American Quanto Lookback Option

**Quanto feature** Payoff is determined by a financial price or index in one currency but the actual payout is realized in another currency.

**Exchange rate** $F_t$ is the domestic currency price of one unit of foreign currency.

Risk neutral process: \[
\frac{dF_t}{F_t} = (r_d - r_f)\,dt + \sigma_F \,dZ_F
\]

**Stock price** $S_t$ in foreign currency world.

Risk neutral process: \[
\frac{dS_t}{S_t} = (r_f - q)\,dt + \sigma_S \,dZ_F
\]

Risk neutralized drift rate of $S_t$ in domestic currency world: \[
\delta^d_S = r_f - q - \rho \sigma_S \sigma_F, \quad \rho dt = dZ_S dZ_F.
\]
Maximum exchange rate quanto call

The terminal payoff is

\[ \text{terminal payoff} = F_{max}^{[T_0, T]}(S_T - K)^+. \]

Let \( V_M(S, F, \tau; F_{max}) \) be the value of an American maximum rate quanto call option in domestic currency.

\[
V_M(S, F, \tau; F_{max}) = \sup_{\tau^* \in T_{t, T}} E \left\{ e^{-r(\tau^* - t)} F_{max}^{[T_0, \tau^*]} (S_{\tau^*} - K)^+ \big| S_t = S, F_t = F, F_{max}^{[T_0, t]} = F_{max} \right\}
\]

where \( T_{t, T} \) is the set of stopping times between \( t \) and \( T \).
The linear complimentarity formulation for $V_M(S, F, \tau; F_{\text{max}})$ is given by

$$
\frac{\partial V_M}{\partial \tau} - LV_M \geq 0, \quad V_M \geq F_{\text{max}} \max(S - K, 0),
$$

$$
\left(\frac{\partial V_M}{\partial \tau} - LV_M\right) [V_M - F_{\text{max}} \max(S - K, 0)] = 0,
$$

$$
S > 0, 0 < F < F_{\text{max}}, \tau \in [0, T],
$$

$$
\frac{\partial V_M}{\partial F_{\text{max}}} |_{F_{\text{max}}=F} = 0 \quad \text{and} \quad V_M(S, F, 0; F_{\text{max}}) = F_{\text{max}} \max(S - K, 0),
$$

where $L$ is the differential operator defined by

$$
L = \frac{\sigma^2 S}{2} S^2 \frac{\partial^2}{\partial S^2} + \rho \sigma S \sigma F S \frac{\partial^2}{\partial S \partial F} + \frac{\sigma^2 F}{2} F^2 \frac{\partial^2}{\partial F^2} + \delta^d S \frac{\partial}{\partial S} + (r_d - r_f) F \frac{\partial}{\partial F} - r_d.
$$
In the continuation region, $V_M$ satisfies

$$\frac{\partial V_M}{\partial \tau} - LV_M = 0 \quad \text{and} \quad V_M > F_{max} \max(S - K, 0);$$

while in the exercise region, $V_M$ satisfies

$$\frac{\partial V_M}{\partial \tau} - LV_M > 0 \quad \text{and} \quad V_M = F_{max} \max(S - K, 0).$$

Define

$$U_M(S, \xi, \tau) = \frac{V_M(S, F, \tau; F_{max})}{F_{max}}, \quad \text{where} \quad \xi = F/F_{max}.$$
Apparently, $S_{\text{max}}$ does not appear in the governing equation, but only as boundary condition
$$\frac{\partial V_M}{\partial F_{\text{max}}} |_{F=F_{\text{max}}} = 0.$$

**Rationale** We define $M_n = \left[ \int_{T_0}^{t} (S_\xi)^n \, d\xi \right]^{1/n}$, $t > T_0$, where the evolution equation is
$$dM_n = \frac{1}{n(M_n)^{n-1}} \frac{S^n}{S^n} \, dt.$$

Note that
$$M = \max_{T_0 \leq \xi < t} S_\xi = \lim_{n \to \infty} M_n.$$

An extra term
$$\frac{1}{n(M_n)^{n-1}} \frac{S^n}{S^n} \frac{\partial V_M}{\partial M_n}$$
occurs in $dV_M$ when we perform the Ito differentiation. Since
$$\frac{1}{n(M_n)^{n-1}} \to 0 \text{ as } S < M,$$
this term tends to zero as $n \to \infty$. 
Change of numeraire theorem

Consider two numeraires \( N \) and \( M \) with the martingale measures \( Q^N \) and \( Q^M \). Since the value of a contingent claim is given by

\[
V = N(t) E^N \left( \frac{H(T)}{N(T)} | F_t \right) = M(t) E^M \left( \frac{H(T)}{M(T)} | F_t \right),
\]

that is, \( V/N(t) \) is a \( Q^N \)-martingale and \( V/M(t) \) is a \( Q^M \)-martingale. We have

\[
E^N (G(T) | F_t) = E^M \left( G(T) \frac{N(T)/N(t)}{M(T)/M(t)} | F_t \right)
\]

where \( G(T) = H(T)/N(T) \).

Change of numeraire formula

The Radon-Nikodym derivative that changes the martingale measure \( Q^M \) into \( Q^N \) is given by

\[
\frac{dQ^N}{dQ^M} = \frac{N(T)/N(t)}{M(T)/M(t)}.
\]
How to find the risk adjusted drift rate in $F$, the exchange rate?

In domestic currency world, there are two marketed assets:

1. Domestic money market account $B_d$

2. Foreign money market account $B_f$; in domestic terms, it becomes $B_fF$.

$$
\begin{align*}
\frac{dB_d}{B_d} &= r_d B_d \, dt, \quad dB_f = r_f B_f \, dt \\
\frac{dF}{F} &= \mu_F \, dt + \sigma_F \, dZ_F \\
\end{align*}
$$

so that

$$
\frac{d(B_fF)}{B_fF} = (r_f + \mu_F) \, dt + \sigma \, dZ_F.
$$

Suppose the domestic money market account is used as numeraire, the relative price process $(B_fF)' = B_fF/B_d$ follows the process

$$
\frac{d(B_fF)'}{(B_fF)'} = (r_f - r_d + \mu_F) \, dt + \sigma \, dZ_F.
$$
Find the domestic martingale measure $Q^D$ under which the relative price process $(B_fF)'$ is a martingale.

We apply the Girsanov Theorem to transform Itô processes with general drifts into martingales. In this case, we choose

$$dZ_D = dZ_F + \left( \frac{r_f - r_d + \mu_F}{\sigma_F} \right) dt$$

so that

$$\frac{d(B_fF)'}{(B_fF)'} = (r_f - r_d + \mu_F) dt + \sigma_F \left[ dZ_D - \frac{(r_f - r_d + \mu_F) dt}{\sigma_F} \right] = \sigma dZ_D.$$

Hence, $(B_fF)'$ under $Q^D$ is a martingale. Now, the process of $F$ under $Q^D$ is given by

$$\frac{dF}{F} = \mu_F dt + \sigma_F \left[ dZ_D - \frac{(r_f - r_d + \mu) dt}{\sigma_F} \right] = (r_d - r_f) dt + \sigma_F dZ_D.$$
\( S \) = asset price in foreign currency
\( S^* \) = asset price in domestic currency
\( F \) = exchange rate (domestic currency price of one unit of foreign currency)

\[ S^* = FS \]

\( \delta_{S^*}^d \) = risk adjusted drift rate for \( S^* \) in domestic currency = \( r_d - q \)

\( \delta_{F}^d \) = risk adjusted drift rate for \( F \) in domestic currency = \( r_d - r_f \)

By Ito’s lemma: \( \delta_{S^*}^d = \delta_{S}^d + \delta_{F}^d + \rho_{SF}\sigma_S\sigma_F \)

so that

\[ \delta_{S}^d = \delta_{S^*}^d - \delta_{F}^d - \rho_{SF}\sigma_S\sigma_F \]
\[ = (r_d - q) - (r_d - r_f) - \rho_{SF}\sigma_S\sigma_F \]
\[ = r_f - q - \rho_{SF}\sigma_S\sigma_F. \]
Proposition

The normalized price function $U_M(S, \xi, \tau)$ satisfies the following monotonicity properties with respect to $\tau, \xi$ and the strike price $K$.

(a) $\frac{\partial U_M}{\partial \tau} \geq 0$;

(b) $\frac{\partial U_M}{\partial \xi} \geq 0$. 
Proof of Proposition

(a) For any American options, the value of the longer-lived one is always worth at least that of its shorter-lived counterpart, so \( \frac{\partial U_M}{\partial \tau} = \frac{1}{F_{\text{max}}} \frac{\partial V_M}{\partial \tau} \geq 0 \).

(b) For a given value of \( F_{\text{max}} \), \( V_M(S, F, \tau) \) is a non-decreasing function of \( F \) since a higher value of \( F \) would mean at least the same or a higher value of \( F_{[T_0,T]} \) to be realized at expiry compared to the counterpart with a lower value of \( F \). We then have \( \frac{\partial U_M}{\partial \xi} = \frac{\partial V_M}{\partial F} \geq 0 \).
Theorem

Consider the optimal exercise boundary $S^*_M(\xi, \tau)$.

(a) At time close to expiry, $\tau \to 0^+$, we have

$$S^*_M(\xi, 0^+) = \begin{cases} \max \left(1, \frac{r_d}{r_d - \delta_S^d}\right) K & \text{if } r_d > \delta_S^d \\ \infty & \text{if } r_d \leq \delta_S^d \end{cases}$$

(b) $S^*_M(\xi, \tau)$ is monotonically increasing with respect to $\tau$ and $\xi$.

The exercise region and the continuation region are on the right side and the left side of the exercise boundary, respectively.
Proof of Theorem

The monotonicity property: \( \frac{\partial U_M}{\partial \tau} > 0 \) is maintained in the continuation region even when \( \tau \to 0^+ \). First, it is obvious that \( S^*_M(\xi, 0^+) \geq K \). For \( S \in (K, S^*_M(\xi, 0^+)) \), we have \( U_M(S, \xi, 0^+) = S - K \). Since \( U_M(S, \xi, 0^+) \) should satisfy \( \frac{\partial U_M}{\partial \tau} = \hat{L} U_M \), we obtain

\[
\frac{\partial U_M}{\partial \tau} \big|_{\tau=0} = \delta^d_S S - r_d(S - K) = r_d K - (r_d - \delta^d_S) S.
\]

For \( r_d > \delta^d_S \), the condition: \( \frac{\partial U_M}{\partial \tau} \big|_{\tau=0} > 0 \) is satisfied only for \( S < \frac{r_d}{r_d - \delta^d_S} K \).
On the other hand, when \( r_d \leq \delta_S^d \), \( \frac{\partial U_M}{\partial \tau}|_{\tau=0} > 0 \) always holds true. We then conclude that

\[
S^*_M(\xi, 0^+) = \begin{cases} 
\max \left( 1, \frac{r_d}{r_d - \delta_S^d} \right) K & \text{if } r_d > \delta_S^d \\
\infty & \text{if } r_d \leq \delta_S^d
\end{cases}
\]

The above result agrees with the usual result for critical asset price close to expiry for American call options when we visualize \( r_d - \delta_S^d \) as the effective dividend yield of the foreign stock in the domestic currency world.
To show the monotonicity property of $S_M^*(\xi, \tau)$ with respect to $\tau$, we let $\tau_2 > \tau_1$ and consider the evaluation of $U_M(S, \xi, \tau)$ at stock price level $S = S_M^*(\xi, \tau_1)$ and at two times $\tau_1$ and $\tau_2$. By virtue of the monotonicity property of $U_M$ on $\tau$, we have

$$U_M(S_M^*(\xi, \tau_1), \tau_2) > U_M(S_M^*(S, \tau_1), \tau_1) = S_M^*(\xi, \tau_1) - K.$$ 

This implies that the American option remains in the continuation region when $S = S_M^*(\xi, \tau_1)$ and $\tau = \tau_2$. Since the exercise region is on the right side of the continuation region, we deduce that

$$S_M^*(\xi, \tau_2) > S_M^*(\xi, \tau_1), \quad \tau_2 > \tau_1.$$ 

The monotonicity property of $S_M^*(\xi, \tau)$ with respect to $\xi$ can be established by using the monotonicity property of $U_M$ on $\xi$ and following a similar argument as above.
$S_M^*(\xi, \tau)$ changes abruptly at some threshold level of $\xi$. When $\xi$ increases beyond this $\tau$-dependent threshold level, $S_M^*(\xi, \tau)$ increases quite substantially implying that the holder will wait for much significant increase in stock price in order to exercise the quanto lookback call option. When $F$ becomes close to $F_{max}$, $S_M^*(\xi, \tau)$ becomes exceedingly large. This is reasonable since it is much likely that a higher value of $F_{max}$ will be realized later so the holder should restrain from exercising the option prematurely.
Summary of the exercise policies of an American maximum exchange rate quanto call

The critical stock price \( S^*_M(F, \tau; F_{max}) \) at which it is optimal to exercise the option is seen to be monotonically increasing with respect to time to expiry \( \tau \) and exchange rate \( F \) (for fixed realized maximum exchange rate \( F_{max} \)).

It is never optimal to exercise the maximum exchange rate quanto call if the effective dividend yield of the foreign stock in domestic currency world is non-positive.

Also, when \( F \) comes close to \( F_{max} \), it becomes much less likely to exercise the option prematurely.
American joint quanto fixed strike lookback call option

Let $V_J(S, F, \tau; S_{max})$ denote the value of an American joint quanto fixed strike lookback call option in domestic currency. The linear complimentarity formulation for $V_J(S, F, \tau; S_{max})$ is given by

$$
\frac{\partial V_J}{\partial \tau} - LV_J \geq 0, \quad V_J \geq \max(F, F_c) \max(S_{max} - K, 0),
$$

$$
\left(\frac{\partial V_J}{\partial \tau} - LV_J\right) [V_J - \max(F, F_c) \max(S_{max} - K, 0)] = 0, \quad F > 0, 0 < S < S_{max}, \tau
$$

$$
\frac{\partial V_J}{\partial S_{max}} |_{S_{max}=S} = 0 \quad \text{and} \quad V_J(S, F, 0; S_{max}) = \max(F, F_c) \max(S_{max} - K, 0).
$$

In the continuation region, $V_J$ satisfies

$$
\frac{\partial V_J}{\partial \tau} - LV_J = 0 \quad \text{and} \quad V_J > \max(F, F_c) \max(S_{max} - K, 0);
$$

while in the exercise region, $V_J$ satisfies

$$
\frac{\partial V_J}{\partial \tau} - LV_J > 0 \quad \text{and} \quad V_J = \max(F, F_c) \max(S_{max} - K, 0).$$
Theorem

1. The exercise boundary of the American joint quanto fixed strike lookback call option in the $F-\tau$ plane consists of two branches: $F_{up}^*(S, \tau; S_{max})$ and $F_{low}^*(S, \tau; S_{max})$.

2. For fixed values of $\tau, S$ and $S_{max}$, conditional on $S_{max} > K$, the option should be optimally exercised when $F \geq F_{up}^*$ or $F \leq F_{low}^*$.

3. The continuation region lies within $F_{low}^*(S, \tau; S_{max})$ and $F_{up}^*(S, \tau; S_{max})$.

4. The two branches of the exercise boundary intersect at $F = F_c$ at $\tau \to 0^+$.

5. At time close to expiry, conditional on $S_{max} > K$, the option should be optimally exercised for any exchange rate $F$ other than $F_c$. 
The exercise boundaries of an American joint quanto fixed strike lookback call option at different pairs of values of $S$ and $S_{max}$ are plotted.
Observations

Due to the presence of the factor $\max(F, F_c)$ in the payoff function, the exercise boundaries consist of two branches: $F_{up}^*(S, \tau; S_{max})$ and $F_{low}^*(S, \tau; S_{max})$.

Obviously, early exercise is advantageous only when $S_{max} > K$, that is, the option is currently in-the-money.

When $S_{max} > K$ but the value of $F$ is close to the predetermined constant $F_c$, the holder should delay premature exercise since the advantage of taking the maximum of $F$ and $F_c$ is not significant.

Regret of early exercise is low when $F$ is sufficiently above $F_c$ or below $F_c$. 
In the $F$-$\tau$ plane, conditional on $S_{\text{max}} > K$, the continuation region is bounded by the two branches of the exercise boundary.

When $F \geq F_{up}^*$ or $F \leq F_{low}^*$, it becomes optimal to exercise the American joint quanto lookback call.

Therefore, one part of the exercise region is to the right side of the branch $F_{up}^*(S, \tau; S_{\text{max}})$ and the other part is to the left of $F_{low}^*(S, \tau; S_{\text{max}})$. 
When $S_{max}/S$ is quite close to 1, the exercise boundaries become leveled horizontally at large $F$ or small $F$, indicating that it is never optimal to exercise at any exchange rate $F$ when $\tau$ is beyond some threshold value.

When $S_{max}/S$ is sufficiently large, $F_{up}^*(S, \tau; S_{max})$ and $F_{low}^*(S, \tau; S_{max})$ are defined for all $\tau$. 
Proof of Theorem

At $F = F_c$ and $\tau > 0$, conditional on $S_{max} > K$, the option should remain alive. If otherwise, the option value is equal to the exercised payoff. Substituting $V_J = \max(F, F_c)(S_{max} - K)$ into the Black-Scholes equation, we observe that

$$\frac{\partial V_J}{\partial \tau} - LV_J = \left[ \frac{\sigma_F^2}{2} F^2 \delta(F - F_c)(S_{max} - K) + (r_d - r_f)F(S_{max} - K)1_{\{F > F_c\}} ight. $$

$$\left. - r_d \max(F, F_c)(S_{max} - K) \right] \rightarrow -\infty \text{ when } F = F_c,$$

where $\delta(x)$ and $1_A$ are the delta function and indicator function, respectively. Since the condition: $\frac{\partial V_J}{\partial \tau} - LV_J \geq 0$ is not satisfied, the option should not be optimally exercised at $F = F_c$ and $\tau > 0$. The whole vertical line $F = F_c$ in the $F-\tau$ plane lies in the continuation region.
Next, we would like to show that the exercise regions contain the two horizontal line segments: \( \{ \tau = 0, F < F_c \} \) and \( \{ \tau = 0, F > F_c \} \) in the \( F-\tau \) plane. Assume the contrary, suppose there exists a finite interval \( (F_{low}^*(S, 0^+), F_{up}^*(S, 0^+)) \) at \( \tau \to 0^+ \) that lies completely within the continuation region. Let \( F \in (F_{low}^*(S, 0^+), F_{up}^*(S, 0^+)) \); by continuity, the option value evaluated at \( F \) and \( \tau \to 0^+ \) is \( \max(F, F_c)(S_{max} - K) \). Substituting this option value into the Black-Scholes equation, we then have

\[
\left. \frac{\partial V_J}{\partial \tau} \right|_{\tau=0} = \begin{cases} 
- r_f F(S_{max} - K) & \text{for } F > F_c \\
- r_d F(S_{max} - K) & \text{for } F < F_c 
\end{cases}
\]
In both cases, $\frac{\partial V_J}{\partial \tau}|_{\tau=0} < 0$, which is in contradiction to the property: $\frac{\partial V_J}{\partial \tau}|_{\tau=0} \geq 0$. This would then imply the non-existence of such finite interval. Hence, at time close to expiry and conditional on $S_{max} > K$, the option should be optimally exercised for any exchange rate $F$ other than $F_c$.

In the $F-\tau$ plane, the vertical line $F = F_c$ is in the continuation region while the two horizontal line segments: $\{\tau = 0, F < F_c\}$ and $\{\tau = 0, F > F_c\}$ are in the exercise regions. We then deduce that for a fixed value of $\tau$, there exist some critical values $F_{up}^*$ and $F_{low}^*$ ($F_{up}^* > F_c$ and $F_{low}^* < F_c$) such that the option should be optimally exercised when $F \geq F_{up}^*$ or $F \leq F_{low}^*$. 
Due to the monotonic increasing property of the option value with respect to $\tau$, it can be shown that $F_{up}^*(S, \tau; S_{max})$ and $F_{low}^*(S, \tau; S_{max})$ are unique. In other words, the exercise boundary consists of exactly one branch $F_{up}^*(S, \tau; S_{max})$ that lies completely to the right of the vertical line $F = F_c$ and another unique branch $F_{low}^*(S, \tau; S_{max})$ to the left of $F = F_c$.

The two branches $F_{up}^*(S, \tau; S_{max})$ and $F_{low}^*(S, \tau; S_{max})$ intersect at $F = F_c$ when $\tau \to 0^+$.

Further, $F_{up}^*(S, \tau; S_{max})$ and $F_{low}^*(S, \tau; S_{max})$ are, respectively, monotonically increasing and decreasing with respect to $\tau$. 
Summary of the exercise policies of
American joint quanto fixed strike lookback call

The exercise boundary consists of two branches.

Conditional on the option being in-the-money (current realized maximum stock price $S_{max}$ is higher than the strike price $K$), it is optimal to exercise the option only when the exchange rate $F$ is either sufficiently above or below the predetermined constant exchange rate $F_c$.

At time right before expiry, it is optimal to exercise the American joint quanto lookback call at any level of exchange rate $F$ other than $F_c$. 