Guaranteed Minimum Withdrawal Benefit in Variable Annuities

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Agenda

- Product nature of the Guaranteed Minimum Withdrawal Benefit (GMWB)
- Construction of a pricing model based on the stochastic control approach
- Discrete withdrawal model
- Analysis of optimal withdrawal policies
- Conclusions

References

1. Milevsky, M.A. and T.S. Salisbury (2006). Financial valuation of guaranteed minimum withdrawal benefits. *Insurance: Mathematics and Economics*, vol. 38(1), 21-38.

2. Dai, M., Y.K. Kwok and J. Zong (2008). Guaranteed minimum withdrawal benefit in variable annuities. *Mathematical Finance*, vol. 8(6), 561-569.

3. Chen, Z., K. Vetzal and P. Forsyth (2008), The effect of modelling parameters on the value of GMWB guarantee. *Insurance: Mathematics and Economics*, vol. 43(1), 165-173.

4. Bauer, D., A. Kling and J. Russ (2008), A universal pricing framework for guaranteed minimum benefits in variable annuities. *Astin Bulletin*, vol. 38(2), 621-651.

Product Nature of GMWB

- Variable annuities — deferred annuities that are fund-linked.
- The single lump sum paid by the policyholder at initiation is invested in a portfolio of funds chosen by the policyholder — equity participation.
- The GMWB allows the policyholder to withdraw funds on an annual or semi-annual basis until the entire principal is returned.
- In 2004, 69% of all variable annuity contracts sold in the US included GMWB option.

Numerical example

- Let the initial fund value be $\$100,000$ and the withdrawal rate be 7% per annum. Suppose the investment account earns ten percent in the first two years but earns returns of minus sixty percent in each of the next three years.

Year	Rate earned during the year	Fund before with-drawals	Amount withdrawn	Fund after with-drawals	Guaranteed withdrawals remaining balance
1	10%	$110,000$	$7,000$	$103,000$	$93,000$
2	10%	$113,300$	$7,000$	$106,300$	$86,000$
3	-60%	$42,520$	$7,000$	$35,520$	$79,000$
4	-60%	$14,208$	$7,000$	$7,208$	$72,000$
5	-60%	$2,883$	$7,000$	0	$65,000$

- At the end of year five before any withdrawal the value of the fund, $\$2,883$, is not enough to cover the withdrawal payment of $\$7,000$.

The guarantee kicks in:

The value of the fund is set to be zero and the policyholder's ten remaining withdrawal payments are financed under the writer's guarantee. The policyholder's income stream of annual withdrawals is protected irrespective of the market performance.

- If the market does well, then there will be funds left at policy's maturity. However, if performance is bad the investment account balance will have shrunk to zero before the principal is repaid and will remain there.

Numerical example revisited

Suppose the initial lump sum investment of $\$100,000$ is used to purchase 100 units of the mutual fund, so each unit worths $\$1,000$.

- After the first year, the rate of return is 10% so each unit is $\$1,100$. The guaranteed withdrawal of $\$7,000$ represent $\$7,000/\$1,100 = 6.364$ units. The remaining number of units of mutual fund is $100 - 6.364 = 93.636$ units.

- After the second year, there is another rate of return of 10%, so each unit of mutual fund worths $\$1,210$. The withdrawal of $\$7,000$ represents $\$7,000/\$1,210 = 5.785$ units, so the remaining number of units $= 87.851$.

- There is a negative rate of return of 60% in the third year, so each unit of mutual fund worths \$484. The withdrawal of \$7,000 represents $\$7,000/\$484 = 14.463$ units, so the remaining number of unit $= 73.388$.

- Depending on the performance of the mutual fund, the total number of units, withdrawal can be less than 100 (if the fund is performing) or otherwise.
  - In the former case, the holder receives the guaranteed total withdrawal amount of \$100,000 (neglecting time value) plus the remaining units of mutual funds held at maturity.
  - If the mutual fund is non-performing, then the total withdrawal amount of \$100,000 is guaranteed.

How is the benefit funded?

- Percentage deduction from the account balance

  — for a contract with a 7% withdrawal allowance, a typical charge is around 40 to 50 basis points.

- Benefit can also be seen as a guaranteed stream of G per annum plus a call option on the terminal account value W_T. The strike price of the call is zero.

Static withdrawal model – continuous version

- The withdrawal rate G is fixed throughout the life of the policy.
- When the investment account value W_t ever reaches 0, it stays at this value thereafter (absorbing barrier).

  $\tau = \inf\{t : W_t = 0\}$, $\quad \tau$ is the first passage time of hitting 0.

  Under the risk neutral measure Q, the dynamics of W_t is governed by

$$
\begin{aligned}
dW_t &= (r-\alpha)W_t\,dt + \sigma W_t\,dB_t - G\,dt, & t < \tau \\
W_t &= 0, & t \geq \tau \\
W_0 &= w_0,
\end{aligned}
$$

  where α is the proportional annual fee charge on the withdrawal allowance.

  policy value $= E_Q\left[\int_0^T Ge^{-ru}\,du\right] + E_Q[e^{-rT}W_T]$.

To enhance analytic tractability, the restricted account value process W_t is replaced by a surrogate unrestricted process $\widetilde{W}_t$ at the expense of introducing optionality in the terminal payoff (zero strike call payoff). Consider the modified unrestricted stochastic process:

$$\begin{aligned} d\widetilde{W}_t &= (r-\alpha)\widetilde{W}_t\,dt - G\,dt + \widetilde{W}_t\,dB_t \\ \widetilde{W}_0 &= w_0. \end{aligned}$$

Solving for $\widetilde{W}_t$, we obtain

$$\widetilde{W}_t = X_t\left(w_0 - G\int_0^t \frac{1}{X_u}\,du\right)$$

where

$$X_t = e^{\left(r-\alpha+\frac{\sigma^2}{2}\right)t+\sigma B_t}.$$

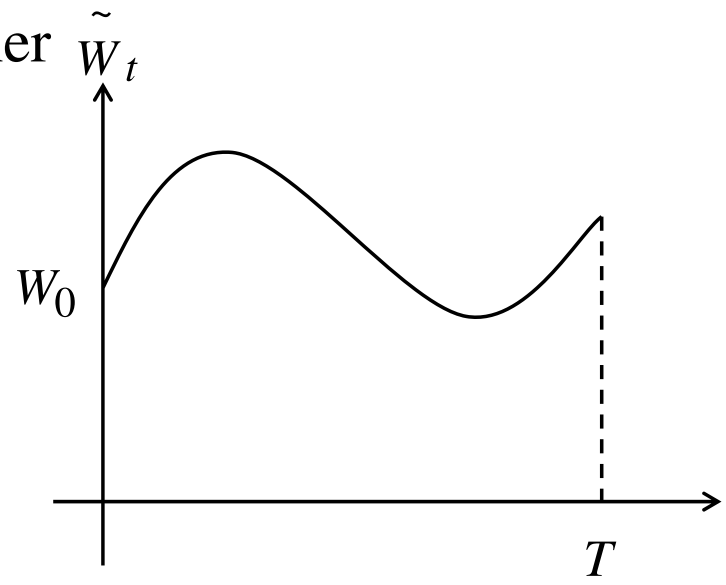
Financial interpretation

Take the initial value of one unit of the fund to be unity for convenience. Here, X_t represents the corresponding fund value process with $X_0 = 1$.

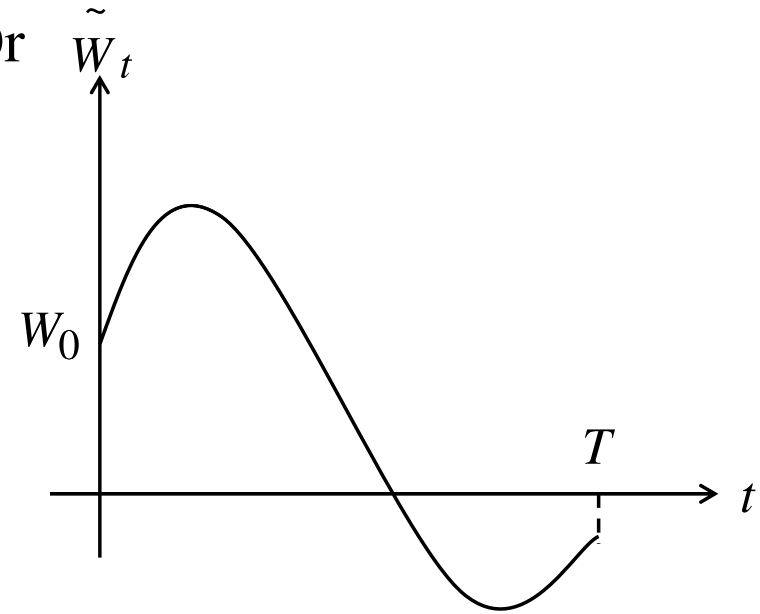
- The number of units acquired at initiation is w_0. The total number of units withdrawn over $(0, t]$ is given by $G \int_0^t \frac{1}{X_u} \, du$.

- Under the unrestricted process assumption, $\widetilde{W}_t$ may become negative when the number of units withdrawn exceeds w_0. However, in the actual case, W_t stays at the absorbing state of zero value once the number of unit withdrawn hits w_0.

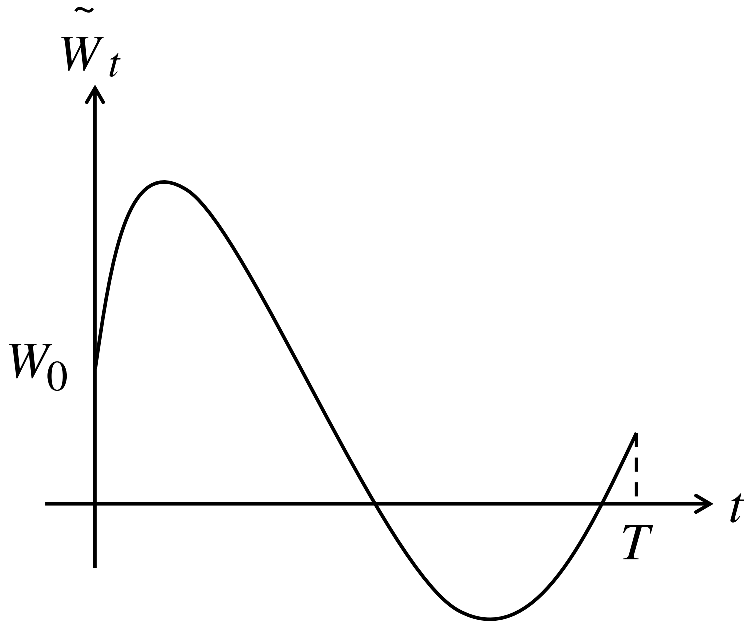
Either



Or



Rule out



Lemma $\tau_0 > T$ if and only if $\widetilde{W}_T > 0$.

$\Longrightarrow$ part. Suppose $\tau_0 > T$, then by the definition of the first passage time, we have $\widetilde{W}_T > 0$.

$\Longleftarrow$ part. Recall that

$$\widetilde{W}_t = X_t \left(w_0 - \int_0^t \frac{G}{X_u} \, du \right)$$

so that

$$\widetilde{W}_t > 0 \quad \text{if and only if} \quad \int_0^t \frac{G}{X_u} \, du < w_0.$$

Suppose $\widetilde{W}_T > 0$, this implies $\int_0^T \frac{G}{X_u} \, du < w_0$. Since $X_u \geq 0$, for any $t < T$, we have

$$\int_0^t \frac{G}{X_u} \, du \leq \int_0^T \frac{G}{X_u} \, du < w_0.$$

Hence, if $\widetilde{W}_T > 0$, then $\widetilde{W}_t > 0$ for any $t < T$.

Mathematical interpretation

Once the process $\widetilde{W}_t$ becomes negative, it will never return to the positive region. This is because when $\widetilde{W}_t = 0$, only the drift term $-G\,dt$ survives, which always pulls $\widetilde{W}_t$ into the negative region.

Relation between W_T and $\widetilde{W}_T$

We observe

$$W_T = \widetilde{W}_T \mathbf{1}_{\{\tau > T\}} = \widetilde{W}_T \mathbf{1}_{\{\widetilde{W}_T > 0\}} = \max(\widetilde{W}_T, 0).$$

Note that $W_T = 0$ if and only if $\tau \leq T$.

The terminal payoff is

$$\max(\widetilde{W}_T, 0) = GX_T \left(\frac{w_0}{G} - \int_0^T \frac{1}{X_u}\, dx \right)^+, \quad x^+ = \max(x, 0).$$

Defining $U_t = \dfrac{G}{w_0} \displaystyle\int_0^t \frac{1}{X_u}\, du$ and observing $T = \dfrac{w_0}{G}$, we obtain

$$E_Q[\widetilde{W}_T^+] = w_0 E_Q[X_T(1 - U_T)^+].$$

Here, U_t represents the fraction of units withdrawn up to time t, which captures the path dependence of depletion of the investment account due to the continuous withdrawal process.

Dynamic continuous withdrawal model

Policyholder has the right to surrender the contract or to withdraw a portion of the account value (partial surrender).

A_t: account balance of the guarantee

Clauses to discourage excessive withdrawals

- Percentage penalty charge applied on the excessive portion of the withdrawal amount.
- An excessive withdrawal may result in a decrease in the guarantee account greater than the withdrawal amount. The guarantee account is reduced to

$$\min(A_t, W_t) - \gamma_i \Delta t \quad \text{if} \quad \gamma_i > G.$$

A_t is a non-negative and non-increasing $\{\mathcal{F}_t\}_{t \geq 0}$-adaptive process. At initiation, $A_0 = w_0$; the withdrawal guarantee becomes insignificant when $A_t = 0$. As withdrawal continues, A_t decreases over the life of the policy until it hits the zero value. At T, A_T becomes zero.

The dynamics of the value of the investment account W_t follows

$$dW_t = (r - \alpha)W_t \; dt + \sigma W_t \, dB_t + dA_t, \quad t < \tau,$$

$$A_t = A_0 - \int_0^t \gamma_s \, ds, \quad 0 \leq \gamma_s \leq \lambda,$$

γ_s is the withdrawal rate process and λ is some upper bound.

Penalty charges are incurred when the withdrawal rate γ exceeds the contractual withdrawal rate G.

Supposing a proportional penalty charge k is applied on the portion of γ above G, then the net amount received by the policyholder is $G + (1 - k)(\gamma - G)$ when $\gamma > G$.

- Let $f(\gamma)$ denote the rate of cash flow received by the policyholder as resulted from the continuous withdrawal process, we then have

$$f(\gamma) = \begin{cases} \gamma & \text{if } 0 \leq \gamma \leq G \\ G + (1-k)(\gamma - G) & \text{if } \gamma > G \end{cases} .$$

The policyholder receives the continuous withdrawal cash flow $f(\gamma_u)$ du over $(u, u+du)$ throughout the life of the policy and the remaining balance of the investment account at maturity.

Rational behavior of policyholder

- He chooses the optimal withdrawal policy dynamically so as to maximize the present value of cash flows generated from holding the variable annuity policy and under the restricted class of withdrawal policies.

- The no-arbitrage value $\overline{V}$ of the variable annuity with GMWB is given by

$$\overline{V}(W, A, t) = \max_{\gamma} \mathsf{E}_t \left[e^{-r(T-t)} \max(W_T, 0) + \int_t^T e^{-r(u-t)} f(\gamma_u) \, du \right].$$

  Here, γ is the *control variable* that is chosen to maximize the expected value of the discounted cash flows.

Hamilton-Jacobi-Bellman (HJB) equation

The governing equation for $\overline{V}$ is found to be

$$\frac{\partial \overline{V}}{\partial t} + \mathcal{L}\overline{V} + \max_{\gamma} h(\gamma) = 0$$

where

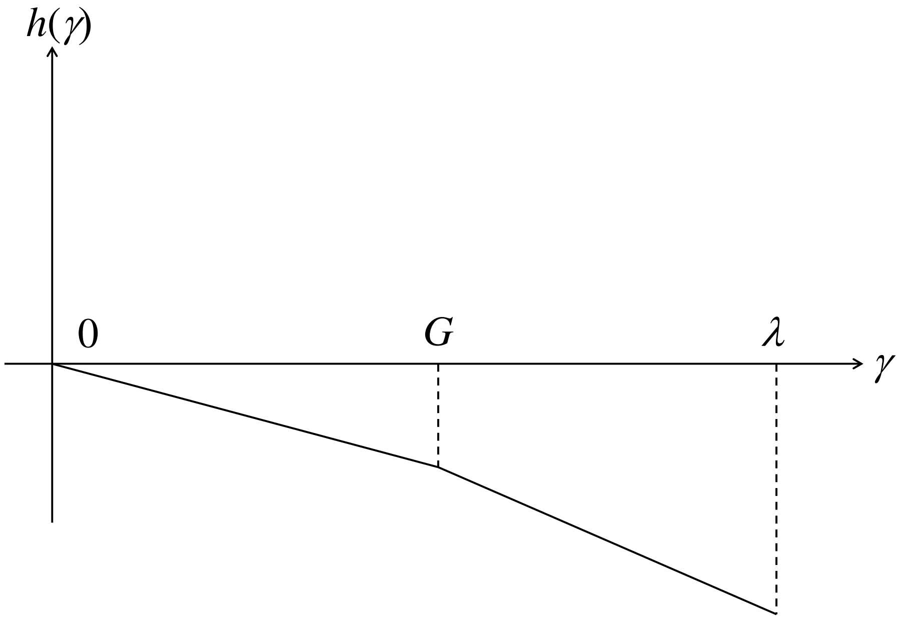
$$\mathcal{L}\overline{V} = \frac{\sigma^2}{2} W^2 \frac{\partial^2 \overline{V}}{\partial W^2} + (r - \alpha) W \frac{\partial \overline{V}}{\partial W} - r\overline{V}$$

$$\begin{aligned} h(\gamma) &= f(\gamma) - \gamma \frac{\partial \overline{V}}{\partial W} - \gamma \frac{\partial \overline{V}}{\partial A} \\ &= \begin{cases} \left(1 - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}\right)\gamma & \text{if } 0 \leq \gamma < G \\ kG + \left(1 - k - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}\right)\gamma & \text{if } \gamma \geq G \end{cases} . \end{aligned}$$

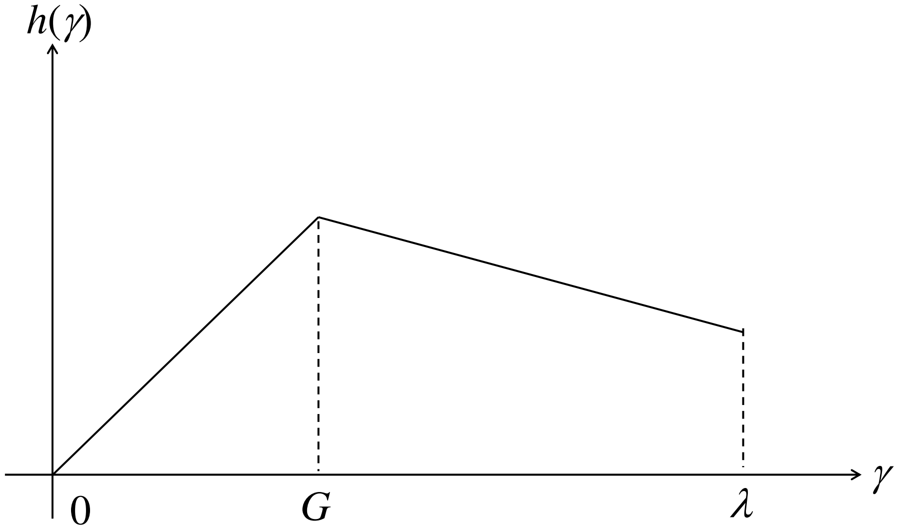
Write $\beta = 1 - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}$, then

$$h(\gamma) = \begin{cases} \beta\gamma & \text{if } 0 < \gamma \leq G \\ \beta\gamma - k(\gamma - G) & \text{if } \gamma \geq G \end{cases} = \begin{cases} \beta\gamma & \text{if } 0 < \gamma \geq G \\ (\beta - k)\gamma + kG & \text{if } \gamma \geq G \end{cases}$$

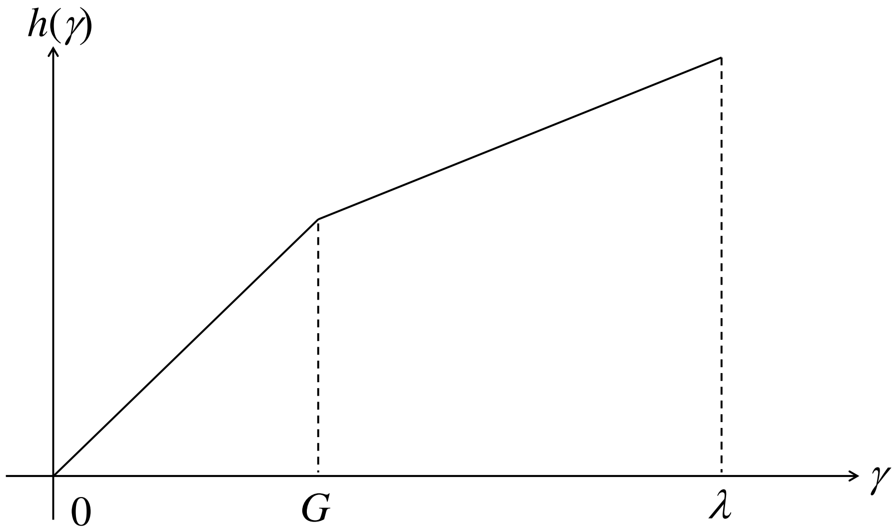
(i) $\beta \leq 0$



(ii) $0 < \beta < k$



(iii) $\beta \geq k$



Penalty approximation

The function $h(\gamma)$ is piecewise linear so its maximum value is achieved at either $\gamma = 0, \gamma = G$ or $\gamma = \lambda$.

Recall $0 \leq \gamma \leq \lambda$. Note that

$$\max_{\gamma} h(\gamma) = \begin{cases} kG + \lambda\left(1 - k - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}\right) & \text{if } 1 - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A} \geq k \\ \left(1 - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}\right) G & \text{if } 0 < 1 - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A} < k \\ 0 & \text{if } 1 - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A} \leq 0 \end{cases} .$$

We obtain the following equation for $\overline{V}$:

$$\frac{\partial \overline{V}}{\partial t} + \mathcal{L}\overline{V} + \min\left[\max\left(1 - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}, 0\right), k\right] G$$
$$+ \ \lambda \max\left(1 - k - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}, 0\right) = 0. \qquad \text{(A)}$$

The set of variational inequalities are given by

$$\frac{\partial \overline{V}}{\partial t} + \mathcal{L}\overline{V} \leq 0 \qquad (i)$$

$$\frac{\partial \overline{V}}{\partial t} + \mathcal{L}\overline{V} + G\left(1 - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}\right) \leq 0 \qquad (ii)$$

$$\frac{\partial \overline{V}}{\partial t} + \mathcal{L}\overline{V} + kG + \lambda\left(1 - k - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}\right) \leq 0 \qquad (iii)$$

and equality holds in at least one of the above three cases.

For example, suppose $1 - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A} \leq 0, \max_\gamma h(\gamma)$ is achieved by taking $\gamma = 0$ (corresponding to zero withdrawal rate). We have equality for (i), and strict inequalities for (ii) and (iii). That is,

$$
\begin{aligned}
&\frac{\partial \overline{V}}{\partial t} + \mathcal{L}\overline{V} = 0 \\
&\frac{\partial \overline{V}}{\partial t} + \mathcal{L}\overline{v} + G\left(1 - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}\right) < 0 \\
&\frac{\partial \overline{V}}{\partial t} + \mathcal{L}\overline{V} + kG + \lambda\left(1 - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A}\right) < 0.
\end{aligned}
$$

This corresponds to the continuation region with no withdrawal.

- Similarly, when $0 < 1 - \frac{\partial \overline{V}}{\partial W} - \frac{\partial \overline{V}}{\partial A} < k$, we have equality for (ii) and strict inequalities for (i) and (iii). This corresponds to the region with withdrawal at rate G.

- When $\frac{\partial \overline{V}}{\partial W} + \frac{\partial \overline{V}}{\partial A} \leq 1 - k$, it is optimal to choose λ as the withdrawal rate. We have strict equality for (iii). Suppose we take $\lambda \to \infty$, then

$$\frac{\partial \overline{V}}{\partial W} + \frac{\partial \overline{V}}{\partial A} = 1 - k$$

in order to satisfy the strict equality in (iii).

Linear complementarity formulation

To obtain $V(W, A, t)$ from $\overline{V}(W, A, t)$, we allow the upper bound λ on γ to be infinite. Conversely, Eq. (A) is visualized as the corresponding penalty approximation

Taking the limit $\lambda \to \infty$, we obtain the following linear complementarity formulation of the value function $V(W, A, t)$:

$$\min\left[-\frac{\partial V}{\partial t} - \mathcal{L}V - \max\left(1 - \frac{\partial V}{\partial W} - \frac{\partial V}{\partial A}, 0\right) G, \frac{\partial V}{\partial W} + \frac{\partial V}{\partial A} - (1-k)\right] = 0,$$

$$W > 0, \quad 0 < A < w_0, \quad t > 0.$$

Remark It can be shown that the case $\gamma = 0$ should be ruled out so that the above formulation can be simplified to

$$\min\left(-\frac{\partial V}{\partial t} - \mathcal{L}V - \left(1 - \frac{\partial V}{\partial W} - \frac{\partial V}{\partial A}\right) G, \frac{\partial V}{\partial W} + \frac{\partial V}{\partial A} - (1-k)\right] = 0.$$

Auxiliary conditions

- At maturity, the policyholder takes the maximum between the remaining guarantee withdrawal net of penalty charge and the remaining balance of the personal account.

$$V(W, A, T) = \max(W, (1-k)A)$$

- When either $A = 0$ or $W \to \infty$, the withdrawal guarantee becomes insignificant. The value of the annuity becomes $We^{-\alpha(T-t)}$.

$$V(W, 0, t) = e^{-\alpha(T-t)}W,$$
$$V(W, A, t) \to e^{-\alpha(T-t)}W \text{ as } W \to \infty.$$

- When $W = 0$, the equity participation of the policy vanishes. The pricing formulation reduces to a simplified optimal control model with no dependence on W.

$$V(0, A, t) = V_0(A, t).$$

Analytic solution to $V_0(A,t)$

- When $W = 0$, the equity participation of the policy vanishes.
- Let $V_0(A,t)$ be the value function of the annuity when $W = 0$, which is the solution to the following linear complementarity formulation

$$\min\left[-\frac{\partial V_0}{\partial t} + rV_0 - \max\left(1 - \frac{\partial V_0}{\partial A}, 0\right) G, \frac{\partial V_0}{\partial A} - (1-k)\right] = 0,$$
$$0 < A < A_0, 0 < t < T,$$
$$V_0(A,T) = (1-k)A \quad \text{and} \quad V_0(0,t) = 0.$$

Solution without the derivative constraint

First, we consider the solution of $V_0(A,t)$ without the inequality constraint: $\frac{\partial V_0}{\partial A} - (1-k) \geq 0$. Together with the observation that $\frac{\partial V_0}{\partial A} \leq 1$, the governing equation for $V_0(A,t)$ is given by

$$\frac{\partial V_0}{\partial t} - G\frac{\partial V_0}{\partial A} - rV_0 + G = 0, \quad 0 \leq t \leq T, 0 \leq A \leq A_0,$$

with auxiliary conditions: $V_0(A,T) = (1-k)A$ and $V_0(0,t) = 0$.

If we define

$$W_0(A,t) = V_0(A,t)e^{r(T-t)} - \frac{G}{r}\Big[e^{r(T-t)} - 1\Big],$$

then $W_0(A,t)$ satisfies the prototype hyperbolic equation:

$$\frac{\partial W_0}{\partial t} - G\frac{\partial W_0}{\partial A} = 0$$

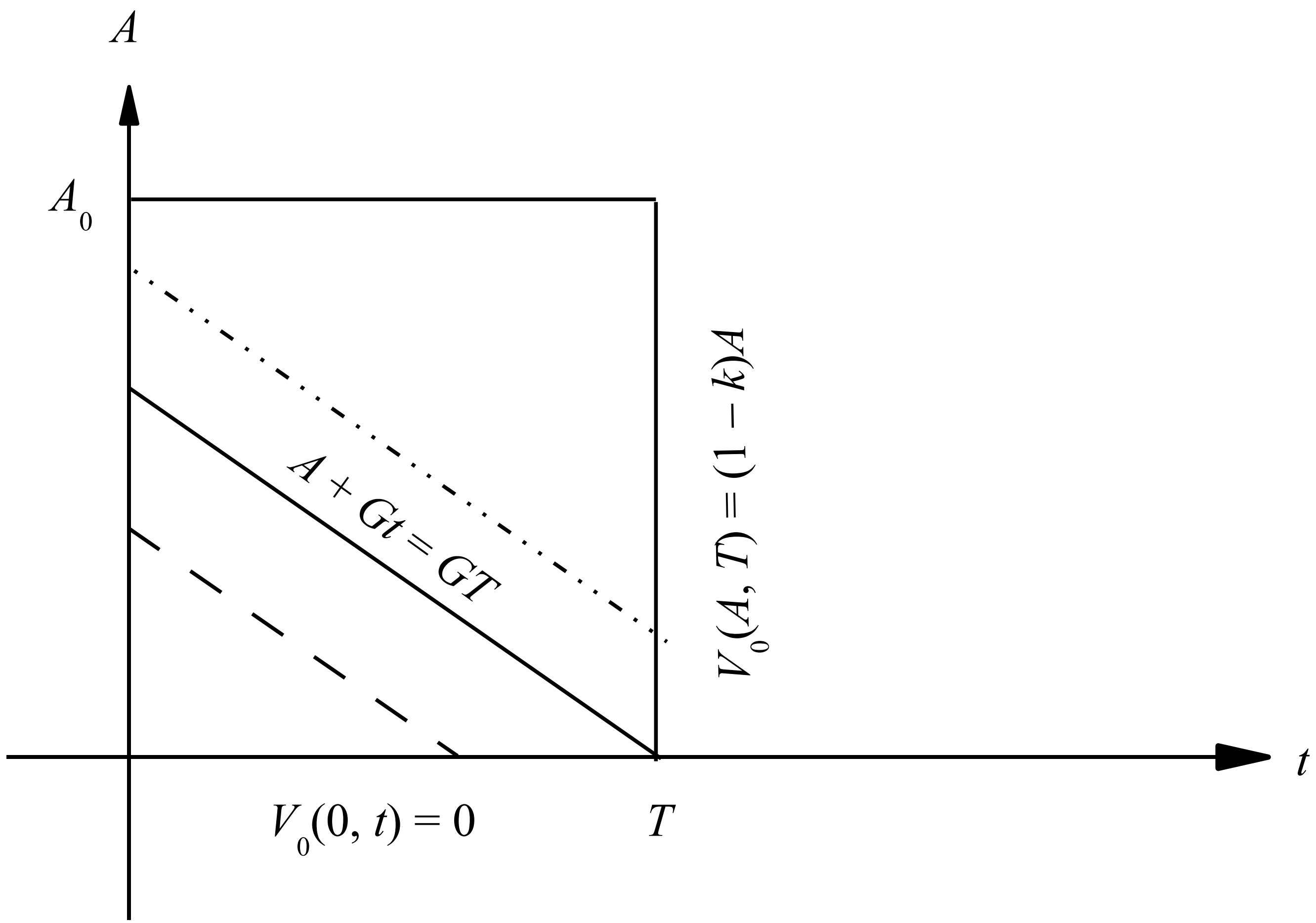
with auxiliary conditions:

$W_0(A,T) = (1-k)A$ and $W_0(0,t) = -\frac{G}{r}\Big[e^{r(T-t)} - 1\Big]$.

The general solution to $W_0(A,t)$ is of the form

$$W_0(A,t) = F(\xi), \quad \xi = t + \frac{A}{G},$$

where F is some function to be determined by the auxiliary conditions. The characteristics of the hyperbolic equation are given by the lines: $\xi = t + \frac{A}{G} = \xi_0$, for varying values of ξ_0.



The characteristic lines are given by $t + \dfrac{A}{G} = \xi_0$ for varying values of ξ_0. For $\xi_0 > T$, the characteristic lines intersect the right vertical boundary: $t = T$; and for $\xi_0 \leq T$, the characteristics lines intersect the bottom horizontal boundary: $A = 0$.

The continuation region is limited to the region:

$$\{(A,t) : A < -\frac{G}{r}\ln(1-k) \text{ and } A < G(T-t)\}.$$

Define

$$\tau^* = \min\Big(-\frac{\ln(1-k)}{r}, T-t\Big),$$

then the solution of $V_0(A,t)$ in the continuation region is given by

$$V_0(A,t) = \frac{G}{r}\Big(1 - e^{-\frac{r}{G}A}\Big) \quad \text{if} \quad A < G\tau^*.$$

In the stopping region, $V_0(A,t)$ satisfies $\frac{\partial V_0}{\partial A} = 1 - k$.

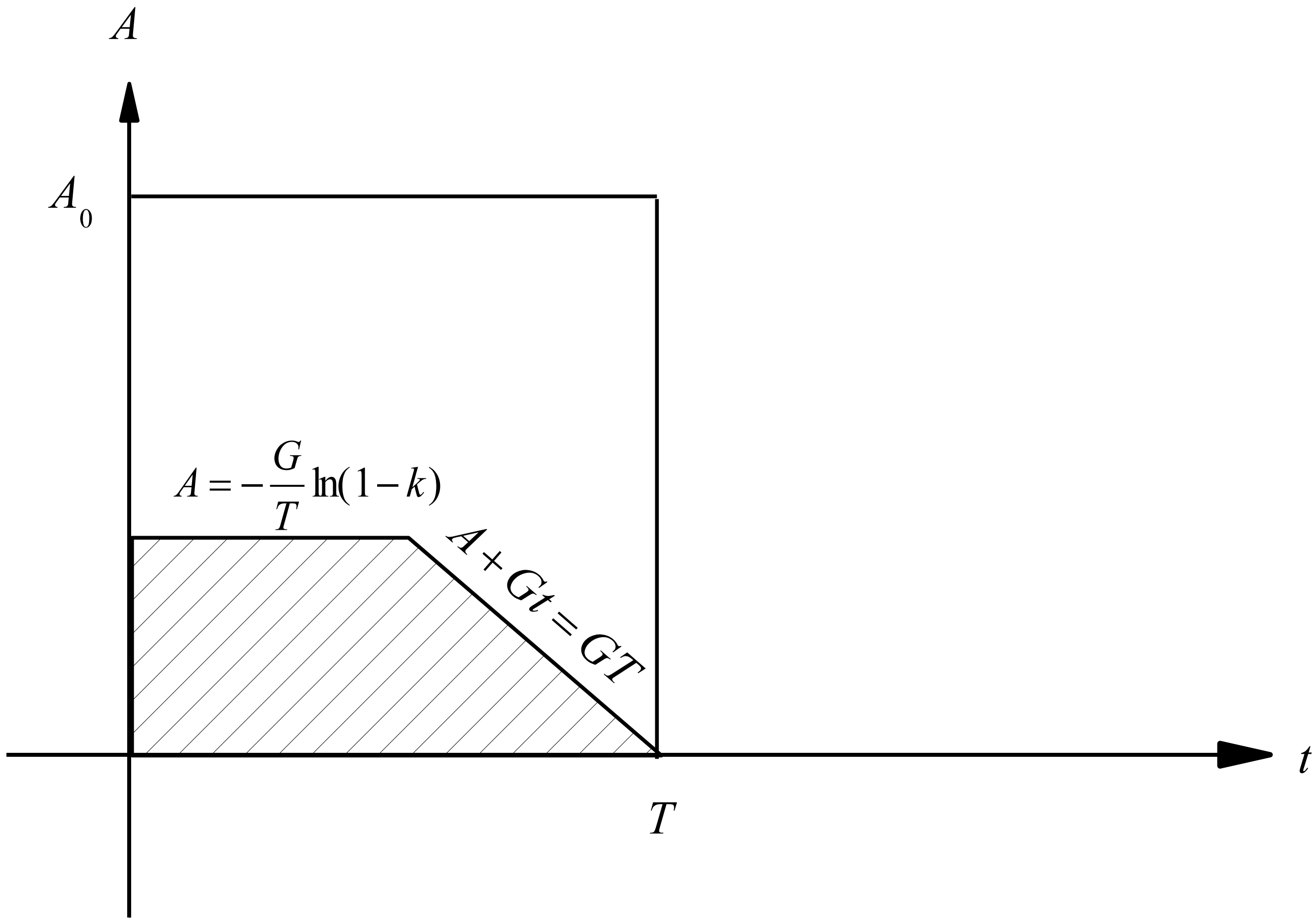
The solution takes the form: $V_0(A,t) = (1-k)A + C(t)$, where $C(t)$ is some arbitrary function.

The solution in the stopping region is given by

$$V_0(A,t) = (1-k)A + \frac{G}{r}(1 - e^{-r\tau^*}) - (1-k)G\tau^*, \quad A \geq G\tau^*.$$

Combining the results, the solution is found to be

$$V_0(A,t) = (1-k)\max(A - G\tau^*, 0) + \frac{G}{r}\Big[1 - e^{-r\min(\tau^*, \frac{A}{G})}\Big].$$



The continuation region lies in the region (shaded part)

$$\{(t,A) : A \le -\frac{G}{r}\ln(1-k) \text{ and } A - G(T-t) \le 0\},$$

with $V_0(A,t) = \dfrac{G}{r}(1 - e^{-\frac{r}{G}A})$.

Optimal withdrawal policies when the investment account becomes zero

- The policyholder strikes the balance between the penalty charge and the time value of the cash flows.

To minimize the penalty charge, the policyholder either withdraws at the rate G or infinite rate (instantaneous withdrawal of finite amount).

- When $A(t)$ is above certain threshold level (time-dependent), the optimal strategy is to withdraw a certain part of $A(t)$ instantaneously, followed by withdrawing the remaining balance at the rate G.

Suppose cash flows are received at the rate G from the current time t to some future time T_0, where $T_0 \le T$, then the corresponding present value is given by

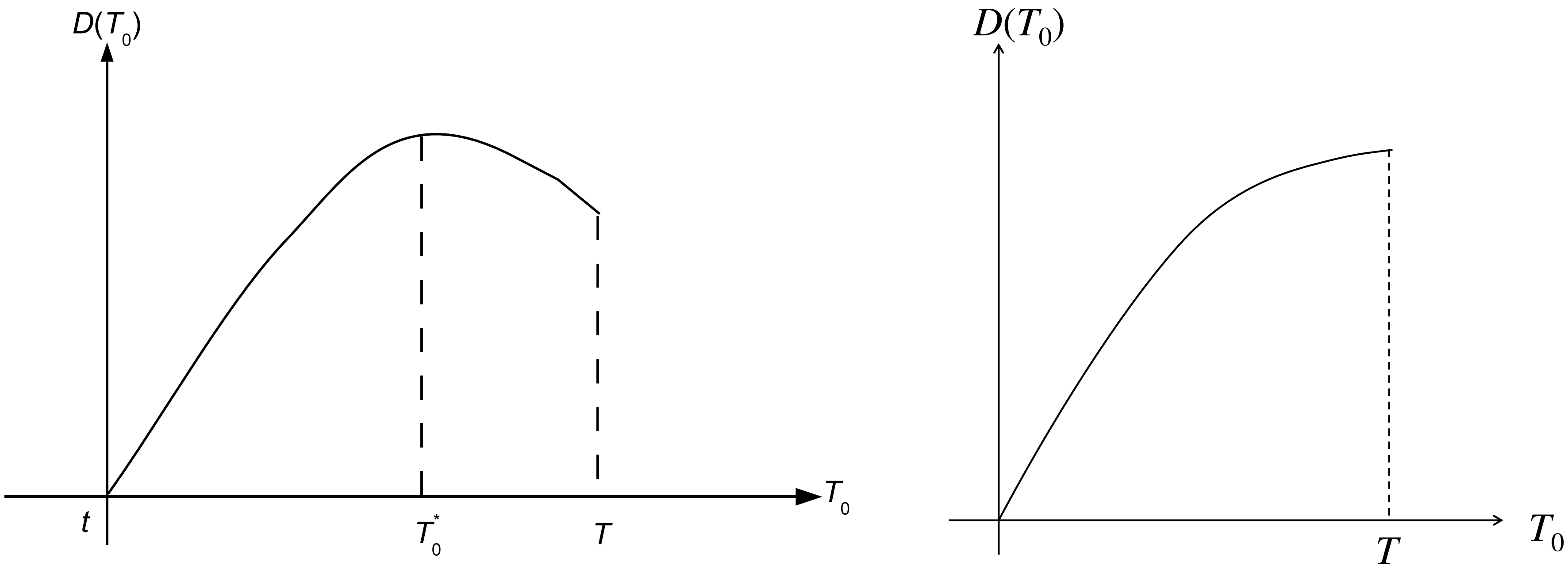
$$\int_t^{T_0} e^{-r(u-t)} G\,du = \frac{G}{r}\left[1 - e^{-r(T_0-t)}\right].$$

However, suppose the lump sum $G(T_0-t)$ is received instantaneously at time t, the net amount received is $(1-k)G(T_0-t)$.

Difference $D(T_0) = \dfrac{G}{r}[1 - e^{-r(T_0-t)}] - (1-k)G(T_0-t), \quad t < T_0 \le T.$

The maximum is achieved at

$$T_0^* = t + \min\left(-\frac{\ln(1-k)}{r}, T-t\right).$$



(i) $T_0^* = t - \frac{\ln(1-k)}{r}$ (ii) $T_0^* = T$

(i) $T_0^* < T$

$A(t) \leq G(T - T_0^*)$ – withdraws at the constant rate G throughout the remaining life; actually $A(t)$ becomes zero by time T_0^*.

$A(t) > G(T - T_0^*)$ – withdraws the finite amount $A(t) - G(T - T_0^*)$ instantaneously, then followed by withdrawing at the rate G until time T_0^* (at that time $A(t)$ becomes zero).

(ii) $T_0^* = T$

$A(t) \leq G(T - t)$ – withdraws at the constant rate G throughout the remaining life;

$A(t) > G(T - t)$ – withdraws the finite amount $A(t) - G(T - t)$ instantaneously, then followed by withdrawing at the rate G throughout the remaining life.

Construction of the finite difference scheme

- The numerical solution of the singular stochastic control formulation poses a difficult computational problem.

- Instead of solving the singular stochastic control model directly, we solve for the penalty approximation model in which the allowable control is bounded.

- Since the governing equation is a degenerate diffusion equation with only the first order derivative of A appearing, upwind discretization must be used to deal with the first order derivative terms in the differential equation.

Two-level implicit schemes

$$\begin{aligned}\frac{\overline{V}_{j,k}^{n+1}-\overline{V}_{j,k}^{n}}{\triangle\tau} &= \theta\mathcal{L}_h\overline{V}_{j,k}^{n+1}+(1-\theta)\mathcal{L}_h\overline{V}_{j,k}^{n}\\ &\quad+\min\Big(\max\Big(1-\frac{\overline{V}_{j,k}^{n+1}-\overline{V}_{j-1,k}^{n+1}}{\triangle W}-\frac{\overline{V}_{j,k}^{n+1}-\overline{V}_{j,k-1}^{n+1}}{\triangle A},0\Big),k\Big)G\\ &\quad+\lambda\ \max\Big(1-k-\frac{\overline{V}_{j,k}^{n+1}-\overline{V}_{j-1,k}^{n+1}}{\triangle W}-\frac{\overline{V}_{j,k}^{n+1}-\overline{V}_{j,k-1}^{n+1}}{\triangle A},0\Big),\end{aligned}$$

where θ is a weighting factor, $0<\theta\leq 1$. When $\theta=1$, we have the fully implicit scheme; while $\theta=\frac{1}{2}$ corresponds to the Crank-Nicholson scheme.

- Due to the presence of the mildly non-linear penalty term in the differential equation, a non-linear algebraic system of equations has to be solved at each time step. Newton type iterations are applied to solve the non-linear algebraic equations.

Discrete withdrawal model

- Discrete withdrawal amount is only allowed at time $t_i, i = 1, 2, \cdots, N$.
- Let the discrete withdrawal amount at time t_i be denoted by γ_i.
- Since the account balance of the withdrawal guarantee A_t remains unchanged within the interval $(t_{i-1}, t_i), i = 1, 2, \cdots, N$, the annuity value function $V(W, A, t)$ satisfies the following differential equation which has no dependence on A:

$$\frac{\partial V}{\partial t} + \mathcal{L}V = 0, \quad t \in (t_{i-1}, t_i), \quad i = 1, 2, \cdots, N.$$

The updating of A_t only occurs at the withdrawal dates.

Jump condition across a withdrawal date

- Upon withdrawing an amount γ_i at t_i, the annuity account drops from W_t to $\max(W_t - \gamma_i, 0)$, while the guarantee balance drops from A_t to $A_t - \gamma_i$.

- The jump condition of $V(W, A, t)$ across t_i is given by

$$V(W, A, t_i^-) = \max_{0 \leq \gamma_i \leq A} \{V(\max(W - \gamma_i, 0), A - \gamma_i, t_i^+) + \widehat{f}(\gamma_i)\}.$$

  Here, $\widehat{f}(\gamma_i)$ represents the actual cash amount received by the policyholder subject to a penalty charge under excessive withdrawal.

Discretely monitored path dependent option formulation

The auxiliary conditions for $V(W, A, t)$ remain the same except that the boundary value function $V_0(A, t)$ under discrete withdrawal is governed by

$$\frac{\partial V_0}{\partial t} - rV = 0, \qquad t \neq t_i, \quad i = 1, 2, \cdots, N,$$

$$V_0(A, t^-) = \max_{0 \leq \gamma_i \leq A} \{V_0(A - \gamma_i, t^+) + \widehat{f}(\gamma_i)\}, \quad t = t_i, \quad i = 1, 2, \cdots, N,$$

$$V_0(A, T) = \widehat{f}(A) \quad \text{and} \quad V_0(0, t) = 0.$$

Here, A serves the role as the path dependent variable, which is updated whenever the calendar time sweeps across a fixing date.

N_t	N_W	N_A	annuity value	change in value	ratio of change
32	64	64	96.241		
64	128	128	94.720	-1.521	
128	256	256	93.788	-0.932	1.6
256	512	512	93.506	-0.282	3.3
512	1024	1024	93.419	-0.087	3.3

Examination of the rate of convergence of the Crank-Nicholson scheme for solving the penalty approximation model.

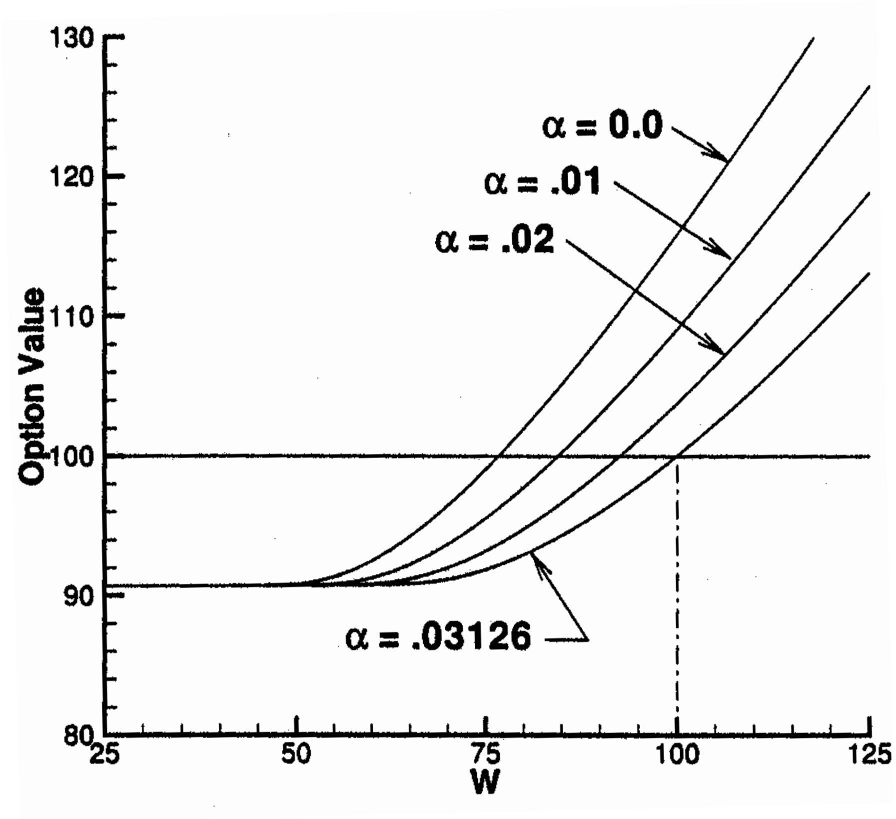
penalty parameter λ	$k = 1\%$	$k = 10\%$
	annuity value	annuity value
10^1	89.515	87.187
10^2	99.924	92.720
10^3	101.884	93.327
10^4	101.028	93.410
10^5	101.043	93.418
10^6	101.045	93.419
10^7	101.045	93.419
10^8	101.045	93.419

Test of convergence of the numerical approximation solution to the annuity value with varying values of the penalty parameter λ and penalty charge k.

Fair insurance fee

contractual rate, g	maturity, $T = 1/g$	$k = 5\%$		$k = 10\%$	
		$\sigma = 20\%$	$\sigma = 30\%$	$\sigma = 20\%$	$\sigma = 30\%$
4%	25.00	103 bp	213 bp	56 bp	133 bp
5%	20.00	125 bp	260 bp	69 bp	162 bp
6%	16.67	145 bp	305 bp	83 bp	192 bp
7%	14.29	165 bp	348 bp	97 bp	221 bp
8%	12.50	185 bp	390 bp	111 bp	251 bp
9%	11.11	202 bp	429 bp	124 bp	277 bp
10%	10.00	219 bp	466 bp	137 bp	304 bp
15%	6.67	296 bp	639 bp	198 bp	434 bp

Impact of the GMWB contractual rate g, penalty charge k and equity volatility σ of the account on the required insurance fee α (in basis points).



The value of the GMWB guarantee as a function of W at $t = 0$, $A = 100$, with respect to various values of the insurance fee α including the fair value $\alpha = 0.03126$. The fair value of the fee occurs when the value of the guarantee V satisfies $V = w_0 = 100$.

- When W is relatively small, α has no effect on the contract value since the guarantee component of the contract dominates the equity component ($A \gg W$). In this case, the contract value is independent of the insurance fee which is imposed on the equity component.

- As the fee increases, the no-arbitrage value of the contract decreases near $W = 100$.

- The value of the contract is precisely $V = 100$ at $W = 100$ when the fair fee is charged.

Why $\gamma = 0$ is ruled out?

- Non-withdrawal amount is subject to a proportional insurance fee α.
- Under the risk neutral valuation framework, the drift rate of W_t is $r - \alpha$, which is less than r.
- Withdrawal is more preferable since the withdrawal amount will have a higher return at the rate r as priced under the risk neutral valuation.

$$V(W+\delta, A+\delta, t) \leq V(W, A, t) + \delta, \qquad \delta > 0,$$

thus giving

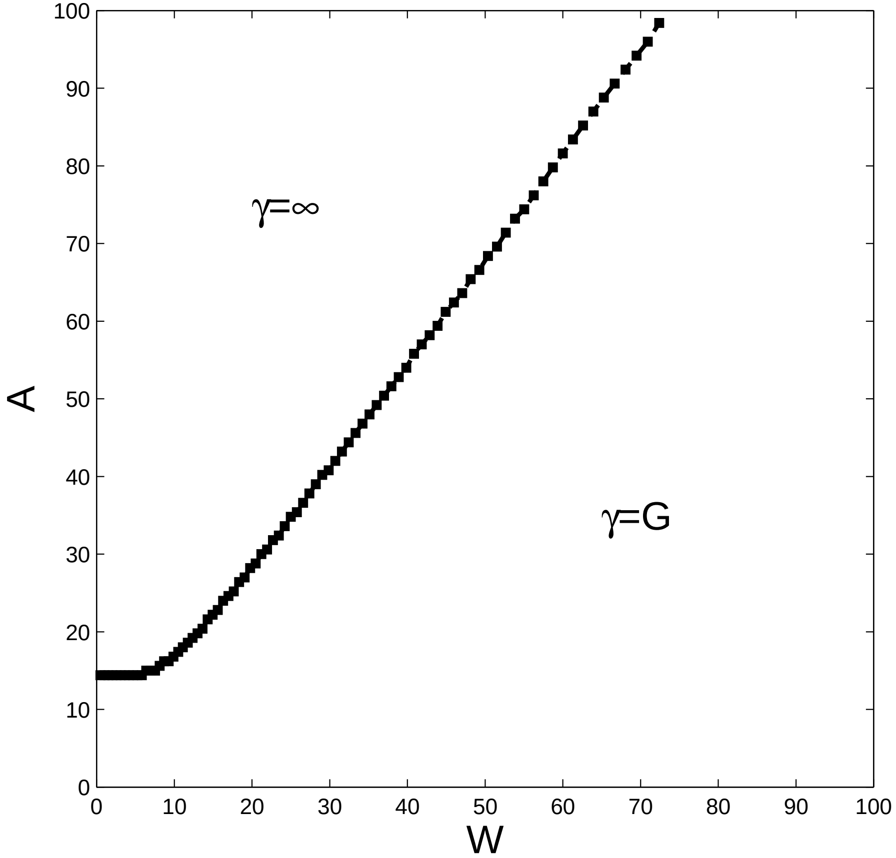
$$\frac{\partial V}{\partial W} + \frac{\partial V}{\partial A} = \lim_{\delta \to 0} \frac{V(W+\delta, A+\delta, t) - V(W, A, t)}{\delta} \leq 1.$$

With the positivity of $1 - \frac{\partial V}{\partial W} - \frac{\partial V}{\partial A}$, the linear complementarity formulation reduces to

$$\min\left(-\frac{\partial V}{\partial t} - \mathcal{L}V - \left(1 - \frac{\partial V}{\partial W} - \frac{\partial V}{\partial V}\right) G, \frac{\partial V}{\partial W} + \frac{\partial V}{\partial A} - (1-k)\right) = 0,$$

thus withdrawal always occurs under the optimal dynamic withdrawal policy.

- When (W_t, A_t) lies within "$\gamma = \infty$" region, the holder should withdraw instantaneously a finite amount until (W_t, A_t) falls to a point on the separating boundary.



Plot of the optimal withdrawal boundary in the (W, A)-plane, separating the "$\gamma = \infty$" region at the top from the "$\gamma = G$" region at the bottom. The boundary intersects the A-axis at $A = -\frac{G}{r}\ln(1-k)$.

Conclusions

- Following the Hamilton-Jacobi-Bellman approach that is commonly used in stochastic control problems, a singular stochastic control model is constructed for pricing variable annuities with guaranteed minimum withdrawal benefit. Here, the withdrawal rate is considered as a control variable.

- We apply the *penalty approach* where an upper bound is placed on the withdrawal rate. We then take the bound to tend to infinity subsequently. This penalty approach leads to an effective numerical approximation methods using the finite difference scheme.

- We have also constructed the numerical scheme for solving the discrete model. The apparent agreement of the numerical results from both versions serves to check for consistency of the two pricing approaches.

Insurance fee

- The insurance fee increases with increasing equity volatility level and contractual withdrawal rate but decreases with a higher penalty charge.

- The insurer should charge a substantially high insurance fee when the poliyholder has the flexibility of *dynamic* withdrawal.

Optimal withdrawal policies

- When there is a penalty on withdrawal above the contractual rate, the policyholder either withdraws a finite amount (infinite withdrawal rate) or withdraws at the contractual rate.

- When it is optimal for the policyholder to choose "withdrawal in a finite amount", he chooses to withdraw an appropriate finite amount instantaneously. This is to make the equity value of the investmnet account and guarantee balance to fall to the level that it becomes optimal for him to withdraw at the contractual rate.