Interaction of optimal policies of call and conversion in convertible bonds

presented by

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Convertible bonds

Combination of bonds and equities – bond plus a conversion option

* bondholder has the right to convert the bond into common shares at some contractual price (conversion number may change over time)

**Holder's perspective:** take advantage of the future potential growth of issuer’s company

**Issuer’s perspective:** raise capital at a lower cost by the provision of conversion privilege to bondholders
Equity perspective on convertibles

- To take advantage of the upside potential growth of the underlying shock (participation into equity).

- Swapping the variable stock dividends in return for fixed coupon payments until the early of the maturity date and the conversion date.

Fixed income perspective on convertibles

- Provides the “bond floor” value.

- Conversion option that allows the investor to exchange the straight bond for fixed number of shares.
• **Call (redeemable) provision** embedded in *convertible bonds* gives the right to the issuer to redeem the bond at a pre-determined *call price*. Upon call, the holder can either convert the bond or redeem at the call price.

• **Notice period requirement**

  When the warrant or bond is called, the holder is given a notice period (say, 30 days) to decide whether to receive the cash or convert into shares.
Issuer’s perspective on the call right

- To have the flexibility to call if they think they can refinance the debt more cheaply.

- To force bondholders to convert debt into equity, which can reduce debt levels and result a beneficial effect on the balance sheet. The issuer has the flexibility to shift debt into equity to reduce the leverage of the firm.

In summary, it is used as a tool by issuer for possible future equity financing and managing the debt / equity balance.
Call protection

**Hard** (or absolute):

To protect the bond from being called for a certain period of time.

**Soft** (or provisional): The issuer is allowed to call only when certain conditions are satisfied.

For example, the closing price of stock has been in excess of 150% of the conversion price on any 20 trading days within 30 consecutive days.
Role of call protection

To preserve the value of the equity option for the bondholders.

While waiting for the stock price to increase, convertibles typically provide more income than the stock. Without the call protection, this income stream could be called away at any time. Hard call protection with the longest possible duration is most desirable for investors.
Redemption at the option of the bond issuer

On or after July 19, 1998, the Issuer may redeem the Bonds at any time in whole or in part at the principal amount of each Bond, together with accrued interest, if for each of 30 consecutive Trading Days, the last of which Trading Days is not less than five nor more than 30 days prior to the day upon which the notice of redemption is first published, the closing price of the Shares as quoted on the Hong Kong Stock Exchange shall have at least 130 percent of the Conversion Price in effect on such Trading Day.
Soft call protection

Parisian feature

The closing price has to be above 130 percent of the conversion price on consecutive 30 trading days.

- On the date of issuance of the notice of redemption, the Issuer looks back 5 to 30 days to check whether the history of the stock price path satisfies the Parisian constraint.

- Effectively, when the Parisian constraint has been satisfied, the Issuer has 5 to 30 days to make the decision on redemption or not.
Questions

1. How does the optimal call policy of issuer interact with the optimal conversion policy of the holder?

2. What is the impact of the notice period requirement on the optimal call policy?
Optimal stopping problems

Pricing of a derivative with embedded early conversion right can be formulated as an optimal stopping problem

\[ V(x,t) = \sup_{t^* \in \mathcal{D}_{x,t}^T} I_{x,t}(t^*), \quad x \in \mathbb{R}^n, \quad 0 \leq t \leq T, \]

where the stochastic state variables \( X_u \) are governed by the stochastic differential equation

\[ dX_u = \alpha(X_u, u) \, du + \sigma(X_u, u) \, dZ_u, \quad X_t = x. \]

\[ I_{x,t}(t^*) = E_{x,t} \left[ e^{-\int_t^{t^*} r(X_u, u) \, du} \psi(X_{t^*}, t^*) \right]. \]

Here, \( \psi \) is the reward function and \( \mathcal{D}_{x,t}^T \) is the set of stopping times, \( t \leq t^* \leq T \).
The holder chooses $t^*$ so as to maximize the expected payoff $I_{x,t}(t^*)$. The optimal stopping time is the infimum among all stopping times such that

$$V(x,t^*) = I_{x,t}(t^*).$$

For game options, the issuer also has the right to pay the penalty function $\phi$ to terminate the contract prematurely. Let $\hat{t}$ denote the optimal stopping time chosen by the issuer. The reward/penalty functional is given by

$$J_{x,t}(t^*,\hat{t}) = E_{x,t} \left[ e^{-\int_{t^*}^{\hat{t}} r(X_{u,u}) \, du} \psi(X_{t^*,t^*}) 1_{\{t^* < \hat{t}\}} ight. + e^{-\int_{t}^{t^*} r(X_{u,u}) \, du} \phi(X_{t^*,t^*}) 1_{\{\hat{t} \leq t^*\}} \right]$$

where $t^*, \hat{t} \in D_{x,t}^T$.

The issuer (holder) chooses $\hat{t}$ ($t^*$) to minimize (maximize) $J_{x,t}(t^*,\hat{t})$. 
Early theoretical results

Ingersoll (1977) and Brennan and Schwartz (1977) conclude that a callable convertible security should be called as soon as its conversion value equals the call price.

Delayed call phenomena

- Empirical evidence suggest that convertible bonds are usually called late:
  \[ S^{\text{actual}}(\tau) > S^{\text{theor}}(\tau). \]

- Corporate finance explanations: safety premium hypothesis, signaling hypothesis, tax advantage, delayed equity financing, etc.

- We argue that part of the “delayed calling” may be attributed to the under-estimation of \( S^{\text{*}}(\tau) \) in earlier contingent claims models.
List of notations

\( V(S, \tau) = \) value of the callable convertible security

\( V_0(S; X) = \) conversion payoff of the convertible security, \( X \) may denote the strike price in a warrant or the par value in a convertible bond

\( V_T(S; X) = \) terminal payoff of the convertible security

\( K = \) pre-determined call price

\( \tau_n = \) length of the notice period

Upon calling, the holder essentially receives a European option with terminal payoff \( \max(V_0(S; X), K) \) and time to expiry \( \tau_n \). The value of this vested European option is denoted by \( c_n(S, \tau_n) \).
Assumptions

1. Usual Black-Scholes pricing framework

\[
\frac{dS}{S} = (r - q) \, dt + \sigma \, dZ
\]

- \( r \) = constant riskless interest rate
- \( q \) = constant dividend yield
- \( \sigma \) = constant volatility

2. Assume zero coupon and no default risk of the issuer.

Black-Scholes pricing model

\[
\mathcal{L} V = \left[ \frac{\partial}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2}{\partial S^2} - (r - q) S \frac{\partial}{\partial S} + r \right] V = 0, \quad 0 < S < \infty, \tau > 0
\]

\[
V(S, 0) = V_T(S; X), \quad \tau = T - t.
\]

Let \( \mathcal{D} = \{(S, \tau) : 0 < S < \infty, \tau > 0\} \) denote the domain of the pricing model.
During the life of the convertible security, the contract may be terminated prematurely either by

1. holder’s early conversion
2. issuer’s premature redemption

Domain $\mathcal{D}$ is divided into three regions: $R_{\text{cont}}, R_{\text{conv}}$ and $R_{\text{call}}$.

I. In the continuation region

$$\mathcal{L}V = 0, \quad V - V_0(S; X) > 0 \quad \text{and} \quad V - c_n(S, \tau_n) < 0.$$ 

II. In the conversion region $R_{\text{conv}}$, the holder chooses to convert optimally.

$$V - V_0(S; X) = 0, \quad \mathcal{L}V > 0 \quad \text{and} \quad V - c_n(S, \tau_n) < 0.$$ 

III. In the call region $R_{\text{call}}$, the security is optimally called by the issuer.

$$V - c_n(S, \tau_n) = 0, \quad \mathcal{L}V < 0 \quad \text{and} \quad V - V_0(S; X) > 0.$$
Variational inequalities formulation

Combination of optimal call by issuer and optimal conversion by holder:

\[
\max(V - c_n(S, \tau_n), \min(\mathcal{L}V, V - V_0(S, X))) = 0
\]

(1)

with terminal payoff

\[
V(S, 0) = \min(V_T(S; X), c_n(S, \tau_n)).
\]

(2)

Observations

Consider the following two mutually exclusive and exhaustive events corresponding to calling or no calling.

1. When \((S, \tau) \notin R_{call}, V < c_n(S, \tau_n)\). With only conversion right, the variational inequalities formulation is

\[
\min(\mathcal{L}V, V - V_0(S; X)) = 0,
\]

so that Eq. (1) satisfied.

2. When \((S, \tau) \in R_{call}, V = c_n(S, \tau_n), \mathcal{L}V < 0\) and \(V - V_0(S; X) > 0\) so that Eq. (1) is again satisfied.
Callable convertible bond

Assume that the face value is $X$ and conversion number is unity. Payoff upon conversion $B_0(S; X) = S$ and terminal payoff

$$B_T(S; X) = \max(X, S) = X + (S - X)^+.$$

The vested European option received by the bondholder upon calling has a life span of $\tau_n$ and terminal payoff $\max(K + X, S)$ so that

$$\tilde{c}_n(S, \tau_n) = (K + X)e^{-r\tau_n} + c_E(S, \tau_n; K + X).$$

The lower bound on the bond value $B(S, \tau)$ is $\max(Xe^{-r\tau}, S)$.

$$\max(Xe^{-r\tau}, S) \leq B(S, \tau) \leq \tilde{c}_n(S, \tau_n).$$

The pricing model constitutes a free boundary value problem with upper and lower obstacle functions.
The price function of a callable convertible bond is plotted against $S$ for varying values of $\tau$. When $\tau = 0.25$ and $\tau = 10$, the bond price curves intersect the conversion value line (shown as dotted-dashed line) and the cap value curve $\tilde{c}_n(S, \tau_n)$ (shown as dashed line), respectively. When $\tau = 1.5$, the price curve of the bond ends at the intersection point of the conversion value line and the cap value curve.
- Since the lower bound is time dependent, $B(S, \tau)$ does not observe monotonicity in $\tau$. Also, we observe the loss of monotonicity in $\tau$ of $\tilde{S}_{conv}^*(\tau)$ and $\tilde{S}_{call}^*(\tau)$.

- Define $\tilde{S}$ to be the solution (unique) to

$$\tilde{c}_n(S, \tau_n) = \max(Xe^{-r\tau}, S)$$

which can be simplified into

$$\tilde{c}_n(S, \tau_n) = S$$

since $\tilde{c}_n(S, \tau_n) > Xe^{-r\tau}$. The upper and lower obstacle functions intersect at $\tilde{S}$.

- $\tilde{S}$ exists when $q > 0$ but solution does not exist when $q = 0$.

- Both $\tilde{S}_{conv}^*(\tau)$ and $\tilde{S}_{call}^*(\tau)$ are bounded above by $\tilde{S}$.

- If we assume $(K + X)e^{-r\tau_n} \geq X$, then $\tilde{S} > X$ since

$$\tilde{S} = c_n(\tilde{S}, \tau_n) > (K + X)e^{-r\tau_n} > X.$$
The critical stock price $\tilde{S}_{\text{conv}}^*(\tau)$ is plotted against $\tau$ for a callable convertible bond. Note that $\tilde{S}_{\text{conv}}^*(\tau)$ stays below $\tilde{S}$ (see the dotted line).
Properties of critical stock price $S_b^*(\tau)$ at optimal conversion

Assume $q > 0$ and no callabel features. $S_b^*(\tau)$ and $S^*(\tau)$ are the optimal exercise price of the convertible bond and American call, respectively.

1. $S_b^*(\tau) \leq S^*(\tau)$ for $\tau \geq 0$,
2. $S_b^*(0^+) = X$,
3. $S_b^*(\infty) = 0$.

Proof

1. Let $\tilde{W}(S, \tau) = B(S, \tau) - X$, then $\tilde{W}(S, \tau)$ satisfies

$$\mathcal{L}\tilde{W} \geq -rX \quad \text{and} \quad \tilde{W} \geq S - X$$

$$(\mathcal{L}\tilde{W} + rX)[\tilde{W} - (S - X)] = 0$$

and $\tilde{W}(S, 0) = (S - X)^+$. By comparison principle, $\tilde{W}(S, \tau) \leq c_A(S, \tau)$. Since the price curve of $\tilde{W}(S, \tau)$ always stays below that of $c_A(S, \tau)$, it intersects the intrinsic value line $S - X$ at a lower critical stock price, so

$$S_b^*(\tau) \leq S^*(\tau) \quad \text{for all} \quad \tau \geq 0.$$
2. $S^*_b(\tau) \geq X e^{-r\tau}$ since the lower bound of $B(S, \tau)$ is $\max(X e^{-r\tau}, S)$. As $\tau \to 0^+$, we have $S^*_b(0^+) \geq X$.

Suppose $S^*_b(0^+) > X$, then there exists $S$ satisfying $X < S < S^*_b(0^+)$ such that the bond remains alive at $\tau \to 0^+$. By continuity, $B(S, 0^+) = S$. Substituting the solution into the Black-Scholes equation, we have $\frac{\partial B}{\partial \tau}(S, 0^+) = -qS < 0$. A contradiction is encountered so $S^*_b(0^+) \leq X$. Combining the results, we obtain $S^*_b(0^+) = X$.

3. When $\tau \to \infty$, the convertible bond becomes essentially equivalent to the American call option with zero strike. Recall that $S^*(\infty) = \frac{\mu_+}{\mu_+ - 1} X$ so that $S^*_b(\infty) = 0$ since $X = 0$. 
Optimal conversion and calling policies and their interaction

Three possible scenarios

1. Conversion right becomes non-effective.

2. Always non-optimal for the issuer to call so that the bond becomes a non-callable convertible bond.

3. Over some part of the bond’s life, optimal call may occur prior to optimal conversion, and vice versa for the other part of the life.

- When \( q = 0 \), the first scenario always occurs.
- When \( q > 0 \), either the second or the third scenario occurs. Which one does occur would depend on the relative magnitude of \( \tilde{S} \) and the maximum value of \( S_b^*(\tau) \) over the bond’s life.
Non-effective conversion right

This occurs if and only if $q = 0$.

1. When $q = 0$, early conversion leads to loss on the insurance value but no gain from the earlier procession of shares.

2. Recall that conversion right becomes non-effective when $\tilde{S}$ is less that $\min S^*_b(\tau)$. However, $S^*_b(\tau) \to 0$ as $\tau \to \infty$ when $q > 0$. Therefore, the conversion right would not be rendered worthless in convertible bond when $q > 0$ since $\tilde{S}$ cannot be less than the minimum value of $S^*_b(\tau)$. 
Non-effective call privilege

Write $S^*_{b,max} = \max_{\tau \in [0,\infty)} S^*_b(\tau)$. The call privilege would be forfeited when $\tilde{S} > S^*_{b,max}$. This occurs when $K$ is of sufficiently high value, $B(S, \tau) < \tilde{c}_n(S, \tau_n)$ for all $\tau$. We then have $\tilde{S}^*_{\text{conv}}(\tau) = S^*_b(\tau)$ for all $\tau$. 
The critical stock price $\tilde{S}_{\text{conv}}^*(\tau)$ of early conversion (shown as dotted-dashed curves) and $\tilde{S}_{\text{call}}^*(\tau)$ of premature calling (shown as solid curve) are plotted against $\tau$ for a callable convertible bond. Both $\tilde{S}_{\text{conv}}^*(\tau)$ and $\tilde{S}_{\text{call}}^*(\tau)$ are bounded above by $\tilde{S}$ (see the dotted line).
Interaction of optimal calling and conversion

1. When $\tilde{S} < S_{b,\text{max}}^*$, both optimal calling and conversion can occur during the life of a callable convertible bond. Recall that $S_{b}^*(\tau)$ starts at $S^*(0^+) = X$, increases to a maximum peak value then decreases to zero as $\tau \to \infty$. Together with $\tilde{S} > X$, we can deduce that when $\tilde{S} < S_{b,\text{max}}^*$, there exist two values of $\tau$ such that $S_{b,\text{max}}^*(\tau) = \tilde{S}$.

2. Let the smaller of these two critical values of $\tau$ be denoted by $\hat{\tau}_{\text{small}}$, and note that $\tilde{S} \geq S_{b}^*(\tau)$ for $\tau \leq \hat{\tau}_{\text{small}}$. 
3. $\tau \leq \hat{\tau}_{small}$, the issuer would not call the convertible bond when $S < S^*_b(\tau)$ since the bond value is less than $\tilde{c}_n(S, \tau_n)$. On the other hand, the bond will be converted into shares optimally by the holder when $S$ reaches $S^*_b(\tau)$.

4. When $\tau$ increases beyond $\hat{\tau}_{small}$, we have $\tilde{S} < S^*_b(\tau)$. In this case, optimal calling by issuer will occur prior to premature conversion. This phenomenon persists for some time interval.

5. Since $S^*_b(\tau)$ tends to zero as $\tau \to \infty$, we expect that optimal conversion prior to optimal calling occurs again when the time to expiry continues to increase beyond some higher threshold value.
Callable American warrants

Payoff functions: $V_0(S; X) = S - X$ and $V_T(S; X) = (S - X)^+$ so that the lower bound on the warrant value is $(S - X)^+$.

For the upper bound, we observe that the payoff of the vested European option at the end of the notice period is

$$\max(S - X, K) = K + (S - K - X)^+$$

so that

$$c_n(S, \tau_n) = Ke^{-r\tau_n} + c_E(S, \tau_n; K + X).$$

The optimal call policy of a callable American warrant is strongly dependent on the dividend policy of the stock price.

1. non-dividend paying
2. continuous dividend yield
3. discrete dividend payments.
Some results on \( S^*(\tau) \) of non-callable American warrant

\[ S^*(\tau) \] is a monotonically increasing function of \( \tau \) with

\[ S^*(0^+) = X \max \left( 1, \frac{r}{q} \right) \quad \text{and} \quad S^*(\infty) = \frac{\mu_+}{\mu_+ - 1} X \]

where \( \mu_+ \) is the positive root of the quadratic equation

\[ \frac{\sigma^2}{2} \mu^2 + \left( r - q - \frac{\sigma^2}{2} \right) \mu - r = 0. \]
\( W(S, \tau) \) = price function of the callable American warrant

\[(S - X)^+ \leq W(S, \tau) \leq c_n(S, \tau_n).\]

Let \( \hat{S} \) denote the unique solution to

\[c_n(S, \tau_n) = S - X.\]

(i) When \( q > 0 \), \( \hat{S} \) always exists. Also, it is an increasing function of \( K \).

(ii) When \( q = 0 \), there is no solution since \( c_n(S, \tau_n) > (S - X)^+ \) for all \( S \).

Both \( S_{\text{conv}}^*(\tau) \) and \( S_{\text{call}}^*(\tau) \) are bounded above by \( \hat{S} \).
Underlying stock is non-dividend paying

- The holder should never exercise the warrant prematurely.

$S^*_{\text{call}}(\tau)$ exhibits the following properties.

1. $S^*_{\text{call}}(\tau)$ does not exist for $\tau \leq \tau_n$.
2. $S^*_{\text{call}}(\tau)$ always exists for $\tau > \tau_n$.
3. $S^*_{\text{call}}(\tau)$ is monotonically decreasing with respect to $\tau$.
4. $S^*_{\text{call}}(\tau) \to \infty$ as $\tau \to \tau^+_n$.
5. $S^*_{\text{call}}(\infty)$ is finite; it is determined by solving

$$N\left(\frac{\ln \frac{S^*_{\text{call}}(\infty)}{K+X} + \left(r - \frac{\sigma^2}{2}\right)\tau_n}{\sigma \sqrt{\tau_n}}\right) = \frac{K}{K+X}.$$  

Proof of (1). Since $W(S, \tau) \leq c_E(S, \tau)$ and for $\tau \leq \tau_n$, we have

$$c_n(S, \tau_n) > c_E(S, \tau) \geq W(S, \tau)$$

so that the issuer never calls prematurely.
The critical stock price $S^{\text{call}}_{\text{call}}(\tau)$ of a callable American warrant on a non-dividend paying stock is plotted against $\tau$. $S^{\text{call}}_{\text{call}}(\tau)$ is a monotonically decreasing function of $\tau$ since $W(S, \tau)$ is a monotonic increasing function of $\tau$. Note that $S^{\text{call}}_{\text{call}}(\tau)$ does not exist for $\tau \leq \tau_n$ and $S^{\text{call}}_{\text{call}}(\tau) \to \infty$ as $\tau \to \tau_n^+$. 
\( \hat{S} \geq S^*(\infty) \) — call right rendered worthless \([K \text{ is of relatively high value}]\)

\[
W(S, \tau) \leq W_{\text{non}}(S, \tau) \leq W_{\text{non}}(S, \infty) \leq c_n(S, \tau_n)
\]

- The call by the issuer would increase the warrant value to \(c_n(S, \tau_n)\), thus premature calling is always non-optimal.
- When the call right is forfeited, then the warrant is terminated prematurely by optimal conversion so that

\[
S_{\text{conv}}^*(\tau) = S^*(\tau).
\]
\( \hat{S} \leq S^*(0^+) \) – conversion right is rendered worthless \([K \text{ is of relatively low value}]

Suppose the conversion region exists, we have \( V = S - X \) where \( S \leq \hat{S} \leq S^*(0^+) \). However, \( \mathcal{L}V = \mathcal{L}(S - X) \leq 0 \) when \( S \leq S^*(0^+) \). This contradicts with the requirement that \( \mathcal{L}V > 0 \) in \( R_{\text{conv}} \). Hence the early conversion right is rendered worthless.

**Properties of \( S_{\text{call}}^*(\tau) \)**

1. \( S_{\text{call}}^*(\tau) \) always exists for \( \tau \geq 0 \) and \( S_{\text{call}}^*(\tau) \) is a decreasing function of \( \tau \).

2. \( S_{\text{call}}^*(\infty) \leq S_{\text{call}}^*(\tau) \leq \hat{S} \)

Here, \( S_{\text{call}}^*(\infty) \) is the solution to

\[
K e^{-r\tau_n} + (\mu_+ - 1)S_{\text{call}}^*(\infty)e^{-q\tau_n} N(d_1) - (K + X)e^{-r\tau_n} N(d_2) = 0,
\]

(3)

where

\[
d_1 = \ln \frac{S_{\text{call}}^*(\infty)}{K + X} + \left( r - q - \frac{\sigma^2}{2} \right) \tau_n \quad \text{and} \quad d_2 = d_1 - \sigma \sqrt{\tau_n}.
\]
The critical stock price $S^*_\text{call}(\tau)$ is plotted against $\tau$ for a callable American warrant, corresponding to $\hat{S} < S^*(0^+)$. In this case, it is always non-optimal for the holder to exercise the warrant prematurely. The asymptotic lower bound $S^*_\text{call}(\infty)$ is found to be 1.4958. For a small time interval near expiry, $S^*_\text{call}(\tau)$ becomes equal to the upper bound $\hat{S}$ (whose value is found to be 1.9983).
Interaction of the optimal call and conversion rights

\[ S^*(0^+) < \hat{S} < S^*(\infty) \]

existence of unique \( \hat{\tau} \) such that \( S^*(\hat{\tau}) = \hat{S} \).

(i) When \( \tau < \hat{\tau} \), the issuer would not call the warrant when \( S < S^*(\tau) \) since \( W(S, \tau) < c_n(S, \tau_n) \).

(ii) When \( \tau > \hat{\tau} \), the warrant is terminated prematurely due to calling.
**Explanation to part (ii)**

- When \( \tau > \hat{\tau} \), the warrant remains unexercised when \( S \) reaches \( \hat{S} \).

- The warrant is always called when \( S > \hat{S} \). Assume the contrary, suppose the warrant is not called when \( S > \hat{S} \), then \( W > S - X > c_n(S, \tau_n) \). This is not possible since \( W \) is bounded above by \( c_n(S, \tau_n) \).

1. When \( \tau \) is slightly above \( \hat{\tau} \), \( S^{\text{call}}_\tau(\tau) = \hat{S} \) for certain period of time.

2. As \( \tau \) increases further, \( S^{\text{call}}_\tau \) then starts to decrease monotonically with \( \tau \). \( S^{\text{call}}_\tau(\tau) \) is bounded from below by \( S^{\text{call}}_\tau(\infty) \), where \( S^{\text{call}}_\tau(\infty) \) is the solution to Eq. (3).

3. The scenario of \( S^{\text{call}}_\tau(\tau) = \hat{S} \) corresponds to the situation where the warrant price curve, intrinsic value line and the curve of \( c_n(S, \tau_n) \) all intersect at \( \hat{S} \).
The price function of a callable American warrant is plotted against $S$ for varying values of $\tau$, corresponding to $S^*(0^+) < \hat{S} < S^*(\infty)$. When $\tau = 0.25$, the price curve intersects the intrinsic value line signifying that premature termination is due to early conversion. When $\tau = 1$ and $\tau = 5$, the price curve intersects the curve of $c_n(S, \tau_n)$ signifying that premature termination is due to calling by issuer.
Figure 1e The critical stock price $S^*_\text{conv}(\tau)$ of early exercise (shown as dotted-dashed curve) and $S^*_\text{call}(\tau)$ of call (shown as solid curve) are plotted against $\tau$ for a callable American warrant, corresponding to $S^*(0^+) < \hat{S} < S^*(\infty)$. There is a time interval within which $S^*_\text{call}(\tau) = \hat{S}$. 
Underlying asset pays discrete dividends

- Holder chooses to exercise only at times right before the dividend dates.

- Within the time interval between successive dividend dates, the warrant may be terminated prematurely by issuer’s call.

- $S_{call}^*$ is monotonically decreasing with respect $\tau$ and $S_{call}^*(\tau)$ does not exist over time interval of width $\tau_n$ right after the dividend date, a phenomena similar to the case $q = 0$. 
Conclusion

1. The calling right allows the issuer to place a cap on the derivative value, where the cap value is the payoff received by the holder upon calling.

2. With the presence of the notice period requirement, the payoff upon calling has dependence on the stock price. The stock price dependence on the cap provides a richer set of patterns of interaction of optimal calling and conversion policies.

3. The critical stock prices at optimal conversion or optimal calling are bounded above by some threshold stock price, whose value is equal to the stock price at which the cap and floor values are equal.

4. The optimal calling and conversion policies and their interaction depend on
   (i) the dividend policy of the underlying stock price and
   (ii) the relative magnitude of the call price (for American warrant) with respect to the strike price or face value (for convertible bond).
5. When the stock is non-dividend paying, premature conversion by the holder is always sub-optimal so that the conversion right is rendered worthless.

6. Conversion privilege also becomes essentially non-effective when the call price is sufficiently low so that optimal calling by issuer always occurs prior to premature conversion.

7. When the call price is too high so that optimal holder's conversion always occurs prior to issuer’s calling, this would render the call right worthless.

8. When the call price assumes value that is intermediate between the upper and lower threshold levels as stated in the above two cases, both optimal conversion and calling would occur during the life of the derivative.