this way, the characteristic function is also analytic.
\( G = \) vertical strip where \( M(z) \) is analytic
\( I = \) set of points at which the real moment generating function \( M(x) \) exists
\( = \) intersection of \( G \) with the real axis
Complex cumulant generating function

We define $\kappa(z) = \log M(z)$ to be the complex cumulant generating function.

- We commonly assume the existence of an analytic form of the cumulant generating function (CGF) so that analytic expressions for the derivatives of the CGF of various orders can be obtained.

- The CGF $\kappa(z)$ is commonly assumed to be finite in some open vertical strip $\{z : \alpha_- < \text{Re}(z) < \alpha_+\}$ in the complex plane that contains the imaginary axis, where $\alpha_- < 0$ and $\alpha_+ > 0$; and both $\alpha_-$ and $\alpha_+$ can be infinite.
In terms of the complex CGF $\kappa(z)$, for $b > 0$, we have

density function: $f_U(u) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \exp(\kappa(z) - zu) \, du$

tail probability: $F_U(u) = P[U > u] = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\exp(\kappa(z) - zu)}{z} \, dz$.

An inversion formula similar to those for densities and tail probabilities also exists for $E[(U - u)^+]$. For $b > 0$, we have

$$E[(U - u)^+] = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\exp(\kappa(z) - zu)}{z^2} \, dz$$

$$= \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \exp(\tilde{\kappa}(z) - zu) \, dz$$

where

$$\tilde{\kappa}(z) = \kappa(z) - \log z^2.$$
Saddlepoint approximation formula for the density function

Consider the Bromwich integral:

\[ f(y) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \exp(\kappa(u) - yu) \, du. \]

Set \( \hat{u} \) be the solution in \( u \) of the equation: \( \kappa'(u) = y \). With regard to the convexity in \( u \) of the function \( \kappa(u) \) (a property that is observed for most Markov processes), solution of the above equation exists and is unique also. Consider the Taylor expansion about \( \hat{u} \), where

\[ \kappa(u) - yu = \kappa(\hat{u}) - y\hat{u} + \frac{1}{2} \left. \frac{\partial^2 \kappa(u)}{\partial u^2} \right|_{u=\hat{u}} (u - \hat{u})^2 + \cdots. \]

We set \( u = \hat{u} + iv, v \in \mathbb{R} \), so that

\[
\begin{align*}
  f(y) &\approx \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(\kappa(\hat{u}) - y\hat{u}) \exp \left( -\frac{1}{2} \left. \frac{\partial^2 \kappa(u)}{\partial u^2} \right|_{u=\hat{u}} v^2 \right) dv \\
  &= \frac{\exp(\kappa(\hat{u}) - y\hat{u})}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{v^2}{2\kappa''(\hat{u})} \right) dv = \frac{\exp(\kappa(\hat{u}) - y\hat{u})}{\sqrt{2\pi} \sqrt{\kappa''(\hat{u})}}.
\end{align*}
\]
Lugannani-Rice formula

Lugannani and Rice (1980) derive a saddlepoint approximation to the tail probability as follows:

\[ P\{X \geq x\} = 1 - \Phi(s) + \phi(s) \left( \frac{1}{\hat{t}\sqrt{n\kappa''(t)}} - \frac{1}{s} \right) + O(n^{-3/2}) \]

with \( s = \text{sgn}(\hat{t})\sqrt{2n|\kappa(t) - xt|} \), and \( \hat{t} \) satisfies \( \kappa'(\hat{t}) = x \). Here, \( \Phi \) and \( \phi \) are the CDF and PDF of the standard normal random variable, respectively.

The proof of the above formula can be performed in a similar manner by approximating \( \kappa(t) - xt \) in an interval containing both \( t = 0 \) and \( t = \hat{t} \) by a quadratic function, where

\[ [\kappa(t) - xt] - [\kappa(t) - x\hat{t}] = \frac{1}{2}(w - \hat{w})^2, \]

with \( -\hat{w}^2/2 = \kappa(\hat{t}) - x\hat{t} \). Finally, we change the integration variable from \( t \) to \( w \).
Pricing options on discrete realized variance

The advantage of the saddlepoint approximation is more appreciated when we consider approximation formulas for pricing highly path dependent derivatives, like options on discrete realized variance.

The terminal payoff of a put option on the discrete realized variance is given by
\[ \max(K - V_d(0, T; N), 0), \]
where \( K \) is the strike price and
\[ V_d(0, T; N) = \frac{A}{N} \sum_{i=1}^{N} \left( \ln \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2. \]

The annualized factor \( A \) is taken to be 252 for daily monitoring and \( \{t_0, t_1, \ldots, t_N\} \) is the set of monitoring instants.

The moment generating function of \( V_d(0, T; N) \) cannot be found readily under stochastic volatility models, unlike that of the continuous realized variance.
Our contributions to the saddlepoint approximation method for pricing options on discrete realized variance

- We develop a viable approach such that the saddlepoint approximation can be derived even when $\kappa(z)$ is defined only in the left half complex plane not including the imaginary axis.

- When the analytic expression of the CGF is not available, we deduce useful analytic approximations using the small time asymptotic approximation of the Laplace transform of the discrete realized variance as a control.
Let $\kappa(\theta)$ and $\kappa_0(\theta)$ denote the CGF of the random discretely sampled realized variance $I$ and $I - K$, respectively, where $K$ represents the fixed strike. The two CGFs are related by $\kappa_0(\theta) = \kappa(\theta) - K\theta$. We write $X = I - K$ and $F_0(x)$ as the distribution function of $X$.

Recall that the Fourier transform of the payoff of a call option on discrete realized variance $E[X1_{\{X>0\}}]$ is $\frac{e^{\kappa_0(t)}}{t^2}$. Consider the following tail expectations expressed in terms of Bromwich integrals:

$$\Xi_1 = E[X1_{\{X>0\}}] = \frac{1}{2\pi i} \int_{\tau_1-i\infty}^{\tau_1+i\infty} \frac{e^{\kappa_0(t)}}{t^2} dt, \quad \tau_1 \in (0, \alpha_+) \text{ where } \alpha_+ > 0;$$

$$\Xi_2 = -E[X1_{\{X<0\}}] = \frac{1}{2\pi i} \int_{\tau_2-i\infty}^{\tau_2+i\infty} \frac{e^{\kappa_0(t)}}{t^2} dt, \quad \tau_2 \in (\alpha_-, 0) \text{ where } \alpha_- < 0.$$

The contour is taken to be along a vertical line parallel to the imaginary axis. We write the integrand as $e^{\kappa_0(t) - 2\ln t}$. 
First order saddlepoint approximation

The first order saddlepoint approximation to $\Xi_j, j = 1, 2$, is given by

$$\Xi_j \approx \hat{\Xi}_j = e^{\kappa_0(\hat{t}_j)/\hat{t}_j^2} \sqrt{2\pi \left[ \frac{2}{\hat{t}_j^2} + \kappa_0^{(2)}(\hat{t}_j) \right]}, \quad j = 1, 2,$$

where $\hat{t}_1 > 0$ ($\hat{t}_2 < 0$) is the positive (negative) root in $(\alpha_-, \alpha_+)$ of the saddlepoint equation:

$$\kappa'_0(t) - 2/t = 0.$$

Note that $\Xi_1 - \Xi_2 = \mu_X$, which is consistent with the put-call parity in option pricing theory.

Suppose both roots $\hat{t}_1$ and $\hat{t}_2$ exist, we can use either the saddlepoint approximation $\hat{\Xi}_1 (\bar{\Xi}_1)$ or $\mu_X + \hat{\Xi}_2 (\mu_X + \bar{\Xi}_2)$ to approximate the value of the call option.
Second order Saddlepoint approximation

By performing the Taylor expansion of $\kappa(t) - xt$ up to the fourth order, we manage to derive the second order saddlepoint approximation formulas. The second order saddlepoint approximation to $\Xi_j$ is given by

$$\tilde{\Xi}_j = \hat{\Xi}_j (1 + R_j), \quad j = 1, 2,$$

where the adjustment term $R_j$ is given by

$$R_j = \frac{1}{8} \frac{\kappa_0^{(4)}(\hat{t}_j) + 12\hat{t}_j^{-4}}{[\kappa_0^{(2)}(\hat{t}_j) + 2\hat{t}_j^{-2}]^2} - \frac{5}{24} \frac{\kappa_0^{(3)}(\hat{t}_j) - 4\hat{t}_j^{-3}}{[\kappa_0^{(2)}(\hat{t}_j) + 2\hat{t}_j^{-2}]^3}, \quad j = 1, 2.$$
Small time asymptotic approximation to MGF

We consider the small time asymptotic approximation to the MGFs of the quadratic variation process \( I(0, T; \infty) = \frac{1}{T} [\ln S_T, \ln S_T] \) and discretely sampled realized variance \( I(0, T; N) \). The asymptotic limit of \( V_d(0, T; N) \) is gamma distributed with shape parameter \( N/2 \) and scale parameter \( 2V_0/N \).

For any \( u \leq 0 \), we obtain

\[
\lim_{T \to 0^+} M_{I(0, T; \infty)}(u) = e^{uV_0},
\]

\[
\lim_{T \to 0^+} M_{I(0, T; N)}(u) = \left(1 - \frac{2V_0u}{N}\right)^{-N/2}.
\]
Difference in the corresponding asymptotic approximation terms as a control

The difference $M_{I(0,T;N)}(u) - M_{I(0,T;\infty)}(u)$ is seen to be almost invariant with respect to $T$, we use the above difference as a control and propose the following approximate MGF formula:

$$\hat{M}_{I(0,T;N)}(u) = M_{I(0,T;\infty)}(u) + \left(1 - \frac{2V_0 u}{N}\right)^{-N/2} - e^{uV_0}, \quad u \in \mathbb{C}_-.$$ 

Under an affine stochastic volatility model, $M_{I(0,T;\infty)}(u)$ can be derived analytically by solving a Riccati system of equations.
Analytic formulas for the approximate cumulant generating function and its higher order derivatives

\[
\hat{\kappa}_I(0,T;N)(u) = \ln \hat{M}_I(0,T;N)(u),
\]

\[
\hat{\kappa}'_I(0,T;N)(u) = \frac{M'_I(0,T;\infty)(u) + f_1(u)}{M_I(0,T;N)(u)},
\]

\[
\hat{\kappa}^{(2)}_I(0,T;N)(u) = \frac{M^{(2)}_I(0,T;\infty)(u) + f_2(u)}{\hat{M}_I(0,T;N)(u)} - \left[ \frac{M'_I(0,T;\infty)(u) + f_1(u)}{\hat{M}_I(0,T;N)(u)} \right]^2 - \sum_{n=1}^{\infty} f_n(u),
\]

where the sequence of functions \( f_n(u) \) is defined by

\[
f_n(u) = V_0^k \frac{N}{2} \left( \frac{N}{2} + 1 \right) \cdots \left( \frac{N}{2} + n \right) \left( 1 - \frac{2V_0u}{N} \right)^{-N/2-n}, \quad n = 1, 2, \ldots.
\]
Heston affine stochastic volatility models with simultaneous jumps (SVSJ)

Under a pricing measure $Q$, the joint dynamics of stock price $S_t$ and its instantaneous variance $V_t$ under the affine SVSJ model assumes the form

$$
\frac{dS_t}{S_t} = (r - \lambda m) \, dt + \sqrt{V_t} \, dW_t^S + (e^{J^S} - 1) \, dN_t,
$$

$$
dV_t = \kappa(\theta - V_t) \, dt + \varepsilon \sqrt{V_t} \, dW_t^V + J^V \, dN_t,
$$

where $W_t^S$ and $W_t^V$ are a pair of correlated standard Brownian motions with $dW_t^S dW_t^V = \rho \, dt$, and $N_t$ is a Poisson process with constant intensity $\lambda$ that is independent of the two Brownian motions.

- $J^S$ and $J^V$ denote the random jump sizes of the log price and variance, respectively.

- These random jump sizes are assumed to be independent of $W_t^S$, $W_t^V$ and $N_t$. 

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**Joint moment generating function**

Let $X_t = \ln S_t$. The joint moment generating function of $X_t$ and $V_t$ is defined to be

$$E[\exp(\phi X_T + b V_T + \gamma)],$$

where $\phi$, $b$ and $\gamma$ are constant parameters.

Let $U(X_t, V_t, t)$ denote the non-discounted time-$t$ value of a contingent claim with the terminal payoff function: $U_T(X_T, V_T)$, where $T$ is the maturity date. Let $\tau = T - t$, $U(X, V, \tau)$ is governed by the following partial integro-differential equation (PIDE):

$$\frac{\partial U}{\partial \tau} = \left( r - m\lambda - \frac{V}{2} \right) \frac{\partial U}{\partial X} + \kappa (\theta - V) \frac{\partial U}{\partial V} + \frac{V}{2} \frac{\partial^2 U}{\partial X^2} + \frac{\varepsilon^2 V}{2} \frac{\partial^2 U}{\partial V^2} + \rho \varepsilon V \frac{\partial^2 U}{\partial X \partial V} + \lambda E \left[ U(X + J^S, V + J^V, \tau) - U(X, V, \tau) \right].$$
Riccati system of ordinary differential equations

The parameter functions $B(\Theta; \tau, q)$, $\Gamma(\Theta; \tau, q)$ and $\Lambda(\Theta; \tau, q)$ satisfy the following Riccati system of ordinary differential equations:

\[
\begin{align*}
\frac{\partial B}{\partial \tau} &= -\frac{1}{2}(\phi - \phi^2) - (\kappa - \rho \varepsilon \phi)B + \frac{\varepsilon^2}{2}B^2 \\
\frac{\partial \Gamma}{\partial \tau} &= r\phi + \kappa \theta B \\
\frac{\partial \Lambda}{\partial \tau} &= \lambda \left( E[\exp(\phi J^S + BJ^V) - 1] - m\phi \right)
\end{align*}
\]

with the initial conditions: $B(0) = b$, $\Gamma(0) = \gamma$ and $\Lambda(0) = 0$.

Canonical jump distributions

Suppose we assume that $J^V \sim \exp(1/\eta)$ and $J^S$ follows

\[J^S|J^V \sim \text{Normal}(\nu + \rho_J J^V, \delta^2),\]

which is the Gaussian distribution with mean $\nu + \rho_J J^V$ and variance $\delta^2$, we obtain

\[m = E[e^{J^S} - 1] = \frac{e^{\nu + \delta^2/2}}{1 - \eta \rho_J} - 1,\]

provided that $\eta \rho_J < 1$. 
Under the above assumptions on $J^S$ and $J^V$, the parameter functions can be found to be

\[
B(\Theta; \tau, q) = \frac{b(\xi_- e^{-\zeta \tau} + \xi_+)}{(\xi_+ + \varepsilon^2 b)e^{-\zeta \tau} + \xi_+ - \varepsilon^2 b},
\]

\[
\Gamma(\Theta; \tau, q) = r\phi \tau + \gamma - \frac{\kappa \theta}{\varepsilon^2} \left[ \xi_+ \tau + 2 \ln \frac{(\xi_+ + \varepsilon^2 b)e^{-\zeta \tau} + \xi_+ - \varepsilon^2 b}{2\zeta} \right],
\]

\[
\Lambda(\Theta; \tau, q) = -\lambda(m\phi + 1)\tau + \lambda e^{\phi \upsilon} + \frac{\delta^2 \phi^2}{2}
\]

\[
\left[ \frac{k_2}{k_4} - \frac{1}{\zeta} \left( \frac{k_1}{k_3} - \frac{k_2}{k_4} \right) \ln \frac{k_3 e^{-\zeta \tau} + k_4}{k_3 + k_4} \right],
\]

with $q = (\phi \ b \ \gamma)^T$ and

\[
\zeta = \sqrt{(\kappa - \rho \varepsilon \phi)^2 + \varepsilon^2 (\phi - \phi^2)},
\]

\[
\xi_\pm = \zeta \mp (\kappa - \rho \varepsilon \phi),
\]

\[
k_1 = \xi_+ + \varepsilon^2 b,
\]

\[
k_2 = \xi_- - \varepsilon^2 b,
\]

\[
k_3 = (1 - \phi \rho \eta)k_1 - \eta(\phi - \phi^2 + \xi_- b),
\]

\[
k_4 = (1 - \phi \rho \eta)k_2 - \eta[\xi_+ b - (\phi - \phi^2)].
\]
### Numerical results

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The prices of put options on the daily sampled realized variance with varying strike prices and maturities under the SVSJ model.

The first order approximation SPA1 is seen to outperform the second order approximation SPA2 for short-maturity or out-of-the-money put options.
Plots of the percentage errors against moneyness for the one-month (20 days) put options on daily sampled realized variance with different model parameters of the SVSJ model.
Conclusion

- We use the small time asymptotic approximation of the Laplace transform of discrete realized variance to obtain analytic approximation formulas using the saddlepoint approximation method for pricing options on discrete realized variance.

- The second order saddlepoint approximation formulas provide sufficiently good approximation (with a small percentage error) to the values of options on discrete realized variance even when the option is deep out-of-the-money and under the choices of extreme parameter values (high values of jump intensity $\lambda$ and volatility of variance $\varepsilon$).