Intensity-based framework and penalty approximation for optimal stopping problems in pricing mortgage loans

Yue Kuen KWOK
Department of Mathematics

Hong Kong University of Science and Technology

This is a joint work with Min Dai and Hong You
Mortgage loans

A mortgagor owes a scheduled stream of cash flow as liabilities. The mortgage loan is amortized over the life of the loan.

Prepayment (refinancing) option

The right to terminate the contract prematurely by paying the remaining principal plus any applicable transaction costs.

Mortgage-backed securities

Provides its owner a share in the cash flows from a pool of mortgages. Valuation and hedging of these securities require a model of mortgage prepayment behavior since this determines the timing of cash flows from the mortgage pool.
Bundle of risks

- Interest rate risk

- Default risk – not significant since the loan is collateralized by the property

- Prepayment risk – reinvestment at lower prevailing interest rate
Empirical features of mortgage prepayment ("inexact science")

1. Seasonality and other macro-economic factors.
2. Some mortgages are (not) prepaid even when their contract rate is below (above) the current mortgage rates.
3. Prepayment appears to be dependent on some burnout factors.

- Burnout refers to the dependence of expected prepayment rates on cumulative historical prepayment levels. The higher the fraction of the pool that has already prepaid, the less likely for those remaining in the pool to prepay.
Absence of default and prepayment risk

The mortgagor takes a loan of $F_0$ at time 0 and assumes the obligation to pay scheduled coupon at the rate $c_t \geq 0$ continuously for the duration $T$ of the contract.

Let $m_t$ denote the mortgage contract rate originated at time 0. In the absence of default and prepayment, the outstanding principal is

$$F(t) = F_0e^{\int_0^t m_\theta d\theta} - \int_0^t c_s e^{\int_s^t m_\theta d\theta} ds.$$
• The *exercise price* of the prepayment option is time varying with value

\[ F(t)(1 + X) \]

where \( F(t) \) is the remaining principal balance and \( X \) is the fraction of transaction cost with respect to the principal.

• *Transaction costs* are interpreted in a wide sense, including monetary as well as psychological costs (convenience to go through the refinancing procedures).
Statement of the problem

1. Pricing formulation of a mortgage loan with prepayment risk.
   • Formulated as optimal stopping problem or free boundary value problem.

2. Intensity-based approach of rational prepayment model.
   • Formulated as the penalty approximation of variational inequalities formulation – intensity of prepayment can be identified as penalty parameter.


4. Construction of second order accurate numerical schemes.
Mathematical setup

Define a filtered probability space \((\Omega, G, \{G_t\}_{t \geq 0}, Q)\), where

- \(\Omega\) is the set of all possible outcomes;
- \(G_t\) is the \(\sigma\)-algebra representing all observations available to borrower at time \(t\);
- \(Q\) is the probability measure on \(G\).

The prepayment time \(\tau\) is the first jump time of a Cox process. It is a stopping time on the filtered probability space, that is, at any arbitrary time \(t\), we can tell whether prepayment has “occurred” based on the given information \(G_t\).
Intensity of the random time

\{\mathcal{F}_t\}_{t \geq 0} is the filtration that models the flow of observations available to the lender prior to the prepayment time \( \tau \). Note that \( \tau \) is not an \( \mathcal{F}_t \)-stopping time.

The process \( \Gamma_t = -\ln[1 - Q(\tau \leq t|\mathcal{F}_t)] \) is called the hazard process of the random time \( \tau \). Equivalently,

\[
Q(\tau \leq t|\mathcal{F}_t) = e^{-\Gamma_t}.
\]

Assuming that the process \( \Gamma_t \) is an absolutely continuous process so that

\[
\Gamma_t = \int_0^t \gamma_\theta \, d\theta
\]

for some process \( \gamma_t \), called the intensity of the random time \( \tau \).
Value of mortgage security

Find the time $t$ liability value $L_t$ that pays a coupon payment continuously with the rate of $c_t$ up to time $\tau \wedge T$ and $Z_\tau$ in a lump sum at time $\tau$, if $\tau \leq T$. The liability value of the mortgage loan is

$$L_t = E\left[\int_t^{\tau \wedge T} c_s e^{-\int_0^s r_\theta d\theta} \, ds + 1_{\{t \leq \tau \leq T\}} Z_\tau e^{-\int_t^\tau r_\theta d\theta} \bigg| G_t\right]$$

$$= 1_{\{\tau > t\}} E\left[\int_t^T (c_s + Z_s \gamma_s) e^{-\int_t^s (r_\theta + \gamma_\theta) d\theta} \, ds \bigg| F_t\right],$$

Taking $Z_t = F(t)$, then

$$L_t = F(t) + E\left[\int_t^T (m_s - r_s) F(s) e^{-\int_t^s (r_\theta + \gamma_\theta) d\theta} \, ds \bigg| F_t\right].$$
Stanton’s model of prepayment behaviors (1995)

Let $\psi(t)$ be the payout by the mortgagor upon prepayment, where $\psi(t) = F(t)(1 + X)$.

- **Prepayment due to exogenous reasons**

  The mortgagor may prepay for exogenous reasons, such as divorce or job relocation or sale of the house. This is modeled by a Poisson process with intensity $\lambda$ (*prepay when it is non-profitable*).

- **Prepayment due to endogenous (financial) strategy**

  The mortgagor may consider to refinance when $L_t > \psi(t)$. The endogenous prepayment is modeled by a Poisson process with intensity $\rho 1_{\{L_t > \psi(t)\}}$, $\rho$ is a constant. Here, $\rho$ is finite which indicates that mortgagor may not prepay even when it is profitable.
The prepayment time $\tau$ is the minimum of these two independent random times, so the prepayment intensity is

$$\gamma_t = \begin{cases} 
\lambda & \text{if } L_t \leq \psi(t) \\
\lambda + \rho & \text{if } L_t > \psi(t)
\end{cases},$$

with dependence on $L_t$.

For simplicity, we take the interest rate to be the single stochastic state variable to model market risk. The short rate $r(t)$ under the risk neutral measure $Q$ is governed by

$$dr = \mu_r(r, t) \, dt + \sigma_r(r, t) \, dZ.$$

Let $c(t)$ be the continuous stream of amortized cash flows.
**Optimal stopping formulation**

Assuming the mortgagor to be fully rational, the optimal endogenous strategy is to exercise the prepayment right when $L_t$ reaches $\psi(t)$ – corresponding to infinite value of $\rho$.

Let $\mathcal{L}$ denote the operator: $\mathcal{L} = \mu_r \frac{\partial}{\partial r} + \frac{\sigma_r^2}{2} \frac{\partial^2}{\partial r^2} - (r + \lambda)$.

The linear complementarity (variational inequalities) formulation of the optimal stopping problem is

$$\frac{\partial L}{\partial t} + \mathcal{L}L + c(t) + \lambda \psi(t) \geq 0$$

$$L - \psi(t) \leq 0$$

$$\left[ \frac{\partial L}{\partial t} + \mathcal{L}L + c(t) + \lambda \psi(t) \right] [L - \psi(t)] = 0. \quad (1)$$
Intensity/penalty formulation

The governing equation for \( L(r,t) \) is derived from

\[
rL dt = E_t[dL + c(t) dt],
\]
where \( E_t \) is the expectation under \( Q \) conditional on the filtration \( G_t \).

\[
\frac{\partial L}{\partial t} + \mu_r \frac{\partial L}{\partial r} + \frac{\sigma_r^2}{2} \frac{\partial^2 L}{\partial r^2} - (r + \lambda)L + c(t) + \lambda \psi(t) = \rho \max(L - \psi(t), 0),
\]
with auxiliary conditions

\[
L(r, T) = 0 \quad \text{and} \quad \lim_{r \to \infty} L(r, t) = 0.
\]

The differential equation formulation in eq. (2) can be visualized as the penalty approximation to the variational inequalities formulation in eq. (1). Here, the penalty parameter \( \rho \) is interpreted as the intensity of endogenous prepayment.
Fair value of the mortgage, $M(r, t)$

The difference between $L(r, t)$ and $M(r, t)$ arises since the mortgagor pays $P(t)(r + t)$ at the time of prepayment while the mortgagor loan equals $P(t)$ upon prepayment.

\[
\frac{\partial M}{\partial t} + \mu_r \frac{\partial M}{\partial r} + \frac{\sigma_r^2}{2} + \frac{\partial^2 M}{\partial r^2} - (r + \lambda)M + c(t) + \lambda P(t) = \rho[M - P(t)] \mathbf{1}_{\{L > \psi(t)\}}.
\]

Given the known solution to $L(r, t)$, the differential equation for $M(r, t)$ is linear.
Monotonicity properties with respect to $\rho$

Let $L_i, i = 1, 2$, be the solution to

$$\left(\frac{\partial}{\partial t} + \mathcal{L}\right)L_i + f_i(t) = \rho_i \max(L_i - \psi, 0), \quad i = 1, 2,$$

sharing the same set of initial-boundary conditions. Suppose $\rho_1 \geq \rho_2 > 0$ and $f_1(t) \leq f_2(t)$, using the comparison principle in partial differential equation, we have

$$L_1 \leq L_2 \quad \text{for all values of } r \text{ and } t.$$
Separating boundary

It is obvious that $\frac{\partial L}{\partial r} \leq 0$. We define

$$r^*(t) = \min\{r : L(r, t) < F(t)(1 + X)\}.$$

Consequently, $r^*(t)$ is a decreasing function of $\rho$. This is because a higher level value of $\rho$ leads to smaller value of $L(r, t)$, so lower level of interest rate is required in order that $L(r, t)$ reaches the value $\psi(t) = F(t)(1 + X)$. 

Plot of the separating boundary $r^*(t)$ against time to expiry $T - t$ with varying values of the penalty parameter $\rho$. With a higher value of $\rho$, $r^*(t)$ assumes a lower value.
Monotonicity property with respect to $X$

Assume $X_1$ and $X_2$ to be constant. If $X_1 < X_2$, then

$$r^*(t; X_1) \geq r^*(t; X_2).$$

With a higher transaction cost, the interest rate has to be lowered further in order to increase the value of liability $L$ to the level of prepayment payout $\psi(t)$.

The proof requires the following property on $F(t)$:

$$-\frac{dF(t)}{dt} + rF(t) \geq 0.$$

This property holds since the mortgage loan is amortized throughout the life.
Plot of the separating boundary $r^*(t)$ against time to expiry $T - t$ with varying values of the transaction cost factor $X$. With a higher value of $X$, $r^*(t)$ assumes a lower value.
Second order accurate numerical schemes for solving the penalty formulation

Let $U^n_j$ denote the discrete numerical approximation to $U(S_j, \tau_n)$, where $S_j = S_0 + j\Delta S, j = 1, 2, \cdots, N_S$, and $\tau_n = n\Delta \tau, n = 1, 2, \cdots, N_\tau$. Define the spatial difference operator $\mathcal{L}_h$ by

$$\mathcal{L}_h U^n_j = \frac{\sigma^2}{2} S_j^2 \frac{U^n_{j+1} - 2U^n_j + U^n_{j-1}}{\Delta S^2} + rS_j \frac{U^n_{j+1} - U^n_{j-1}}{2\Delta S} - rU^n_j.$$

Crank-Nicolson discretization

$$U^{n+1}_j = U^n_j + \frac{\Delta \tau}{2} \mathcal{L}_h U^{n+1}_j + \mathcal{L}_h U^n_j + \xi^{n+1}[\phi(S_j) - U^{n+1}_j]$$
Scheme One

The penalty term is discretized at \((n + \frac{1}{2})^{th}\) time level, so that it now becomes

\[
\xi^{n+\frac{1}{2}} \left[ \phi(S_j) - \frac{U_{j}^{n+1} + U_{j}^{n}}{2} \right]
\]

where

\[
\xi^{n+\frac{1}{2}} = \begin{cases} 
\rho \Delta \tau & \text{if } \phi(S_j) > \frac{U_{j}^{n+1} + U_{j}^{n}}{2} \\
0 & \text{otherwise}
\end{cases}
\]

Since the penalty term is non-linear, one has to solve a non-linear system of algebraic equations at each time step.
Scheme Two

The non-linearity in the penalty term disappears when we replace the implicit term \((U_j^{n+1} + U_j^n)/2\) by the explicit term \((3U_j^n - U_j^{n-1})/2\).

The discretized penalty term is given by

\[
\hat{\xi}^{n+\frac{1}{2}} = \frac{1}{2} \left[ \phi(S_j) - \frac{3U_j^n - U_j^{n-1}}{2} \right],
\]

where

\[
\hat{\xi}^{n+\frac{1}{2}} = \begin{cases} 
\rho \Delta \tau & \text{if } \phi(S_j) > \frac{3U_j^n - U_j^{n-1}}{2} \\
0 & \text{otherwise}
\end{cases}
\]

Note that the new scheme is a three-level scheme.
Examination of the rate of convergence of numerical calculations of pricing a mortgage loan using Stanton’s scheme and our proposed modified Crank-Nicolson schemes.

<table>
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<tr>
<th>$N_\tau$</th>
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<th>Stanton’s Scheme</th>
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<th>Scheme Two</th>
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Proof of convergence of Scheme One

Let $U^n = (U_1^n \cdots U_{N_S}^n)$ and $\Phi = (\phi(S_1) \cdots \phi(S_{N_S}))$, then Scheme One can be expressed as

$$U^{n+1} - U^n + A \frac{U^{n+1} + U^n}{2} = \Lambda^{n+\frac{1}{2}} \left( \Phi - \frac{U^{n+1} + U^n}{2} \right),$$

where $A$ is the matrix representing the difference operator $\mathcal{L}_h$ and $\Lambda^{n+\frac{1}{2}}$ is a diagonal matrix whose diagonal entry $\Lambda^{n+\frac{1}{2}}_{jj}$ is

$$\Lambda^{n+\frac{1}{2}}_{jj} = \begin{cases} \rho \Delta \tau & \text{if } \phi(S_j) > \frac{U_j^{n+1} + U_j^n}{2}, \\ 0 & \text{otherwise} \end{cases}, \quad j = 1, 2, \cdots, N_S.$$
To establish the convergence of the above numerical scheme, it is necessary to show that $U^n$ satisfies the following set of discrete variational inequalities:

$$U^{n+1} - U^n + A\left(\frac{U^{n+1} + U^n}{2}\right) \geq 0$$

$$\frac{U^{n+1} + U^n}{2} - \Phi \geq -\frac{C}{\rho \Delta \tau}$$

$$\left[U^{n+1} - U^n + A\left(\frac{U^{n+1} + U^n}{2}\right) = 0\right] \lor \left(\left|\frac{U^{n+1} + U^n}{2} - \Phi\right| \leq \frac{C}{\rho \Delta \tau}\right),$$

where $C$ is a positive constant independent of $\rho \Delta \tau, \Delta \tau$ and $\Delta S$. 
Concluding remarks

• When optimality of exercising the prepayment right is assumed, the pricing formulation constitutes an optimal stopping problem.

• Suppose the propensity of mortgagor’s prepayment is modeled by the intensity of a Poisson process, the pricing formulation resembles the penalty approximation approach of solving the linear complementarity formulation of an optimal stopping problem.

• We have deduced several monotonicity properties of separating boundaries with respect to the intensity of prepayment and level of transaction cost.

• Two versions of second order finite difference schemes for solving the penalty approximation have been proposed.