Efficient Option Pricing using the Fast Fourier Transforms

presented by

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Agenda

• Overview of the Fourier transform approach to option pricing
  – Generalized Fourier transform
  – Characteristic functions
  – Lévy models and stochastic volatility model
  – Option pricing in the Fourier domain

• Fourier space-time stepping method
  – Characteristic component of a Lévy process
  – Applications to path dependent options

• Markov chain approximation method
Fourier transform and characteristic functions

- A piecewise continuous real function $f(x)$ over $(-\infty, \infty)$ is said to be $L^1$-integrable if it satisfies

$$\int_{-\infty}^{\infty} |f(x)| \, dx < \infty.$$  

The Fourier transform of $f$ as defined by

$$\mathcal{F}_f(u) = \int_{-\infty}^{\infty} e^{iux} f(x) \, dx$$

exists, provided that $f$ is $L^1$-integrable.

- The Fourier inversion of $\mathcal{F}_f(u)$ is known to be

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \mathcal{F}_f(u) \, du.$$
Generalized Fourier transform

Unfortunately, the Fourier transform of most payoff function does not exist. For example, the call payoff \((x - K)^+\) is not integrable. Its Fourier transform is found to be

\[
\int_{-\infty}^{\infty} e^{izx}(e^x - K)^+ \, dx = -\frac{K^{iz+1}}{z(z - i)},
\]

which has singularities at \(z = 0\) and \(z = i\).

It is necessary to consider the generalized Fourier transforms where the frequency argument \(u\) is taken to be complex in general.
**Parseval relation:** Consider the inner product of two complex-valued $L^1$ functions, then

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx$$

$$= \frac{1}{2\pi} \langle \mathcal{F}f(u), \mathcal{F}g(u) \rangle .$$

European option prices can be expressed as the inner product of density function and terminal payoff. The Parseval relation dictates that option prices can be computed via taking inner product of their corresponding Fourier transforms.
Characteristic function

For \( z \in \mathbb{C} \) and \( a < \text{Im} \ z < b \), the characteristic function of the random process \( X_t \) is defined as

\[
\phi_t(z) = E[e^{izX_t}].
\]

Let \( p_t(x) \) be the transition probability density for a random process to reach \( X_t = x \) after the elapse of time \( t \), \( X_0 = 0 \). For \( a < \text{Im} \ z < b \), the characteristic function of the random process is the generalized Fourier transform of the transition density

\[
\phi_t(z) = \mathcal{F}[p_t(x)] = \int_{-\infty}^{\infty} e^{izx} p_t(x) \, dx, \quad a < \text{Im} \ z < b.
\]
Lévy Processes

An adapted real-valued stochastic process $X_t$, with $X_0 = 0$, is called a Lévy process if it observes the following properties:

1. **Independent increments**
   For every increasing sequence of times $t_0, t_1, \ldots, t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.

2. **Time-homogeneous**
   The distribution of $\{X_{t+s} - X_s; t \geq 0\}$ does not depend on $s$.

3. **Stochastically continuous**
   For any $\epsilon > 0$, $P\left[|X_{t+h} - X_t| \geq \epsilon\right] \to 0$ as $h \to 0$.

4. **Cadlag process**
   It is right continuous with left limits as a function of $t$. 
• Lévy processes are a combination of a linear drift, a Brownian process, and a jump process. They are general enough to include a wealth of patterns and so they account for smile and skew effects in option pricing.

• When the Lévy process $X_t$ jumps, its jump magnitude is non-zero. The Lévy measure $w(dx)$ gives the arrival rate of jumps of size $(x, x + dx)$. The Lévy measure $w$ of $X_t$ defined on $\mathbb{R} \setminus \{0\}$ dictates how the jump occurs.
**Characteristic exponent**

The characteristic function of a Lévy process can be described by the Lévy-Khinchine representation

\[
\phi_X(u) = E[e^{iuX_t}] = \exp \left( aitu - \frac{\sigma^2}{2} tu^2 + t\int_{\mathbb{R}\setminus\{0\}} \left( e^{iux} - 1 - iux\mathbf{1}_{|x|\leq 1} \right) w(dx) \right) = \exp(t\psi_X(u)),
\]

where \( \int_{\mathbb{R}} \min(1, x^2) w(dx) < \infty, w(0) = 0, a \in \mathbb{R}, \sigma^2 \geq 0. \)

- We identify \( a \) as the drift rate and \( \sigma \) as the volatility of the diffusion process. Here, \( \psi_X(u) \) is called the characteristic exponent of \( X_t \).
• Actually, $X_t \overset{d}{=} tX_1$. All moments of $X_t$ can be derived from the characteristic function since it generalizes the moment-generating function to the complex domain.

• To ensure that the reinvested relative price $e^{qt}S(t)/B(t)$ is a martingale under $Q$, we need to ensure that

$$\phi_X(-i) = e^{(r-q)t}$$

so that

$$\mu = r - q - \frac{\sigma^2}{2} - \int_{\mathbb{R}} \left[ e^x - 1 - x1_{|x|<1} \right] w(dx).$$
• In the finite-activity models, we have $\int_{\mathbb{R}} w(dx) < \infty$. In the infinite-activity models, we observe $\int_{\mathbb{R}} w(dx) = \infty$ and the Poisson intensity cannot be defined.

<table>
<thead>
<tr>
<th>Model</th>
<th>Characteristic Function $\phi_{X_t}(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Finite-activity models</strong></td>
<td></td>
</tr>
<tr>
<td>Geometric Brownian motion</td>
<td>$\exp{iu\mu t - \frac{1}{2} \sigma^2 t u^2}$</td>
</tr>
<tr>
<td>Lognormal jump diffusion</td>
<td>$\exp\left{iu\mu t - \frac{1}{2} \sigma^2 t u^2 + \lambda t (e^{in\eta j} - \frac{1}{2} \sigma_j^2 u^2 - 1)\right}$</td>
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<td>Double exponential jump diffusion</td>
<td>$\exp\left{iu\mu t - \frac{1}{2} \sigma^2 t u^2 + \lambda t \left(\frac{1-\eta^2}{1+\eta^2} e^{iu\kappa} - 1\right)\right}$</td>
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<tr>
<td><strong>Infinite-activity models</strong></td>
<td></td>
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<tr>
<td>Variance gamma</td>
<td>$\exp(iu\mu t) \left(1 - iu\nu \theta + \frac{1}{2} \sigma^2 \nu u^2\right)\frac{\nu}{\nu^2}$</td>
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<tr>
<td>Normal inverse Gaussian</td>
<td>$\exp\left{iu\mu t + \delta t \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}\right}$</td>
</tr>
</tbody>
</table>
Option pricing in the Fourier domain

- When the characteristic function of the underlying asset price is known in closed form, option values can also be obtained using a single integration.

- Under the risk neutral measure $Q$, suppose the underlying asset price process assumes the form

$$S_t = S_0 \exp(rt + X_t), \quad t > 0,$$

where $X_t$ is a Lévy process and $r$ is the riskless interest rate.

Let $F_{V_T}$ denote the generalized Fourier transform of the terminal payoff function $V_T(x)$, where $x = \log S_T$. We have

$$F_{V_T}(z) = \int_{-\infty}^{\infty} e^{izx} V_T(x) \, dx,$$

where the dummy frequency argument $z$ is in general complex.
The European option value can be expressed as

\[ V(S_0, T) = e^{-rT} E_Q[V_T(x)] \]

\[ = \frac{e^{-rT}}{2\pi} E_Q \left[ \int_{i\mu - \infty}^{i\mu + \infty} e^{-izx} F_{V_T}(z) \, dz \right], \quad \mu = \text{Im } z. \]

Write \( Y = \log S_0 + rT \), we observe

\[ e^{-izx} = e^{-iz(Y+X_T)} \]

so that

\[ E_Q[e^{-izx}] = e^{-izY} E_Q[e^{i(-z)X_T}] = e^{-izY} \phi_{X_T}(-z), \]

where \( \phi_{X_T}(z) \) is the characteristic function of \( X_T \).
The European option value admits the following form as a complex integral:

\[ V(S_0, T) = \frac{e^{-rT}}{2\pi} \int_{i\mu - \infty}^{i\mu + \infty} e^{-izY} \phi_{XT}(-z) \mathcal{F}_{VT}(z) \, dz. \]

Let \( S_V \) denote the infinite horizontal strip in the complex \( z \)-plane where \( F_{VT}(z) \) is analytic. Similarly, \( \phi_{XT}(z)[\phi_{XT}(-z)] \) is analytic in the strip \( S_X \) (\( S^*_X \), conjugate strip of \( S_X \)).

The integrand is analytic in \( S_V \cap S^*_X \) and the contour path of integration must be chosen to lie completely inside the domain of analyticity.
Black-Scholes type formula

For the $T$-maturity European call option with terminal payoff $(S_T - K)^+$, its value is given (Lewis, 2001) by

$$C(S, T; K) = -Ke^{-rT} \int_{i\mu-\infty}^{i\mu+\infty} \frac{e^{-iz\kappa} \phi_X(-z)}{z^2 - iz} dz$$

$$= -Ke^{-rT} \left[ \int_{i\mu-\infty}^{i\mu+\infty} e^{-iz\kappa} \phi_X(-z) \frac{i}{z} dz ight]$$

$$- \int_{i\mu-\infty}^{i\mu+\infty} e^{-iz\kappa} \phi_X(-z) \frac{i}{z - i} dz$$

$$= S \left[ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{e^{iu\kappa} \phi_X(u - i)}{iu\phi_X(-i)} \right) du \right]$$

$$- Ke^{-rT} \left[ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{e^{iu\kappa} \phi_X(u)}{iu} \right) du \right],$$

where $\kappa = \log \frac{S}{K} + rT$. Here, $k$ is seen to be natural moneyness measure.
Proof

Suppose $S_X^* = \{z : \alpha \leq \Im z \leq \beta\}$, $\alpha < 0$ and $\beta > 1$, we choose $\mu \in (1, \beta)$. To evaluate the first integral, we move the contour to be along the real axis with an indentation at $z = 0$. We let $z = -u$ and consider the principal value of the Fourier integral

$$P \int_{-\infty}^{\infty} e^{iuk \phi_T(u)} \frac{i}{u} \, du.$$
We have

\[-\frac{Ke^{-rT}}{2\pi} \int_{i\mu-\infty}^{i\mu+\infty} e^{-izk} \phi_{X_T}(-z) \frac{i}{z} \, dz\]

\[= \frac{2\pi i}{2} Ke^{-rT} i + \frac{Ke^{-rT}}{2\pi} P \int_{-\infty}^{\infty} e^{iku} \phi_{T}(u) \frac{i}{u} \, du\]

\[= -Ke^{-rT} \left[ \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \text{Re} \left( \frac{e^{iku} \phi_{X_T}(u)}{iu} \right) \, du \right],\]

where the pole at \( z = 0 \) has residue \( -\frac{Ke^{-rT} i}{2\pi} \).

For the second integral, we choose the contour path to be \( z = i - u \) with an indentation at the pole \( z = i \). Note that the pole at \( z = i \) has residue \( \frac{Si}{2\pi} \).
Remark

- These integrands are integrable since $\phi_{X_T}(u)$ is an analytic function in a neighbourhood of $u = 0$ and $u = -i$. Due to the presence of the singularity at $u = 0$ in the integrand function, we cannot apply the FFT directly to evaluate the integrals.

- In the expansion of the integrals as Taylor series in $u$, the leading term in the expansion for both integral is $O\left(\frac{1}{u}\right)$. This is the source of the divergence, which arises from the discontinuity of the payoff function at $S_T = K$. As a consequence, the Fourier transform of the payoff function has large high frequency terms.
Carr-Madan's fast Fourier transform algorithm

- Carr and Madan (1999) propose to dampen the high frequency terms by multiplying the payoff by an exponential decay function.

- Let \( C_T(k) \) denote the price of a European call option with maturity \( T \) and strike \( K = e^k \). We have

\[
C_T(k) = \int_k^\infty e^{-rT}(e^s - e^k)q_T(s) \, ds,
\]

where \( q_T \) is the risk neutral density of \( s_T = \ln S_T \).
Formulation

We consider a modified function

\[ c_T(k) = e^{\alpha k} C_T(k) \]

which is square integrable for a suitable \( \alpha > 0 \) (choice of \( \alpha \) depends on the model for the asset price process). By taking the Fourier transform of \( c_T \) with respect to the log-strike price, we obtain

\[ C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty \frac{e^{-rT} \phi_T(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v} \, dv, \]

where \( \phi_T \) is the Fourier transform of \( q_T \).

- The damping parameter \( \alpha \) ensures that the damped call price is \( L^1 \)-integrable. Usually \( \alpha = 3 \) works well for various models.

- The denominator has only imaginary roots while integration is provided along real \( v \). Therefore, the integrand is well-behaved.
• A sufficient condition for $c_T$ to be square-integrable is given by $\psi_T(0)$ being finite. This is equivalent to
\[ E[S_T^{\alpha+1}] < \infty. \]

• Using Fourier inversion, the option price is obtained in terms of $\psi_T$
\[
C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty e^{-ivk}\psi(v) \, dv
\]
\[ \approx \frac{e^{-\alpha k}}{\pi} \sum_{j=0}^{N-1} e^{-iv_j k}\psi(v_j)\eta, \]
where $v_j = \eta j, j = 0, 1, \cdots, N - 1$, and $\eta > 0$ is the distance between the points of the integration grid.

• The FFT technique provides an efficient algorithm for computing the sums
\[
w_u = \sum_{j=0}^{N-1} e^{-i2\pi ju/N} x_j \quad \text{for } u = 0, 1, \cdots, N - 1.
\]
Control variate approach

- For strike prices that are far from the at-the-money level and for short maturities, the integrand may become highly oscillatory.

- To stabilize the numerical Fourier inversion, we use the Black-Scholes model as a control variate

\[ V^M = (V^M - V^{BS}) + V^{BS}. \]

This is done by subtracting the Black-Scholes characteristic function from the integrand and adding the Black-Scholes price. This works well if the characteristic functions of both models are close and we know an appropriate volatility level for the Black-Scholes model.

Rationale Remove a part of the FFT error by substituting it with the exact Black-Scholes solution.
Truncation and sampling errors

- Truncation errors – truncation of the infinite Fourier domain into a finite interval
- Sampling error – numerical approximation of the integral by a specified set of grid points

To avoid severe pricing errors, its application requires careful decision regarding the choice of the parameters. A comparison between the choices of $\Delta u$ and $\Delta k$ in the FFT calculations is called for here since $\Delta u \Delta k = \frac{2\pi}{N}$. That is, the choice of a finer grid $\Delta u$ leads to a larger spacing $\Delta k$ on the log strike.
Lee (2004) derives analytic error bound in the numerical evaluation of these Fourier integrals as $N$-point sums. He shows how these bounds lead to algorithms that make efficient choices of quadrature parameters and compute option prices with guaranteed numerical accuracy.

Application of the fractional FFT algorithm allows the grid sizes $\Delta u$ and $\Delta k$ to be chosen independently.
Fourier space-time (FST) stepping method

- For option pricing under Lévy models, the option price function is governed by a partial integral-differential equation (PIDE) where the integral terms in the equation arise from the jump components in the underlying Lévy process.

- The robustness of the FST method is shown with regard to its symmetric treatment of the jump terms and diffusion terms in the PIDE and the ease of incorporation of various forms of path dependence in the option models.

Advantages

- Unlike the usual finite difference schemes, the FST method does not require time stepping calculations between successive monitoring dates in pricing Bermudan options and discretely monitored barrier options.

- The FST method does not require the analytic expression for the Fourier transform of the terminal payoff of the option so it can deal easier with more exotic forms of the payoff functions.

- The FST method can be easily extended to multi-asset option models with exotic payoff structures and pricing models that allow regime switching in the underlying asset returns.
Algorithm

Let $S(t)$ denote a $d$-dimensional price index vector of the underlying assets in a multi-asset option model whose $T$-maturity payoff is denoted by $V_T(S(T))$. Suppose the underlying price index follows an exponential Lévy process, where

$$S(t) = S(0)e^{X(t)},$$

and $X(t)$ is a Lévy process.

Let the characteristic component of $X(t)$ be the triplet $(\mu, M, \nu)$, where $\mu$ is the non-adjusted drift vector, $M$ is the covariance matrix of the diffusion components, and $\nu$ is the $d$-dimensional Lévy density.
The Lévy process $X(t)$ can be decomposed into its diffusion and jump components as follows:

$$X(t) = \mu(t) + M W(t) + J^l(t) + \lim_{\epsilon \to 0} J^\epsilon(t),$$

where the large and small components are

$$J^l(t) = \int_0^t \int_{|y| \geq 1} y m(dy \times ds)$$

$$J^\epsilon(t) = \int_0^t \int_{\epsilon \leq |y| < 1} y [m(dy \times ds) - \nu(dy \times ds)],$$

respectively. Here, $W(t)$ is the vector of standard Brownian processes, $m(dy \times ds)$ is a Poisson random measure counting the number of jumps of size $y$ occurring at time $s$, and $\nu(dy \times ds)$ is the corresponding compensator.
Once the volatility and Lévy density are specified, the risk neutral drift can be determined by enforcing the risk neutral condition:

$$E_0[e^{X(1)}] = e^r,$$

where $r$ is the riskfree interest rate.

The governing partial integral-differential equation (PIDE) is

$$\frac{\partial V}{\partial t} + \mathcal{L}V = 0$$

with terminal condition: $V(X(T), T) = V_T(S(0)e^{X(T)})$, where $\mathcal{L}$ is the infinitesimal generator of the Lévy process. We have

$$\mathcal{L}f(x) = \left(\mu^T \frac{\partial}{\partial x} + \frac{\partial}{\partial x} M \frac{\partial}{\partial x}\right) f(x)$$

$$+ \int_{\mathbb{R}^n \setminus \{0\}} \left\{[f(x + y) - f(x)] - y^T \frac{\partial}{\partial x} f(x) \mathbf{1}_{|y|<1}\right\} \nu(dy).$$
By the Lévy-Khintchine formula, the characteristic component of the Lévy process is given by

\[ \psi_X(u) = i \mu^T u - \frac{1}{2} u^T M u \]

\[ + \int_{\mathbb{R}^n} \left( e^{iu^T y} - 1 - iu^T y \mathbf{1}_{|y| < 1} \right) \nu(dy), \]

that is, \( F[\mathcal{L}f(x)] = \psi_X(u) F[f](t, u). \)

By taking the Fourier transform on both sides of the PIDE, the PIDE is reduced to a system of ordinary differential equations parametrized by the \( d \)-dimensional frequency vector \( u \).

When we apply the Fourier transform to the infinitesimal generator \( \mathcal{L} \) of the process \( X(t) \), the Fourier transform can be visualized as a linear operator that maps spatial differentiation into multiplication by the factor \( iu \), where

\[ F[\partial_x^n f](t, u) = iu F[\partial_x^{n-1} f](t, u) = \cdots = (iu)^n F[f](t, u). \]
In the Fourier domain, \( \mathcal{F}[V] \) is governed by the following system of ordinary differential equations:

\[
\frac{\partial}{\partial t} \mathcal{F}[V](u, t) + \psi_X(u) \mathcal{F}[V](u, t) = 0
\]

with terminal condition: \( \mathcal{F}[V](u, T) = \mathcal{F}_{VT}(u, T) \). The option pricing problem is solved directly in the Fourier space.

- If there is no embedded optionality feature like the knock-out feature or early exercise feature between \( t \) and \( T \), then the above differential equation can be integrated in a single time step.

- By solving the PIDE in the Fourier domain and performing Fourier inversion afterwards, the price function of a European vanilla option with terminal payoff \( V_T \) can be formally represented by

\[
V(x, t) = \mathcal{F}^{-1} \left\{ \mathcal{F}[V_T](u, T) e^{\psi_X(u)(T-t)} \right\} (x, t).
\]
• Let $v_T$ and $v_t$ denote the $d$-dimensional vector of option values at maturity $T$ and time $t$, respectively, that are sampled at discrete spatial points in the real domain.

• The numerical evaluation of $v_t$ via the discrete Fourier transform and inversion can be formally represented by

$$v_t = \mathcal{FFT}^{-1}[\mathcal{FFT}[v_T] e^{\psi X(T-t)}],$$

where $\mathcal{FFT}$ denotes the multi-dimensional FFT transform. It is not necessary to know the analytic representation of the Fourier transform of the terminal payoff function. This is a great advantage for non-standard payoffs.
Summary

- Derivation of the PIDE for the option price.
- Transform the PIDE into ODE in the Fourier space and solve the ODE analytically.
- Apply FFT to switch between the real and Fourier space efficiently.
- Applicable to pricing of a wide range of path dependent multi-state derivatives.
<table>
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<th>Value</th>
<th>Change</th>
<th>log$_2$(ratio)</th>
<th>Time(ms)</th>
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Convergence results for pricing of a European put ($S = 100.0, K = 100.0, T = 10$) under the Merton jump-diffusion model ($\sigma = 0.15, \lambda = 0.1, \tilde{\mu} = -1.08, \tilde{\sigma} = 0.4, r = 0.05, q = 0.02$). Reference price of 18.0034. The order of convergence is 2 in space.
Discretely monitored barrier options

Consider the pricing of a discretely monitored barrier option where the knock-out feature is activated at the set of discrete time points $\mathcal{X}$.

- Between times $t_n$ and $t_{n+1}$, $n = 1, 2, \ldots, N$, the barrier option behaves like a European vanilla option so that the single step integration can be performed from $t_n$ to $t_{n+1}$.

- At time $t_n$, we impose the contractual specification of the knock-out feature. Say, the option is knocked out when $S$ stays above the up-and-out barrier $B$.

- Let $R$ denote the rebate paid upon the occurrence of knock-out, and $v^n$ be the vector of option values at discrete spatial points.
The time stepping algorithm can be succinctly represented by

\[ v^n = H_B(\mathcal{F} \mathcal{F} T^{-1}[\mathcal{F} \mathcal{F} T[v^{n+1}] e^{\psi X(t_{n+1} - t_n)}]), \]

where the knock-out feature is imposed by defining \( H_B \) to be

\[ H_B(v) = v \mathbf{1}_{\{x < \log \frac{B}{S(0)}\}} + R \mathbf{1}_{\{x \geq \log \frac{B}{S(0)}\}}. \]

No time stepping is required between two successive monitoring dates.
The boundary conditions are applied in the real space while the time stepping procedure is performed in the Fourier space.
**Discrete Barrier Option**

<table>
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<tr>
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</table>

- Option: Up-and-out barrier call $S = 100, K = 105, T = 0.5, B = 115, R = 0.5$ with daily monitoring

- Model: Kou jump-diffusion with mean reversion
  $\sigma = 0.3, \lambda = 4.0, \eta_p = 0.95, \eta = 0.1, \theta = 92.0, \kappa = 5.0, r = 0.06$

- Monte Carlo: 0.58289924 – 95% CI width of 0.0028937@56 sec.
Pricing of American/Bermudan style options

- Early exercise feature
  - American options can be exercised at any time.
  - Bermudan options can only be exercised at certain dates in the future.

- Define the set of exercise dates as
  \[ T = \{t_1, t_2, \cdots, t_M\} \]
  and \( 0 = t_0 \leq t_1 \). Let \( R(t, S(t)) \) denote the exercise payoff.

- Write \( C \) as the continuation value of the option, and \( V \) as the value of the option immediately before the exercise opportunity.
Dynamic programming procedure

The Bermudan option price can be found via backward induction as

\[
\begin{align*}
V(t_M, S(t_M)) &= R(t_M, S(t_M)) \\
C(t_m, S(t_m)) &= e^{-r\Delta t} E_t m[V(t_{m+1}, S(t_{m+1}))], \quad m = M - 1, \ldots, 1, \\
V(t_m, S(t_m)) &= \max\{C(t_m, S(t_m)), R(t_m, S(t_m))\}.
\end{align*}
\]

Let \( f(y|S(t_m)) \) represents the probability density describing the transition from \( S(t_m) \) at \( t_m \) to \( y \) at \( t_{m+1} \), then

\[
C(t_m, S(t_m)) = e^{-r\Delta t} \int_{-\infty}^{\infty} V(t_{m+1}, y) f(y|S(t_m)) \, dy.
\]

- The overall complexity is \( O(MN^2) \) for an \( M \)-times exercisable Bermudan option with \( N \) grid points used to discretize the price of the underlying asset.
Suppose the asset price process observes the independent increments property, then
\[ f(x_{m+1}|x_m) = f(x_{m+1} - x_m). \]
Write \( z = x_{m+1} - x_m \), then
\[ C(t_m, S(t_m)) = e^{-r\Delta t} \int_{-\infty}^{\infty} V(t_{m+1}, S(t_m) + z)f(z)\,dz. \]
Define the convolution between two integrable functions \( f(x) \) and \( g(x) \) as
\[ h(x) = f * g(x) = \int_{-\infty}^{\infty} f(y)g(x - y)\,dy, \]
then
\[ \mathcal{F}_h = \mathcal{F}_f \mathcal{F}_g. \]
Let \( c(t_m, x) = e^{\alpha x + r\Delta t}C(t_m, x) \) be the dampen continuation value with a damping factor \( \alpha > 0 \), then by virtue of the convolution formula:
\[
\mathcal{F}_x \{ c(t_m, x) \}(u) = \int_{-\infty}^{\infty} e^{iux} e^{\alpha x} \int_{-\infty}^{\infty} V(t_{m+1}, x + z)f(z)\,dz\,dx \\
= \mathcal{F}_y \{ v(t_{m+1}, y) \}(u) \phi(-(u - i\alpha)).
\]
Here, \( \phi(u) \) is the characteristic function of the random variable \( z \). Consider the dampen option value

\[
v(t, y) = e^{\alpha y} V(t, y) = e^{\alpha y} \max\{C(t, y), R(t, y)\},
\]
whose Fourier transform is given by

\[
\hat{v}(t, u) = \mathcal{F}_y \{\hat{v}(t, y)\}(u) = \int_{-\infty}^{\infty} e^{iuy} v(t, y) \, dy.
\]
Finally, the dampen continuation value can be obtained from Fourier inversion:

\[
c(t_m, x) = \mathcal{F}_u^{-1}\{\hat{v}(t_{m+1}, u)\phi(-(u - i\alpha))\}(x).
\]

The basic essence is the calculation of the above convolution:

\[
c(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \hat{v}(u)\phi(-(u - i\alpha)) \, du.
\]
Pseudo code of CONV method

\[ V(t_M, x) = R(t_M, x) \] for all \( x \).

For \( m = M - 1 \) to 0

1. Dampen \( V(t_{m+1}, x) \) with \( e^{\alpha x} \) and take its Fourier transform
2. Calculate \( \mathcal{F}\{e^{\alpha y}V(t_{m+1}, y)\}(u)\phi(-(u - i\alpha)) \)
3. Calculate \( C(t_m, x) \) by applying Fourier inversion to the above convolution and undamping

\[ V(t_m, x) = \max\{R(t_m, x), C(t_m, x)\} \]

Next \( m \)
Calculation of hedge parameters

\[ \Delta = \frac{\partial V}{\partial S} = \frac{1}{S} \frac{\partial V}{\partial x}, \quad \Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{1}{S^2} \left( -\frac{\partial V}{\partial x} + \frac{\partial^2 V}{\partial x^2} \right). \]

Define

\[ \mathcal{F}\{e^{\alpha x} V(t_0, t)\} = e^{-r\Delta t} A(u), \text{ where} \]
\[ A(u) = \mathcal{F}\{e^{\alpha y} V(t_1, y)\} \phi(-u + i\alpha), t_1 > 0. \text{ Then} \]
\[ \Delta = \frac{e^{-\alpha x} e^{-r\Delta t}}{S} \left[ \mathcal{F}\{-iuA(u)\} - \alpha \mathcal{F}^{-1}\{A(u)\} \right] \]

and

\[ \Gamma = \frac{e^{-\alpha x} e^{-r\Delta t}}{S^2} \left[ \mathcal{F}^{-1}\{(-iu)^2 A(u)\} - (1 + 2\alpha) \mathcal{F}^{-1}\{-iuA(u)\} \right. \]
\[ \left. + \alpha(\alpha + 1) \mathcal{F}^{-1}\{A(u)\} \right]. \]

Since differentiation is exact in the Fourier space, the rate of convergence of the Greeks will be the same as that of the option value (advantage over finite difference / lattice tree / Monte Carlo calculations).
Markov chain approximation method

- The trinomial tree model is inadequate to approximate Lévy process since such process cannot be fully characterized by the first two order moments. A remedy is to increase the number of branches.

Drawbacks

- The number of nodes grows rapidly with the number of time steps.

- The multinomial tree may not be a recombining tree.

- The calculations of the probabilities of the branching processes by moment matching may be very tedious.
FFT network trees

- As an extension of the multinomial tree model using the Markov property of Lévy processes, we approximate a Lévy process by a finite state Markov chain (Duan and Simonato, 2001).

- The number of states is fixed in advance and remains unchanged at all time points.

- The FFT network requires the computation of probabilities from $S_i$ to $S_j$ for all $i$ and $j$. 
A network model with 3 time steps and 7 states.
Transition probability matrix

- The transition probabilities are defined as
  \[ P[X_{t+\Delta t} = x_j | X_t = x_i] = p_{ij} \]
  where \( p_{ij} \geq 0 \) and \( \sum_{j=1}^{N} p_{ij} = 1, i = 1, 2, \cdots, N \).
  Once the transition probabilities are known, we can perform option valuation using the usual discounted expectation approach.

- The characteristic function for \( x_j \) is
  \[ \phi_j(u) = E[e^{iuX_t} | X_t = x_j] = \int_{-\infty}^{\infty} e^{iu x_k} f(x_k | x_j) \, dx_k, \]
  where \( f(x_k | x_j) \) denotes the probability density function of \( X_{t+\Delta t} = x_k \) conditional on \( X_t = x_j \).
• Conversely, if the characteristic function is known, \( f(x_k|x_j) \) can be recovered by

\[
f(x_k|x_j) = \mathcal{F}^{-1}\{\phi_j(u)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux_k} \phi(u) \, du.
\]

One may apply FFT to perform the Fourier inversion.

• Let \( \tilde{f}(x_k|x_j) \) denote the numerical approximation to \( f(x_k|x_j) \), then

\[
p_{jk} \approx \frac{\tilde{f}(x_k|x_j)}{\sum_{k=1}^{N} \tilde{f}(x_k|x_j)}.
\]

• The incorporation of various path dependent features can be performed using the forward shooting grid techniques.
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<th>Price</th>
<th>Time (sec.)</th>
<th>Diff.</th>
<th>Price</th>
<th>Time (sec.)</th>
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American Down-and-out Call Option
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American Geometric Asian Call Option Prices

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<td>Diff.</td>
<td>Price</td>
</tr>
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American Arithmetic Asian Call Option Prices
Summary

- FFT finds its applications when we consider option pricing in the Fourier domains – the underlying asset price processes follow the Lévy processes or stochastic volatility models.

- Dampen techniques are commonly adopted to avoid singularity in the integrand function that arises from the terminal payoff structure of options.

- CONV method uses the independent increments property of Lévy processes and the convolution formula of Fourier transform. Hedge parameters can be calculated at ease.

- In the Markov chain approximation method, the larger number of branches in the FFT-based network approach can provide better accuracy to approximate the Lévy process with jumps when compared to the usual trinomial tree approach.
• The Fourier space time stepping method solves the governing partial integral-differential equation of option pricing. Unlike usual finite difference schemes, no time stepping procedures are required between successive monitoring instants in option models with discretely monitored features.

• Both the FFT-based network method and the Fourier space time stepping techniques allow greater flexibility in the construction of the numerical algorithms to handle various form of path dependence of the underlying asset price processes through the incorporation of the auxiliary conditions that arise from modeling the embedded optionality features.

• Successful numerical implementation requires the ingenious choice of parameters, like the damping coefficient, sampling size, and interval of truncation, etc.