Pricing theory of financial derivatives

financial economics
(arbitrage free)

analytic tools
(martingale)

computation methods
(lattice methods,
Monte Carlo simulation)
One-period securities model

$S$ denotes the price process \( \{S(t) : t = 0, 1\} \), where

\[
S(t) = (S_1(t) \quad S_2(t) \quad \cdots \quad S_M(t)).
\]

Here, \( M \) is the number of securities.

At \( t = 1 \), there are \( K \) possible state of the world.

\[
S(1; \Omega) = \begin{pmatrix}
S_1(1; \omega_1) & S_2(1; \omega_1) & \cdots & S_M(1; \omega_1) \\
S_1(1; \omega_2) & S_2(1; \omega_2) & \cdots & S_M(1; \omega_2) \\
\vdots & \vdots & \ddots & \vdots \\
S_1(1; \omega_K) & S_2(1; \omega_K) & \cdots & S_M(1; \omega_K)
\end{pmatrix}.
\]

Discounted price process: \( S^*(t) = S(t)/S_0(t), \quad t = 0, 1 \).

\( H = (h_0 \quad h_1 \quad \cdots \quad h_M) \) represents the holding of securities in a portfolio;

\( H \) is considered as the trading strategy.
Value process of the portfolio

\[ V_t = h_0 S_0(t) + \sum_{m=1}^{M} h_m S_m(t), \quad t = 0, 1. \]

Discounted value process

\[ V_t^* = \frac{V_t}{S_0(t)} \]

\[ = h_0 + \sum_{m=1}^{M} h_m S_m^*(t), \quad t = 0, 1. \]

Numerical example

2 risky securities and 3 possible states:

\[ S^*(0) = \begin{pmatrix} 4 & 2 \\ 4 & 3 \\ 3 & 2 \\ 2 & 4 \end{pmatrix} \]
**Arbitrary opportunity**

An arbitrary opportunity is some trading strategy $\mathcal{H}$ that has the properties:

(i) $V_0 = 0$,  
(ii) $V_1(\omega) \geq 0$ and $EV_1(\omega) > 0$,  
(iii) $\mathcal{H}$ is self-financing.

A probability measure $Q$ on $\Omega$ is a risk neutral probability measure if it satisfies

(i) $Q(\omega) > 0$ for all $\omega \in \Omega$,

(ii) $E_Q[\Delta S^*_m] = 0, m = 1, \ldots, M$, where $E_Q$ denotes the expectation under $Q$.

From $E_Q[\Delta S^*_m] = 0$, we have

$$S^*_m(0) = \sum_{k=1}^{K} Q(\omega_k) S^*_m(1; \omega_k).$$

The current discounted security price is given by the expectation of the discounted security payoff one period later.
No arbitrage theorem

No arbitrage opportunities exist if and only if there exists a risk neutral probability measure $Q$.

**Numerical example**

\[ S^*(0) = (4 \ 2) \text{ and } S^*(1; \Omega) = \begin{pmatrix} 4 & 3 \\ 3 & 2 \\ 2 & 4 \end{pmatrix}. \]

\[
\begin{align*}
4 &= 4Q(\omega_1) + 3Q(\omega_2) + 2Q(\omega_3) \\
2 &= 3Q(\omega_1) + 2Q(\omega_2) + 4Q(\omega_3) \\
1 &= Q(\omega_1) + Q(\omega_2) + Q(\omega_3).
\end{align*}
\]

We obtain $Q(\omega_1) = Q(\omega_2) = 2/3$ and $Q(\omega_3) = -1/3$, so $Q$ does not exist. Can we find a trading strategy $(h_1, h_2)^T$ such that $V_0^* = 4h_1 + 2h_2 = 0$ but $V_1^*(\omega_k) > 0, k = 1, 2, 3$ (with at least one strict inequality)?

Yes, suppose we take $h_1 = -2$ and $h_2 = 4$, we then have

\[
V_1^*(\omega_1) = 4, V_1^*(\omega_2) = 2 \text{ and } V_1^*(\omega_3) = 12.
\]
Suppose we change $S^*(0)$ to $S^*(0) = (3 \ 3)$, then

\[
\begin{align*}
3 & = 4Q(\omega_1) + 3Q(\omega_2) + 2Q(\omega_3) \\
3 & = 3Q(\omega_1) + 2Q(\omega_2) + 4Q(\omega_3) \\
1 & = Q(\omega_1) + Q(\omega_2) + Q(\omega_3).
\end{align*}
\]

We obtain $Q(\omega_1) = Q(\omega_2) = Q(\omega_3) = 1/3$. This implies the existence of a risk measure $Q$, and there will be no arbitrage opportunity.
The region above the two bold lines represents arbitrage trading strategies. The trading strategies that lies on the dotted line: \(4h_1 + 2h_2 = 0, h_1 < 0\) are dominant trading strategies.
The proof of the “no arbitrage theorem” requires the “Separating Hyperplane Theorem”.

If $A$ and $B$ are two non-empty disjoint convex sets in a vector space $V$, then they can be separated by a hyperplane.
Valuation of contingent claims

- A contingent claim is a random variable $X$ that represents the time $T$ payoff from a “seller” to a “buyer”.

- A contingent claim is said to be marketable or attainable if there exists a self-financing trading such that

$$V_T(\omega) = X(\omega) \text{ for all } \omega \in \Omega.$$ 

Risk neutral valuation principle

The time $t$ value of a marketable contingent claim $X$ is equal to $V_t$, the time $t$ value of the portfolio which replicates $X$.

$$V_t^* = V_t/B_t = E_Q[X/B_T|F_t], \quad t = 0, 1, \cdots, T$$

for all risk neutral probability measure $Q$. 
Martingales

The process \( Z \) is said to be a \textit{martingale} if

\[
E[Z_{t+s}|F_t] = Z_t \quad \text{for all } s \text{ and } t \geq 0.
\]

- A martingale is the mathematical formalization of the concept of a fair game.

- If prices of derivative securities can be modelled as martingales, this implies that no market participant can consistently make (or lose) money by trading in derivatives.
Martingales pricing theory

A risk neutral probability measure (martingale measure) is a probability measure $Q$ such that

1. $Q(\omega) > 0$ for all $\omega \in \Omega$; and

2. the discounted price process $S_n^*$ is a martingale under $Q, n = 1, \cdots, N$.

$$E_Q[S_n^*(t + s)|F_t] = S_n^*(t), \quad t \text{ and } s \geq 0.$$  

Theorems

1. There are no arbitrage opportunities if and only if there exists a martingale measure $Q$.

2. If $Q$ is a martingale measure and $\mathcal{H}$ is a self financing trading strategy, then $V^*$, the discounted value process corresponding to $\mathcal{H}$, is a martingale under $Q$. 
Completeness theorem

If the market is complete, that is, all contingent claims can be replicated, then equivalent martingale measures are unique.

Fundamental theorem of asset pricing

In an arbitrage free complete market, there exists a unique equivalent martingale measure.

Risk neutral pricing formula

In an arbitrage free complete market, arbitrage prices of contingent claims are their discounted expected values under the risk neutral (equivalent martingale) measure.
Market price of risk

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} - \rho_V V = 0
\]

where \( \rho_V \) is the expected rate of return of \( V \).

We may write formally the stochastic process of \( V \) as

\[
\frac{dV}{V} = \rho_V \ dt + \sigma_V \ dZ.
\]

Since the option and the stock are hedgeable, they share the same market price of risk

\[
\frac{\rho_V - r}{\sigma_V} = \frac{\mu - r}{\sigma}
\]

from which we can deduce

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial V^2} + r S \frac{\partial V}{\partial S} - r V = 0.
\]
Riskless hedging principle

Consider the writer of a derivative whose underlying asset has the following price process

$$\frac{dS}{S} = \mu dt + \sigma dZ.$$

Let $V(S,t)$ denote the price of the derivative. Black-Scholes (1973) derived the governing equation for $V(S,t)$ by following the dynamic hedging principle. Form the portfolio $\pi$ that contains $\Delta$ units of the underlying asset and shorts one unit of the derivative

$$\pi = -V + \Delta S.$$

By Ito's lemma:

$$d\pi = -\left[ \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} dt \right] + \Delta dS.$$
Suppose we choose $\Delta = \frac{\partial V}{\partial S}$ so that the stochastic components are cancelled.

$$d\pi = - \left( \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.$$

The riskfree portfolio should earn the riskfree interest rate, otherwise there is arbitrage opportunity. We then have

$$d\pi = r\pi dt \quad \Rightarrow \quad - \left( \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right) dt = r \left( -V + \frac{\partial V}{\partial S} S \right) dt,$$

and obtain the *Black-Scholes equation*

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0.$$

Note that the expected rate of return $\mu$ does not appear in the equation.

*Risk neutral valuation*

$$V(S, t) = e^{-r(T-t)} E^*[V_0(S_T)].$$

The option price is the discounted expectation of the terminal payoff under the risk neutral measure.
European call price formula

Terminal payoff: \( c(S, T) = \max(S - X, 0) \), where \( X \) is the strike price

\[
c(S, t) = SN(d_1) - X e^{-r(T-t)} N(d_2), \text{ where}\]
\[
d_1 = \frac{\ln \frac{S}{X} + \left( r + \frac{\sigma^2}{2} \right) (T-t)}{\sigma \sqrt{T-t}} \quad \text{and} \quad d_2 = \frac{\ln \frac{S}{X} + \left( r - \frac{\sigma^2}{2} \right) (T-t)}{\sigma \sqrt{T-t}}.
\]

The formula dictates that a call option can be replicated by holding \( \Delta = N(d_1) \) units of asset and shorting cash amount \( X e^{-r(T-t)} N(d_2) \). The hedge ratio \( \Delta \) changes with respect to time and asset value. The replication requires the dynamic hedging procedure. The writer charges the cost of constructing the replicating portfolio.